The Piecewise Deterministic Process Associated to EEQT

Ph. Blanchard

Faculty of Physics and BiBoS, University of Bielefeld

Universitätstr. 25, D-33615 Bielefeld

A. Jadczyk

Inst. of Theor. Physics, University of Wrocław

Pl. Maxa Borna 9, PL-50 204 Wrocław

R. Olkiewicz *

Faculty of Physics and BiBoS, University of Bielefeld

Universitätstr. 25, D-33615 Bielefeld

Abstract

In the framework of event enhanced quantum theory (EEQT) a probabilistic construction of the piecewise deterministic process associated with a dynamical semigroup is presented. The process describes sample histories of individual systems and gives a unique algorithm generating time series of pointer readings in real experiments.

Key words: quntum measurements, open systems, completely positive semigroups, piecewise deterministic processes.

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*permanent address: Institute of Theoretical Physics, University of Wrocław, PL 50-204 Wrocław, Poland

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I. INTRODUCTION

One of the primary aims of quantum measurement theory is to understand the mechanism by which potential properties of quantum systems become actual. This is not an abstract or philosophical problem. Nowadays it is possible to carry out prolonged observations of individual quantum systems. These observations provide us with time series of data, and a complete theory must explain the mechanism by which these time series are being generated; must be able to "simulate" the natural process of events generation. There are several methods of approaching this problem. John Bell[1] for instance, sought a solution in hidden variable theories of Bohm and Vigier, his own idea of beables, and also in the spontaneous localization idea of Ghirardi, Rhimini and Weber[2]. More recently, in a series of papers, two of us (Ph. B. and A.J.)[3–5] proposed a formalism that goes in a similar direction but avoids introducing other hidden variables beyond the wave function itself. Our "Event Enhanced Quantum Theory"(in short: EEQT) describes a consistent mode of coupling between a quantum and a classical system, in which a classical system is one described by an Abelian algebra. We suggest that a measurement process is, by definition, a coupling of a quantum and a classical system, where transfer of information about quantum state to the classical recording device is mathematically modelled by a dynamical semigroup (i.e. semigroup of completely positive and trace preserving maps) of the total system. It is instructive to see that such a transfer of information cannot, indeed, be accomplished by a Hamiltonian or, more generally, by any automorphic evolution[1]. To this end consider a system described by

[1]For a discussion of this fact in a broader context of algebraic theory of superselection sectors – cf. Landsman[6,*Sec.*4.4]. Cf. also the no–go result by Ozawa[7]

a von Neumann algebra $\mathcal{A}$ with centre $\mathcal{Z}$. Then $\mathcal{Z}$ describes the classical degrees freedom of the system. Let ω be a state of $\mathcal{A}$, and let $\omega|_Z$ denote its restriction to $\mathcal{Z}$. Let α_t be an automorphic time evolution of $\mathcal{A}$, and denote $\omega^t = \alpha^t(\omega)$, where the dual evolution of states is given by $\alpha^t(\omega)(A) = \omega(\alpha_t(A))$. Each α_t is an automorphism of the algebra $\mathcal{A}$, and so it leaves its centre invariant: $\alpha_t : \mathcal{Z} \to \mathcal{Z}$. The crucial observation is that, because the evolution of states of $\mathcal{Z}$ is dual to the evolution of the observables in $\mathcal{Z}$, and we have $\alpha^t(\omega)|_{\mathcal{Z}} = \alpha^t|_{\mathcal{Z}}(\omega|_{\mathcal{Z}})$, the restriction $\omega^t|_{\mathcal{Z}}$ depends only on $\omega^0|_{\mathcal{Z}}$. In other words the future state of the the classical subsystem depends only on the past state of that subsystem and – not on its extension to the total system. This shows that no information transfer from the total system to its classical subsystem is possible – unless we use more general, non–automorphic evolutions. The idea of describing a quantum measurement as a two-way coupling between quantum system and a classical system occured before to several authors – we mention only the classical papers by Sudarshan[8] – but never within the completely positive semigroup approach.

