and

$$\mathbb{P}\left(\tau_k = -\mathbf{e}_{\underline{i}}\right) = \frac{1}{2(n+1)^k} \sum_{\substack{\ell_0+\cdots+\ell_n=k \\ \ell_j \text{ odd if } 1\leq j\in\underline{i}, \ell_j \text{ even if } 1\leq j\notin\underline{i}}} \binom{k}{\ell_0,\ldots,\ell_n} - \frac{1}{2}\mathbb{E}\left(\langle\tau_k, \mathbf{e}_{\underline{i}}\rangle\right), \quad (2.28)$$

where

$$\mathbb{E}\left(\langle\tau_k, \mathbf{e}_{\underline{i}}\rangle\right) = \begin{cases} \dfrac{1}{(n+1)^k} \displaystyle\sum_{\substack{0\leq j\leq k \\ j \text{ even}}} \binom{k}{j}(2p-n)^{j/2} & \text{if } \underline{i} = \emptyset, \\ \dfrac{1}{(n+1)^k} \displaystyle\sum_{\substack{1\leq j\leq k \\ j \text{ odd}}} \binom{k}{j}(2p-n)^{(j-1)/2} & \text{if } |\underline{i}| = 1, \\ 0 & \text{in all other cases.} \end{cases} \quad (2.29)$$

Proof. The result follows from the following observation:

$$\begin{aligned} \mathbb{P}\left(\tau_k = \mathbf{e}_{\underline{i}}\right) = t_{\underline{i}}^+(k) &= \frac{1}{2}\left[\left(t_{\underline{i}}^+(k) + t_{\underline{i}}^-(k)\right) + \left(t_{\underline{i}}^+(k) - t_{\underline{i}}^-(k)\right)\right] \\ &= \frac{1}{2}\left[\mathbb{P}(\tau_k = \pm\mathbf{e}_{\underline{i}}) + \left\langle\langle\tau_k\rangle, \mathbf{e}_{\underline{i}}\right\rangle\right] \quad (2.30) \end{aligned}$$

Similarly,

$$\mathbb{P}\left(\tau_k = -\mathbf{e}_{\underline{i}}\right) = t_{\underline{i}}^-(k) = \frac{1}{2}\left[\left(t_{\underline{i}}^+(k) + t_{\underline{i}}^-(k)\right) - \left(t_{\underline{i}}^+(k) - t_{\underline{i}}^-(k)\right)\right]. \quad (2.31)$$

□

Theorem 2.10. *The random walk $(\tau_k)_{k\geq 0}$ satisfies*

$$\tau_k \xrightarrow{\mathcal{D}} \mathcal{U}\left(\{\pm\mathbf{e}_{\underline{i}}\}\right). \quad (2.32)$$

Proof. Because $\langle\tau_k\rangle \to 0$, sufficiently large values of k give

$$\mathbb{P}\left(\tau_k = \mathbf{e}_{\underline{i}}\right) = \frac{1}{2(n+1)^k} \sum_{\substack{\ell_0+\cdots+\ell_n=k \\ \ell_j \text{ odd if } 1\leq j\in\underline{i}, \ell_j \text{ even if } 1\leq j\notin\underline{i}}} \binom{k}{\ell_0,\ldots,\ell_n} + o(\varepsilon). \quad (2.33)$$

Moreover,

$$\mathbb{P}\left(\tau_k = -\mathbf{e}_{\underline{i}}\right) = \frac{1}{2(n+1)^k} \sum_{\substack{\ell_0+\cdots+\ell_n=k \\ \ell_j \text{ odd if } 1\leq j\in\underline{i}, \ell_j \text{ even if } 1\leq j\notin\underline{i}}} \binom{k}{\ell_0,\ldots,\ell_n} + o(\varepsilon). \quad (2.34)$$

Passing to binary representations of subsets $\underline{i}$, each blade $\mathbf{e}_{\underline{i}} \in \mathcal{C}\ell_{p,q}$ is uniquely associated with a vertex of the n-dimensional hypercube. By identifying each pair $\pm\mathbf{e}_{\underline{i}}$, the walk (τ_k) induces a walk on the n-dimensional hypercube.