On Quantum Iterated Function Systems

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Abstract

Quantum Iterated Function System on a complex projective space is defined by a family of linear operators on a complex Hilbert space. The operators define both the maps and their probabilities by one algebraic formula. Examples with conformal maps (relativistic boosts) on the two-sphere - CP(1) are discussed in some details.

1 Introduction

Iterated Function Systems generate fractal sets due to non-commutativity of maps. In quantum theory position and momentum operators do not commute (which leads to the well known Heisenberg's uncertainty relations), and also different components of spin do not commute. This suggests that fractal patterns and chaos may arise as a result of certain quantum measurement processes. In the present note I will briefly describe the present status of this new research avenue and point at some open questions. While no knowledge of quantum theory will be required for the reader, comments in the footnotes and in the Conclusions will be addressed to those readers who would like to have a broader view of the subject. The flagship example of an iterated function system (in short: IFS) is the Sierpinski fractal. generated by random application of 3×3 matrices $A[i], i = 1, 2, 3$ to the vector:

$$v_0 = \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} \tag{1}$$

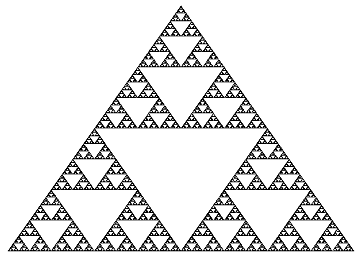

Figure 1: The classical fractal: Sierpinski Triangle generated by an Iterated Function System.

where $A[i]$ is given by

$$A[i] = \begin{pmatrix} 0.5 & 0 & ax_i \\ 0 & 0.5 & ay_i \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

and $ax_1 = 0.0, ay_1 = 0.0, ax_2 = 0.5, ay_2 = 0.0, ax_3 = 0.25, ay_3 = 0.5$. (Our 3×3 matrices encode affine transformations - usually separated into a 2×2 matrix and a translation vector.) At each step one of the three transformations $A[i], i = 1, 2, 3$ is selected with probability $p[i] = 1/3$. After each transformation the transformed vector is plotted on the (x, y) plane.

2 Algebraic preliminaries.

The important property of the maps $A[i]$ is that they are contractions. Siepinski triangle, as well as another well known example, the fern [1], live on a 2-dimensional plane. Quantum iterated function systems (QIFS) live complex projective spaces, the simplest one being $CP(1)$ - a 2-dimensional sphere S^2. Affine transformations form a natural group of transformations acting on the plane. What is the natural group of transformations acting on the sphere? One would think it is the rotation group $O(3)$. But rotations are volume preserving and they would not mimick contractions. The next candidate in line is the Lorentz group $O(3,1)$. It is not so well known that the Lorentz group acts on the sphere in a natural way. One way to see that this is the case is to notice that the Lorentz group is the group preserving the space-time metric $s^2 = -x_0^2 + x_1^2 + x_2^2 + x_3^2$ and thus the light cone $C = \{x = (x^0, x^1, x^2, x^3) : -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 0\}$, and therefore, because it acts as linear transformations, also the projective light cone $\mathbb{P}C$, that is the set of equivalence classes $\hat{x} : x \in C$, with respect to the equivalence relation $R \subset (C \setminus \{0\}) \times (C \setminus \{0\})$ where xRy iff $x = \lambda y, \lambda \neq 0$. Each equivalence class has a unique representative with $x^0 = 1$, so $\mathbb{P}C$ can be identified with the sphere $S^2 = \{\mathbf{n} \in \mathbb{R}^3 : \mathbf{n}^2 = 1\}$. The Lorentz group $O(3,1)$ consists of 4×4 real matrices $\Lambda = (\Lambda^\mu{}_\nu)$ satisfying $\Lambda^T \eta \Lambda = \eta$, where $\eta = (\eta_{\mu\nu}) = diag(-1, 1, 1, 1)$ is the diagonal metric matrix. The action $S^2 \ni \mathbf{n} \mapsto \Lambda(\mathbf{n})$ of $O(3,1)$ on S^2 is given explicitly by the formula:

$$\Lambda(\mathbf{n})^i = \frac{\Lambda^i{}_0 + \Lambda^i{}_j n^j}{\Lambda^0{}_0 + \Lambda^0{}_j n^j}, \tag{3}$$