EEQT has several points of contact with other approaches. The mathematical model was a result of our studies of the papers of Jauch[9,10], Hepp[11], Piron[12–14] , Gisin[15,16] and Araki[17] , and also of the papers by Primas (cf.[18,19]). It was then found that our master equation describing a coupled quantum–classical system is of the type already well known to statisticians. In his monographs[20,21] dealing with stochastic control and optimization M. H. A. Davis, having in mind mainly queuing and insurance models, described a special class of piecewise deterministic processes that was later found to fit perfectly the needs of quantum measurement theory, and that reproduced the master equation postulated originally by the two of us in[3].

In[22] it was shown that the special class of couplings between a classical and quantum system leads to a unique piecewise deterministic process with values on E-the pure state space of the total system. That process consists of random jumps, accompanied by changes of a classical state, interspersed by random periods of Schrödinger-type deterministic evolution. The process, although mildly nonlinear in quantum wave function ψ, after averaging, recovers the original linear master equation for statistical states. The action of the dynamical semigroup T_t is given in term of the process in the following way

$$T_t(P_x) \;=\; \int P(t,\, x,\, dy) P_y,$$

where $P(t, x, dy)$ is the transition probability function of the process and $y \to P_y$ is a tautological map, which assigns to every point $y \in E$ a one-dimensional projector P_y. The main objective of this paper is to provide a probabilistic construction of the process and discuss some of its properties and applications. The paper is organised as follows. In sec. II the formalism for classical-quantum interactions is presented. In sec. III the probabilistic construction of the PD process is described and some of its properties are analysed. In sec. IV the classical part of the process is discussed. We also present an example of direct photodetection. Concluding remarks are given in sec. V.

II. THE FORMALISM

We start by recalling the theorem by Christensen and Evans that describes the most general form of a generator of a completely positive semigroup of transformations of an algebra with an notrivial centre. The theorem generalizes the classical results of Gorini, Kossakowski and Sudarshan[23] and of Lindblad[24] to the case of arbitrary $C^\star$–algebra, and

it states that eseentially the Linblad form of the generator holds also for this more general case. We quote the theorem for the convenience of the reader[25]:

Theorem 1 (Christensen – Evans) *Let $\alpha_t = \exp(Lt)$ be a norm–continuous semigroup of CP maps of a $C^\star$– algebra of operators $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$. Then there exists a CP map ϕ of $\mathcal{A}$ into the ultraweak closure $\bar{\mathcal{A}}$ and an operator $K \in \bar{\mathcal{A}}$ such that the generator L is of the form:*

$$L(A) = \phi(A) + K^\star A + AK\,. \tag{1}$$

Let us apply this theorem to the case of $\mathcal{A}$ being a von Neumann algebra, and the maps α_t being *normal*. Then ϕ can be also taken normal. We also have $\bar{\mathcal{A}} = \mathcal{A}$, so that $K \in \mathcal{A}$. Let us assume that $\alpha_t(I) = I$ or, equivalently, that $L(I) = 0$. It is convenient to introduce $H = i(K - K^\star)/2 \in \mathcal{A}$, then from $L(I) = 0$ we get $K + K^\star = -\phi(I)$, and so $K = -iH - \phi(1)/2$. Therefore we have

$$L(A) = i\,[H, A] + \phi(A) - \{\phi(1), A\}/2, \tag{2}$$

where $\{\,,\,\}$ denotes anticommutator.

We now apply the above formalism to the hybrid system which is a direct product of the classical and quantum mechanical one. The physical idea behind such a model is that a quantum measurement is to be defined as a particular coupling between a quantum and a classical system. We continuously observe the classical system, notice changes of its pure states (we call these changes "events") and from these we deduce properties of the coupled quantum system. Details can be found in Ref.[4,5]. One can think of events as 'clicks' of a

particle counter, sudden changes of the pointer velocity, changing readings on an apparatus LCD display. The concept of an event is of course an idealization - like all concepts in a physical theory. Let us consider the simplest situation corresponding to a finite set of possible events. The space of pure states of our classical system C, denoted by $\mathcal{S}_c$, has m states, labeled by $\alpha = 1, \ldots, m$. Statistical states of C are probability measures on $\mathcal{S}_c$ – in our case just sequences $p_\alpha \geq 0, \sum_\alpha p_\alpha = 1$.