(we will always use Einstein's summation convention over repeated indices). The group $O(3,1)$ has four connected components. We will need only the connected component of the identity $SO_+(3,1)$ consisting of those matrices Λ in $O(3,1)$ for which $\det(\Lambda) = 1$ and $\Lambda^0{}_0 > 0$. The group $SO_+(3,1)$, though connected, is not simply connected. It's simply connected double covering group is the group $SL(2,\mathbb{C})$ of 2×2 complex matrices of determinant 1. By polar decomposition every matrix $A \in SL(2,\mathbb{C})$ can be uniquely decomposed $A = PU$ into a positive part P and a unitary part $U \in SU(2)$.[1] The group $SU(2)$ is the double covering of the rotation group $SO(3)$. Nontrivial positive matrices in $SL(2,\mathbb{C})$ have two eigenvalues $\lambda_1 < 1$ and $\lambda_2 = 1/\lambda_1 > 1$. It is the positive matrices in $SL(2,\mathbb{C})$ that will generate our iterated function systems.

To describe the $2:1$ group homomorphism $A \mapsto \Lambda(A)$ from $SL(2,\mathbb{C})$ to $SO_+(3,1)$, and also to describe algberaically the action of $SL(2,\mathbb{C})$ on S^2 it is convenient to use the Pauli spin matrices σ_0, σ_1, σ_2, σ_3 defined by

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The homomorphism $SL(2,\mathbb{C}) \to SO_+(3,1)$ is then given by the formula:

$$\Lambda(A)^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu A \sigma_\nu A^\star), \tag{4}$$

where $A^\star$ denotes the Hermitian conjugate of A. Every Hermitian 2×2 matrix X can be uniquely represented as $X = x^\mu \sigma_\mu$, with x^μ real. For every $\epsilon \in [0,1]$, and every unit length vector $\mathbf{n} \in S^2$ let

$$P(\mathbf{n}, \epsilon) = \frac{1}{2}(I + \epsilon\sigma(\mathbf{n})), \tag{5}$$

[1] In Relativity the positive matrices represent "Lorentz boosts."

where $\sigma(\mathbf{n}) \doteq n^1\sigma_1 + n^2\sigma_2 + n^3\sigma^3$. It is easy to see that a Hermitian matrix $X \neq I$ is an idempotent if and only if it is of the form $X = P(\mathbf{n}, 1)$ for some $\mathbf{n} \in S^2$. We will write $P(\mathbf{n}) \doteq P(\mathbf{n}, 1)$. It is also easy to check that a matrix P is positive if and only if it is of the form $P = c\ P(\mathbf{n}, \epsilon)$, for some $c > 0,\ \epsilon \in [0, 1],\ \mathbf{n} \in S^2$. Notice that $\det(P) = 1$ if and only if $\epsilon < 1$ and $c = 2(1 - \epsilon^2)^{-1/2}$. We will use the matrices $P(\mathbf{n}, \epsilon)$, with the same ϵ but different vectors $\mathbf{n}$ to generate IFS-s on S^2.

The formula (3) describing the action of the Lorentz group on S^2 is not the most convenient one for our needs. Another way of describing the same action is by noticing that, for $\mathbf{r} \in S^2$, the following identity holds[2]

$$P(\mathbf{n}, \epsilon)P(\mathbf{r})P(\mathbf{n}, \epsilon) = \lambda(\epsilon, \mathbf{n}, \mathbf{r})P(\mathbf{r}'), \tag{6}$$

where $\lambda(\epsilon, \mathbf{n}, \mathbf{r}) \geq 0$ is a given by:

$$\lambda(\epsilon, \mathbf{n}, \mathbf{r}) = \frac{1 + \epsilon^2 + 2\epsilon(\mathbf{n} \cdot \mathbf{r})}{4}, \tag{7}$$

while

$$S^2 \ni \mathbf{r}' = \frac{(1 - \epsilon^2)\mathbf{r} + 2\epsilon(1 + \epsilon(\mathbf{n} \cdot \mathbf{r}))\mathbf{n}}{1 + \epsilon^2 + 2\epsilon(\mathbf{n} \cdot \mathbf{r})} \tag{8}$$

where $(\mathbf{n} \cdot \mathbf{r})$ denotes the scalar product

$$\mathbf{n} \cdot \mathbf{r} = n_1 r_1 + n_2 r_2 + n_3 r_3. \tag{9}$$

The map $\mathbf{r} \mapsto \mathbf{r}'$ is the same as the one described in Eq. (3), with $\Lambda = \Lambda\left(2P(\mathbf{n}, \epsilon)/\sqrt{1 - \epsilon^2}\right)$. Notice that that the dilation coefficient $2/\sqrt{1 - \epsilon^2}$ is not important here, because it would cancel out anyway in Eq.(3). The transformation $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ implemented by Λ can be explicitly described by the formula known from texts on special relativity:

$$\begin{aligned} x^{0\prime} &= \cosh(\alpha)x^0 + \sinh(\alpha)(\mathbf{x} \cdot \mathbf{n}), \\ \mathbf{x}' &= \mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + [\sinh(\alpha)x^0 + \cosh(\alpha)(\mathbf{x} \cdot \mathbf{n})]\mathbf{n}, \end{aligned} \tag{10}$$

where the "velocity" $\beta = \tanh(\alpha) = 2\epsilon/(1 + \epsilon^2)$.[3] What is important for us, is the fact that the coefficient $\lambda(\epsilon, \mathbf{n}, \mathbf{r})$ in Eq.(6) is positive, and thus can be interpreted as a (relative) probability associated with the transformation $\mathbf{r} \mapsto \mathbf{r}'$. In other words: relative probabilities associated to maps

[2] A more general formula is discussed in Sec. 5.4, Eq. (22)

[3] Notice that for $\epsilon \to 1$, $\beta \to 1$ - the velocity of light. In this limit the maps $\mathbf{r} \mapsto \mathbf{r}'$ degenerate to $\mathbf{r} \mapsto \mathbf{n}$ and become non-invertible.

implemented by $P(\mathbf{n},\epsilon)$ are naturally associated with the maps. It should be noticed that positivity of λ is guaranteed by the algebraic properties of the operators involved. Indeed, because $P(\mathbf{n},\epsilon) = P(\mathbf{n},\epsilon)^\star$, and because $P(\mathbf{r}) = P(\mathbf{r})^\star = P(\mathbf{r})^2$ is an orthogonal projection, the right hand side in Eq.(6) can be represented as $A^\star A$, with $A = P(\mathbf{r})P(\mathbf{n},\epsilon)$, and is, therefore, automatically positive.

3 Quantum Iterated Function Systems on S^2.

Given a sequence $\mathbf{n}_i$, $i = 1, \ldots, N$ of vectors in S^2 we associate with this sequence an iterated function system $\{w_i, p_i\}$ on S^2, with place dependent prababilities, defined as follows:

$$w_i(\mathbf{r}) = \mathbf{r}' = \frac{(1-\epsilon^2)\mathbf{r} + 2\epsilon(1+\epsilon(\mathbf{n}_i\cdot\mathbf{r}))\mathbf{n}_i}{1+\epsilon^2+2\epsilon(\mathbf{n}_i\cdot\mathbf{r})}, \tag{11}$$

$$p_i(\mathbf{r}) = \frac{\lambda(\epsilon,\mathbf{n}_i,\mathbf{r})}{\sum_{j=1}^{N}\lambda(\epsilon,\mathbf{n}_j,\mathbf{r})}. \tag{12}$$

System $\{w_i, p_i\}$ defined by these formulae will be called a *Quantum Iterated Function Systems* or *QIFS* . [4] The formula for the probabilities simplifies whenever

$$\sum_{i=1}^{N}\mathbf{n}_i = 0. \tag{13}$$

In the following we will always assume that the vectors $\mathbf{n}_i$ defining the transformations w_i add to zero. In this case p_i are given by:

$$p_i(\mathbf{r}) = \frac{1+\epsilon^2+2\epsilon(\mathbf{n}_i\cdot\mathbf{r})}{N(1+\epsilon^2)}. \tag{14}$$

In [2] we examined QIFS corresponding to several most symmetric configurations, where the vectors $\mathbf{n}_i$ were placed at the vertices of regular polyhedra: tetrahedron (4), octahedron (6), cube (8), icosahedron (12), dodecahedron (20), double tetrahedron (8), icosidodecahedron (30). In each case numerical simulation of the Markov process, starting with a random original point, lead to fractal-like patterns on the sphere. For ϵ close to 1 the operators $P(\mathbf{n},\epsilon)$ are close to projections, therefore the attraction centers are very distinctive. For ϵ close to 0 the operators $P(\mathbf{n},\epsilon)$ induce maps close to the