The algebra of observables of C is the algebra $\mathcal{A}_c$ of complex functions on $\mathcal{S}_c$ – in our case just sequences $f_\alpha, \alpha = 1, \ldots, m$ of complex numbers. We use Hilbert space language even for the description of the classical system. Thus we introduce an m-dimensional Hilbert space $\mathcal{H}_c$ with a fixed basis, and we realize $\mathcal{A}_c$ as the algebra of diagonal matrices $F = \mathrm{diag}(f_1, \ldots, f_m)$. Statistical states of C are then diagonal density matrices $\mathrm{diag}(p_1, \ldots, p_m)$, and pure states of C are vectors of the fixed basis of $\mathcal{H}_c$. Events are ordered pairs of pure states $\alpha \to \beta$, $\alpha \neq \beta$. Each event can thus be represented by an $m \times m$ matrix with 1 at the (α, β) entry, zero otherwise. There are $m^2 - m$ possible events. Let us point out that important here is the discretness of the classical system not its finitness. We can easily generalize the above to the case when the classical points form, for example, the set of natural numbers. Then the classical algebra becomes l^∞ (uniformly bounded sequences) while statistical states are positive elements from l^1 (summable sequences).

We now come to the quantum system. Let Q be the quantum system whose bounded observables are from the algebra $\mathcal{A}_q$ of bounded operators on a Hilbert space $\mathcal{H}_q$. In this paper we will assume $\mathcal{H}_q$ to be *finite dimensional* . Pure states of Q are unit vectors in $\mathcal{H}_q$; proportional vectors describe the same quantum state. They form a complex projective space $\mathbf{C}P(\mathcal{H}_q)$ over $\mathcal{H}_q$. Statistical states of Q are given by non–negative density matrices

$\hat{\rho}$, with Tr $(\hat{\rho}) = 1$.

Let us now consider the total system $T = Q \times C$. For the algebra $\mathcal{A}_t$ of observables of T we take the tensor product of algebras of observables of Q and C: $\mathcal{A}_t = \mathcal{A}_q \otimes \mathcal{A}_c$. It acts on the tensor product $\mathcal{H}_q \otimes \mathcal{H}_c = \oplus_{\alpha=1}^{m} \mathcal{H}_\alpha$, where $\mathcal{H}_\alpha \approx \mathcal{H}_q$. Thus $\mathcal{A}_t$ can be thought of as algebra of *diagonal* $m \times m$ matrices $A = (a_{\alpha\beta})$, whose entries are quantum operators: $a_{\alpha\alpha} \in \mathcal{A}_q$, $a_{\alpha\beta} = 0$ for $\alpha \neq \beta$. Statistical states of $Q \times C$ are given by $m \times m$ diagonal matrices $\rho = \mathrm{diag}(\rho_1, \ldots, \rho_m)$ whose entries are positive operators on $\mathcal{H}_q$, with the normalization Tr $(\rho) = \sum_\alpha$ Tr $(\rho_\alpha) = 1$. Duality between observables and states is provided by the expectation value $< A >_\rho = \sum_\alpha$ Tr $(A_\alpha \rho_\alpha)$.

We will now generalize slightly our framework. Indeed, there is no need for the quantum Hilbert spaces $\mathcal{H}_\alpha$, corresponding to different states of the classical system, to coincide. We will allow them to be different in the rest of this paper. Intuitively such a generalization corresponds to the idea that a phase transition can accompany the event. We denote $n_\alpha = dim(\mathcal{H}_\alpha)$.

We consider now dynamics. It is normal in quantum theory that classical parameters enter quantum Hamiltonian. Thus we assume that quantum dynamics, when no information is transferred from Q to C, is described by Hamiltonians $H_\alpha : \mathcal{H}_\alpha \longrightarrow \mathcal{H}_\alpha$, that may depend on the actual state of C (as indicated by the index α). We will use matrix notation and write $H = \mathrm{diag}(H_\alpha)$. Now take the classical system. It is discrete here. Thus it can not have continuous time dynamics of its own.