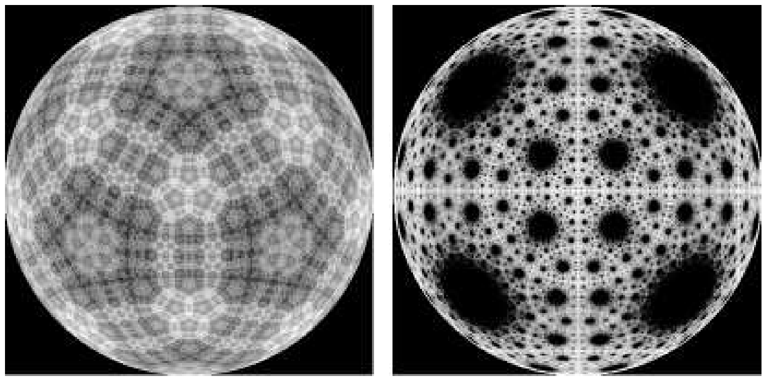

Figure 2: Quantum Dodecahedron ($\epsilon = 0.78$) and Quantum Octahedron ($\epsilon = 0.58$) The darker the place, the smaller probability of it being visited.

identity map - the patterns are fuzzy. Typical patterns are shown in Fig. 1.

It seems that the fractal dimension depends on the value of ϵ. The Hausdorff dimension of the limit set, for the tetrahedral case, has been numerically estimated in Ref. [3] and shown to decrease from 1.44 to 0.49 while ϵ increases from 0.75 to 0.95.

4 Transfer (Markov) Operator and Invariant Measure

Stenflo [4] gives a useful, brief review of the problem of the existence and uniqueness of the invariant measures, which is quite useful in our case. We will follow the notation and the terminology of [4]. The transfer operator T for the system is defined by the formula:

$$(Tf)(\mathbf{r}) = \sum_{i=1}^{N} p_i(\mathbf{r}) f(w_i(\mathbf{r})), \tag{15}$$

where $f \in C(S^2)$ - the space of all continuous functions on S^2. By the Riesz representation theorem T induces the dual operator $T^\star : \mu \mapsto T^\star\mu$ on the space $M(S^2)$ of Borel probability measures on S^2 via the formula:

$$\int_{S^2} (Tf)(\mathbf{r})\, d\mu(\mathbf{r}) = \int_{S^2} f(\mathbf{r})\, d(T^\star\mu)(\mathbf{r}).$$

[4] A short justification for the term "quantum" will be given in the closing section of this paper.

Since S^2 is compact, there always exists an invariant probability measure μ that is invariant, i.e. $T^\star\mu = \mu$. Numerical simulations of QIFS seem to indicate that such a measure is also unique, and that it is concentrated on a unique attractor set, though different for different $\epsilon \in (0,1)$. As each of the normalized operators $2P(\mathbf{n}_i,\epsilon)/\sqrt{1-\epsilon^2} \in SL(2,\mathbb{C})$ has two eigenvalues, $(1+\epsilon)/(1-\epsilon)$, and $(1-\epsilon)/(1+\epsilon)$, - the standard contraction arguments do not apply. In Ref. [4] Stenflo states the following theorem, attributed to Barnsley et al. [5]

Theorem 1. *Let $\{(X,d), p_i(x), w_i(x), i \in S = \{1,2,\ldots,N\}\}$ be an IFS with place-dependent probabilities, with all w_i being Lipshitz continuous, and all p_i being Dini-continuous, and bounded away from* 0. *Suppose*

$$\sup_{x\neq y} \sum_{i=1}^{N} p_i(x) \log\left(\frac{d\left(w_i(x), w_i(y)\right)}{d(x,y)}\right) < 0. \tag{16}$$

Then the generated Markov chain has a unique invariant probability measure.

In our case $w_i(x)$ and $p_i(x)$ are analytic, with $p_i(\mathbf{r}) \geq \frac{1-\epsilon^2}{N(1+\epsilon^2)}$ We made a numerical estimation of the LHS of the inequality (16) for the Quantum Octahedron, with $\epsilon = 0.58$, and obtained the value < -0.266, thus assuring the uniqueness of the invariant measure in this particular case, with d being the natural, rotation invariant, *arc* distance on S^2.

Let μ_0 be the natural, rotation-invariant, normalized measure on S^2. Then, for any finite n, the measure $T^{\star n}\mu_0$ is continuous with respect to μ_0 and therefore can be written as

$$T^{\star n}\mu_0(\mathbf{r}) = f_n(\mathbf{r})\mu_0(\mathbf{r}).$$

The sequence of functions $f_n(\mathbf{r})$ gives a convenient graphic representation of the limit invariant measure. In our case, the functions f_n can be computed explicitly via the following reccurence formula:

$$f_{n+1}(\mathbf{r}) = \sum_{i=1}^{n} p_i\left(w_i^{-1}(\mathbf{r})\right) \frac{d\mu_0\left(w_i^{-1}(\mathbf{r})\right)}{d\mu_0(\mathbf{r})} f_n\left(w_i^{-1}(\mathbf{r})\right) \tag{17}$$

or, explicitly:

$$f_{n+1}(\mathbf{r}) = \frac{(1-\epsilon^2)^4}{N(1+\epsilon^2)} \sum_{i=1}^{N} \frac{f_n\left(w_i^{-1}(\mathbf{r})\right)}{(1+\epsilon^2-2\epsilon\mathbf{n}_i\cdot\mathbf{r})^3} \tag{18}$$

where

$$w_i^{-1}(\mathbf{r}) = \frac{(1-\epsilon^2)\mathbf{r} - 2\epsilon(1-\epsilon\mathbf{n}_i\cdot\mathbf{r})\mathbf{n}_i}{1+\epsilon^2-2\epsilon\mathbf{n}_i\cdot\mathbf{r}} \tag{19}$$

Fig. (3) shows a plot of $\log(f_5(\mathbf{r})+1)$ for Quantum Octahedron, $\epsilon = 0.58$, using the stereographic projection $\mathbf{n} \mapsto z = \frac{n^1 - in^2}{1-n^3}$ from S^2 to the complex plane. It should be noticed that via the stereographic projection the maps $\mathbf{r} \mapsto w_i(\mathbf{r})$ become fractional, and thus conformal, transformations of the complex plane: $z \mapsto w_i(z) = \frac{az+b}{cz+d}$, with $a = 1+\epsilon n_i^3$, $b = \epsilon(n_i^1 - \mathrm{i}n_i^2)$, $c = \epsilon(n_i^1 + \mathrm{i}n_i^2)$, $d = 1-\epsilon n_i^3$.

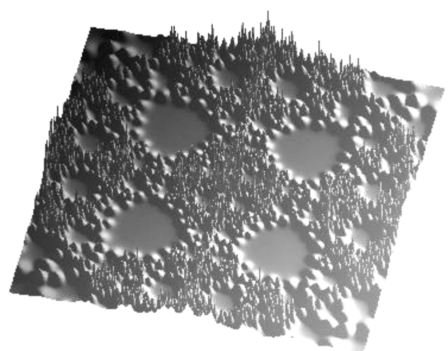

Figure 3: Plot of $f_5(\mathbf{r})$ for Quantum Octahedron ($\epsilon = 0.58$)

5 Concluding Remarks

In this section we will place QIFS within a larger field of piecewise-deterministic Markov processes and their connection to dissipative dynamics of mixed quanto-classical dynamical systems.

5.1 Classical dynamics

Usually classical dynamics is described by a 1-parameter group ϕ_t of differomorphisms of a smooth manifold X. In classical mechanics X is a symplectic manifold, the "phase space" of the system, and the flow ϕ_t is generated by a Hamiltonian vector field on X. States of the system are simply points of X, statistical states are probabilistic measures on X. The set of all statistical states is convex, its extremal elements are called pure states. These are Dirac measures - concentrated at points of X. The flow ϕ_t on X gives rise to a flow on the space of "observables", that is functions on M, and to a flow on the space "statistical states", that is on the space $M(X)$ of probabilistic measures on X. If X is discrete, then we can't have a continuous flow on X, but we can still have a continuous family of transformations acting on observables and on statistical states.