The *coupling* of Q to C is specified by a matrix $V = (g_{\alpha\beta})$, where $g_{\alpha\beta}$ are linear operators: $g_{\alpha\beta} : \mathcal{H}_\beta \longrightarrow \mathcal{H}_\alpha$. We assume $g_{\alpha\alpha} = 0$. This condition expresses the simple fact: we do not need dissipation without receiving information (i.e without an event). It plays a crucial

role in the prove of uniqueness of the piecewise deterministic process that is associated to our master equation. Although the present paper is concerned with the existence and with the important mathematical properties of the process, the uniqueness is important from the point of view of the physical interpretation. It tells us that in our case, contrary to the situation encountered in quantum optics master equations, all the relevant information is contained in the master equation - so that there is a unique process describing the random laws governing the individual system under observation. More on this uniqueness vs. non–uniqueness problem can be found in Ref.[22].

To transfer information from Q to C we need a non–Hamiltonian term which provides a completely positive (CP) coupling. As in Ref.[4,5] we consider couplings for which the evolution equation for observables and for states is given by the Lindblad form:

$$\dot{A}_\alpha = i[H_\alpha, A_\alpha] + \sum_\beta g^\star_{\beta\alpha} A_\beta g_{\beta\alpha} - \frac{1}{2}\{\Lambda_\alpha, A_\alpha\}, \tag{3}$$

or equivalently:

$$\dot{\rho}_\alpha = -i[H_\alpha, \rho_\alpha] + \sum_\beta g_{\alpha\beta} \rho_\beta g^\star_{\alpha\beta} - \frac{1}{2}\{\Lambda_\alpha, \rho_\alpha\}, \tag{4}$$

where

$$\Lambda_\alpha = \sum_\beta g^\star_{\beta\alpha} g_{\beta\alpha}. \tag{5}$$

The above equations describe statistical behavior of ensembles. Individual sample histories are described by the following algorithm:

Suppose that at time t_0 the system is described by a normalized quantym state vector ψ_0 and a classical state α. Then choose a uniform random number $p \in [0,\, 1]$, and proceed with the

continuous time evolution by solving the modified Schrödinger equation

$$\dot{\psi}_t \;=\; (-iH_\alpha \;-\; \frac{1}{2}\Lambda_\alpha)\psi_t$$

with the initial wave function ψ_0 *until* $t \;=\; t_1$*, where* t_1 *is determined by*

$$\int_{t_0}^{t_1} (\psi_t,\, \Lambda_\alpha \psi_t) dt \;=\; p$$

Then jump. When jumping, change $\alpha \rightarrow \beta$ *with probability*

$$p_{\alpha\rightarrow\beta} \;=\; \|g_{\beta\alpha}\psi_{t_1}\|^2/(\psi_{t_1},\, \Lambda_\alpha\psi_{t_1})$$

and change

$$\psi_{t_1} \rightarrow \psi_1 \;=\; g_{\beta\alpha}\psi_{t_1}/\|g_{\beta\alpha}\psi_{t_1}\|.$$

Repeat the steps replacing t_0, ψ_0, α *with* t_1, ψ_1 *and* β*.*

This leads to a stochastic process, in which the randomness appears as point events i.e. there is a sequence of random occurences at random times $T_1 < T_2 < \ldots$, but there is no additional component of uncertainty between these times. It consists of a mixture of deterministic motion and random jumps. A class of such processes is called piecewise deterministic processes (PDP)[26]. The motion between jumps is determined by a complete vector field X on the pure state space E of the total system. The jump mechanism is determined by two further components: a jump rate λ and a transition kernel Q. The vector field X generates a flow $\phi(t,\, x)$ in E, which is given by $\phi(t,\, x) = \gamma_x(t)$, where $\gamma_x(t)$ is the integral curve of X starting at point $x \in E$. The jump rate is a measurable function $\lambda : E \rightarrow \mathbf{R}_+ \cup \{0\}$ such that for any $x \in E$ the mapping $t \rightarrow \lambda \circ \phi(t,\, x)$ is integrable at least near $t \;=\; 0$. The set of those $x \in E$ for which $\lambda(x) \;=\; 0$ we denote by E_0. The transition kernel $Q : \mathcal{B}(E) \times E \rightarrow [0,\, 1]$ satisfies the following conditions:

a) $Q(E,\, x) \,=\, 1 \quad \forall x \in E$,

b) $Q(\{x\},\, x) \,=\, 0$ if $x \in E \setminus E_0$ and $Q(\{x\},\, x) \,=\, 1$ for $x \in E_0$,

c) $\forall \Gamma \in \mathcal{B}(E)$ the map $x \to Q(\Gamma,\, x)$ is measurable.