5.2 Quantum dynamics

Quantum theory is usually formulated in terms of linear operators acting on a separable complex Hilbert space $\mathfrak{H}$. Observables are represented by Hertmitian elements of the algebra $\mathfrak{A} = L(\mathfrak{H})$ of all bounded linear operators on $\mathfrak{H}$. Statistical states are positive, normalized, ultra-weakly continuous, functionals on $\mathfrak{A}$. They are represented by positive, trace class operators ρ, $\mathrm{Tr}\,(\rho) = 1$, with $\rho(A) \doteq \mathrm{Tr}\,(\rho A)$, $A \in \mathfrak{A}$. Pure states are represented by ρ of the form $\rho = P$, where P is an orthogonal projection onto a 1-dimensional subspace of $\mathfrak{H}$. The space of pure states can be thus identified with the space of one-dimensional subspaces of $\mathfrak{H}$. If $\mathfrak{H}$ is finite-dimensional, $\mathfrak{H} \approx \mathbb{C}^n$, then the space of pure states is the complex projective plane CP^{n-1}. Quantum dynamics is usually described in terms of a 1-parameter group of unitary operators $U(t) : t \in \mathbb{R}$. It acts on observables via automorphism $\alpha_t : A \mapsto U(t)^{-1} A U(t)$.

5.3 Mixed quanto-classical dynamics

We will consider the simple case, where the classical system is finite $X = \{1, \ldots, N\}$. For each $\alpha \in X$ consider the Hilbert space $\mathfrak{H}_\alpha = \mathbb{C}^{n_\alpha}$ and let $M(n_\alpha)$ be the algebra of $n_\alpha \times n_\alpha$ complex matrices.[5] The observables of the coupled system are now functions $\alpha \mapsto A_\alpha \in M(n_\alpha)$ on X with values in $M(n_\alpha)$. A pure state of the system is a pair (α, P), where $\alpha \in \{1, \ldots, N\}$ and P is a Hermitian projection matrix onto a one-dimensional subspace in $\mathbb{C}^{n_\alpha}$. It is not possible to couple the classical and the quantum degrees of freedom via reversible, unitary dynamics. A 1-parameter semi-group of completely positive maps of the algebra $\mathfrak{A} = \oplus_{\alpha=1}^{N} M(n_\alpha)$ is being used

[5] In all examples studied so far the dimensions n_α were the same for all α. But such a restriction is not necessary.

instead. We are interested in semi-groups with generators of Lidblad's type (also known as "dynamical semigroups"), in particular with generators of the form:

$$L(A)_\alpha = \mathrm{i}[H_\alpha, A_\alpha] + \sum_{\beta\neq\alpha} g^\star_{\beta\alpha} A_\beta \; g_{\beta\alpha} - \frac{1}{2}(\Lambda_\alpha A_\alpha + A_\alpha \Lambda_\alpha), \tag{20}$$

where $g_{\beta\alpha} \in L(\mathfrak{H}_\alpha, \mathfrak{H}_\beta)$ and

$$\Lambda_\alpha = \sum_{\beta\neq\alpha} g^\star_{\beta\alpha} g_{\beta\alpha} \in L(\mathfrak{H}_\alpha). \tag{21}$$

We always assume that the diagonal terms vanish: $g_{\alpha\alpha} = 0$. It has been shown in [6] that there is a one–to–one correspondence between semigroups with generators of the above type and piecewise-deterministic Markov processes on the space of pure states of the system.

5.4 From dynamical semigroups to QIFS

Here we are not concerned with the continuous time evolution between jumps, so let us extract from the Ref. [6], and also slightly reformulate, the jump process alone. It is determined by the operators $g_{\alpha\beta}$ alone, and it is an iterated function system, with place-dependent probabilities that are also determined by $g_{\alpha\beta}$-s. Let (α, P) be pure state, with P being an orthogonal projection on a unit length vector $\psi \in \mathfrak{H}_\alpha$. Observe that for each $\beta \neq \alpha$ we have:

$$g_{\beta\alpha} P g^\star_{\beta\alpha} = \lambda(\alpha, \beta; P) Q \tag{22}$$

where

$$\lambda(\alpha, \beta; P) = \|g_{\beta\alpha}\psi\|^2 \geq 0 \tag{23}$$

and, if $\lambda(\alpha, \beta; P) > 0$, then Q is a projection operator on the vector $g_{\beta\alpha}\psi/\|g_{\beta\alpha}\psi\|$ in $\mathfrak{H}_\beta$. The probabilities $p(\alpha, \beta; P)$ are defined as

$$p(\alpha, \beta; P) = \frac{\lambda(\alpha, \beta; P)}{\sum_{\beta\neq\alpha} \lambda(\alpha, \beta; P)}. \tag{24}$$

Assume now that all Hilbert spaces $\mathfrak{H}_\alpha \equiv \mathfrak{H}$ are identical. Assume that $X = 2^N$ - the set of N bits, and that $g_{\alpha\beta} = g_i \neq 0$ when α differs from β only at one, the i-th bit, otherwise $g_{\alpha\beta} = 0$. We will just have a family of operators g_i and a jump process on pure states P, that is one-dimensional orthogonal projections in $\mathfrak{H}$. The maps and their probabilities are determined by:

$$g_i P g_i^\star = \lambda(i; P) Q, \tag{25}$$

with $\lambda(i;P) = \|g_i\psi\|^2$, $p_i(P) = \lambda(i;P)/\sum_j \lambda(j,P)$, and Q being the orthogonal projection on the subspace spanned by the vector $g_i\psi$. We have an iterated function system on the complex projective space $CP(n-1)$ (equivalently, on the grassmanian of one-dimensional subspaces of $\mathbb{C}^n$), with place dependent probabilities. Both, maps and probabilities, are determined by the set of linear operators $g_i, \quad i \in \{1, 2, \ldots, N\}$

5.5 A short history of QIFS

The idea of coupling a classical and a quantum system via dynamical semigroup has been originally described in [7]. The first model of a QIFS, on S^2, with index α being also continous $\alpha = \mathbf{n}$, with values in S^2 and, using the notation of Sec. $g_\mathbf{n}$ defined as $g_\mathbf{n} = \exp\left(i\pi\sigma(\mathbf{n})\right)$. These maps were unitary, thus measure preserving, and did not give rise to a fractal attractor. The tetrahedral model was first introduced in [9]. It was then examined analytically and modelled numerically in a PhD Thesis by G. Jastrzebski [3]. The model was further exploited in [10], where it has been described in some details, and where the Lyapunov exponent of the semigroup generator has been computed. The term QIFS has been introduced about that time on sci.physics.research newsgroup on internet. Recently the term QIFS has been adopted in [11] for another class of maps, namely for maps on the space of all statistical states of a quantum system, with arbitrarily assigned probabilities.

References

[1] M. F. Barnsley, *Fractals everywhere*, (Academic Press, San Diego 1988)

[2] A. Jadczyk and R. Öberg, Quantum Jumps, EEQT and the Five Platonic Fractals, Preprint: http://arXiv.org/abs/quant-ph/0204056

[3] G. Jastrzebski, Interacting classical and quantum systems. Chaos from quantum measurements, Ph.D. thesis, University of Wrocław, 1996 (in Polish)

[4] Ö. Stenflo, Uniqueness of invariant measures for place-dependent random iterations of functions, Published in IMA Vol. Math. Appl., 132 (2002), 13–32, Preprint: http://www.math.su.se∼stenflo/IMA.pdf

[5] M. F. Barnsley, S. G. Demko, J. H. Elton and J. S. Geronimo, Invariant measures for Markov processes arising from iterated function

systems with place-dependent probabilities, *Ann. Inst. H. Poincaré Probab. Statist* **24** (1988) 367-294, (Erratum: **25** (1989) 589–590

[6] A. Jadczyk, G. Kondrat and R. Olkiewicz, On uniqueness of the jump process in quantum measurement theory, *J. Phys. A* **30** (1996) 1-18; Preprint http://arXiv.org/abs/quant-ph/9512002

[7] Ph. Blanchard and A. Jadczyk, On the Interaction Between Classical and Quantum Systems *Phys. Lett. A* **175** (1993), 157-164; Preprint http://arXiv.org/abs/quant-ph/9512002

[8] A. Jadczyk, Topics in Quantum Dynamics, in Proc. First Caribb. School of Math. and Theor. Phys., Saint–Francois–Guadeloupe 1993, *Infinite Dimensional Geometry, Noncommutative Geometry, Operator Algebras and Fundamental Interactions*, ed. R.Coquereaux et al., (World Scientific, Singapore 1995); Preprint http://arXiv.org/abs/hep-th/9406204

[9] Jadczyk, A.: 'IFS Signatures of Quantum States', IFT Uni Wroclaw, internal report, September 1993.

[10] Ph. Blanchard, A. Jadczyk and R. Olkiewicz, Completely Mixing Quantum Open Systems and Quantum Fractals, *Physica D: Nonlinear Phenomena* , **148** (3-4) (2001) 227–241; Preprint http://arXiv.org/abs/quant-ph/9909085

[11] A. Lozinski, K. Zyczkowski and W. Slomczynski, Quantum Iterated Function Systems, Phys. Rev. E **68** (2003), 046110; Preprint http://arXiv.org/abs/quant-ph/0210029