Here $\mathcal{B}(E)$ denotes the Borel σ-algebra on E. In our case $E \,=\, \dot{\bigcup}\mathbf{C}P_\alpha$, $\alpha \,=\, 1,\, 2, ..., m$ and we have the following formulas for X, λ and Q:

$$Xf(\psi,\, \alpha) \,=\, \frac{d}{dt} f(\frac{\exp(-iH_\alpha \,-\, \frac{1}{2}\Lambda_\alpha)\psi}{\|\exp(-iH_\alpha \,-\, \frac{1}{2}\Lambda_\alpha)\psi\|},\, \alpha)|_{t=0}$$

$$\lambda(\psi,\, \alpha) \,=< \psi,\, \Lambda_\alpha \psi >$$

$$Q(d\phi,\, \beta;\, \psi,\, \alpha) \,=\, \frac{\|g_{\beta\alpha}\psi\|^2}{\lambda(\psi,\, \alpha)} \delta(\phi \,-\, \frac{g_{\beta\alpha}\psi}{\|g_{\beta\alpha}\psi\|}) d\phi$$

if $(\psi,\, \alpha) \in E \setminus E_0$ and δ denotes the Dirac measure.

The triple $(X,\, \lambda,\, Q)$ is called local characteristic of the process. Its infinitesimal generator is given by

$$\mathcal{L}f(x) \,=\, Xf(x) \,+\, \lambda(x) \int\limits_E [f(y) \,-\, f(x)] Q(dy,\, x)$$

and produces sample paths exactly such as described by the above algorithm.

III. THE PD PROCESS

In this section we present the detailed construction of the process introduced in sec. II and investigate some of its properties. General references on stochastic processes are[27,28]. Probabilistic concepts can be found in[29,30].

At first we construct a probabilistic space $(\Omega,\, \mathcal{A})$ (compare[31] for a similar construction for Markov decision processes). Let Ω be a set of all sequences $(t_0, x_0; t_1, x_1; \ldots)$, which are finite or infinite, and such that $t_0 = 0$, $t_n \leq t_{n+1}$, $t_n \in \dot{\mathbf{R}}_+ \,=\, [0, \infty]$, $x_n \in E$ for all $n \in \mathbf{N} \cup \{0\}$.

If a sequence is finite i.e. $\omega = (t_0, x_0; \ldots, t_n, x_n)$ then we put

$$t_{n+1} = t_{n+2} = \ldots = \infty, \qquad x_{n+1} = x_{n+2} = \ldots = x_n$$

It follows that Ω can be embeded into an infinite product space $\prod_{n=0}^{\infty} \Omega_n$, where $\Omega_0 = \{0\} \times E$ and $\Omega_n = \dot{\mathbf{R}}_+ \times E$. On each Ω_n we have a natural σ-algebra $\mathcal{A}_n$ given by $\mathcal{B}(\dot{\mathbf{R}}_+) \otimes \mathcal{B}(E)$. We define a σ-algebra $\mathcal{A}$ on Ω as $(\otimes_{n=0}^{\infty} \mathcal{A}_n)|_\Omega$.

Now let us construct a family of probabilistic measures P_x on $(\Omega, \mathcal{A})$ with respect to an initial state $x \in E$. They will be determined by the deterministic drift ϕ, the jump rate λ and the transition kernel Q. Because we want to use the Ionescu Tulcea theorem[29] we have to define transition kernels between $(\Omega_n, \mathcal{A}_n)$ and $(\Omega_{n+1}, \mathcal{A}_{n+1})$. We do it step by step.

On Ω_0 we take the Dirac measure $P_0 = \delta_x$. Let $\Lambda(t, x) := \int_0^t \lambda(\phi(s, x))ds$ and let us define

$$F_x(t_1) \;=\; 1 - \exp(-\Lambda(t_1, x))$$

$$K_x(t_1, dx_1) \;=\; Q(dx_1, \phi(t_1, x))$$

As the transition kernel between $(\Omega_0, \mathcal{A}_0)$ and $(\Omega_1, \mathcal{A}_1)$ we take

$$P_0^1(x, B_1 \times \Gamma_1) \;=\; \int_{B_1} \int_{\Gamma_1} K_x(t_1, dx_1) dF_x(t_1)$$

for any $B_1 \in \mathcal{B}(\dot{\mathbf{R}}_+)$ and any $\Gamma_1 \in \mathcal{B}(E)$. In the second step we define

$$F_{(t_1,x_1)}(t_2) \;=\; \begin{cases} 0, & \text{if } t_1 > t_2 \\ 1 - \exp(-\Lambda(t_2 - t_1, x_1)), & \text{if } t_1 \leq t_2 \end{cases}$$

$$K_{(t_1,x_1)}(t_2, dx_2) \;=\; Q(dx_2, \phi(t_2 - t_1, x_1))$$

and put

$$P_2^1(t_1, x_1; B_2 \times \Gamma_2) \;=\; \int_{B_2} \int_{\Gamma_2} K_{(t_1,x_1)}(t_2, dx_2) dF_{(t_1,x_1)}(t_2)$$

It is clear that P_2^1 is a transition kernel between $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$. In the similar way we construct higher kernels P_{n+1}^n. By Ionescu Tulcea theorem there is a unique probabilistic measure P_x on $(\prod_{n=0}^{\infty} \Omega_n,\ \otimes_{n=0}^{\infty} \mathcal{A}_n)$ such that for every measurable rectangle $A = A_0 \times A_1 \times \ldots \times A_n \times \Omega_{n+1} \times \ldots$ the following identity

$$P_x[A] \ = \ \delta_x(A_0) \int\limits_{A_1} P_0^1(x;\, dt_1,\, dx_1) \cdots \int\limits_{A_n} P_n^{n-1}(t_{n-1},\, x_{n-1};\, dt_n,\, dx_n)$$

is satisfied. It is clear from the above formula that P_x is concentrated on $\Omega_x = \{\omega \in \Omega : x_0 = x\}$, $x \in E$. Moreover P_x is measurable with respect to x.

To investigate properties of the above measure let us define a sequence of measurable random variables

$$T_n : \Omega_x \to \dot{\mathbf{R}}_+ \quad T_n(\omega) \ = \ t_n, \quad X_n : \Omega_x \to E \quad X_n(\omega) \ = \ x_n$$

The distributions of T_0 and X_0 are Dirac measures concentrated in $\{0\}$ and $\{x\}$ respectively. The distribution dF_{T_1} of T_1 is given by

$$P_x[T_1 \leq t] \ = \ 1 - \exp(-\Lambda(t,\, x))$$

and the conditional expectation of X_1 given T_1 equals to

$$E_x[1_{\{X_1 \subset \Gamma\}} | T_1] \ = \ Q(\Gamma,\, \phi(T_1,\, x))$$

Here $1_{\{\cdot\}}$ denotes an idicator function of a given set. The above equation can be also written as

$$dF_{X_1|T_1}(y|t) \ = \ Q(dy,\, \phi(t,\, x)),$$

where the left hand side is the conditional distribution of X_1. For arbitrary $n \in \mathbf{N}$ we have the following formulas:

$$E_x[1_{\{T_{n+1} \leq t\}} | T_n,\, X_n] \ = \ \begin{cases} 0 & \text{if } t < T_n \\ \\ 1 - \exp(-\Lambda(t - T_n,\, X_n)) & \text{if } t \geq T_n \end{cases}$$

$$E_x[1_{\{X_{n+1}\subset\Gamma\}}|X_n,\, T_{n+1}] \;=\; Q(\Gamma,\, \phi(T_{n+1},\, X_n))$$

It follows that $P_x[T_1 = 0] \;=\; 0$ so $T_1 > 0$ a.s. Because, after a jump, process starts again so $T_n < T_{n+1}$ a.s. for every n. This fact can be also derived from the following equality:

$$P_x[T_{n+1} - T_n > s] \;=\; E_x[\exp(-\Lambda(s,\, X_n))]$$

It means that a set of paths with two or more simultaneous jumps has zero probability. Moreover, because $Q(\{x\},\, x) \;=\; 0$ for every $x \in E \setminus E_0$ so with probability one the process can not jump to the state it is deterministically approaching. There are no jumps from the set E_0 at all.

Let us calculate some physically interesting probabilities. For example the probability that there is no jump up to time t equals to

$$P_x[T_1 > t] \;=\; \exp(-\int\limits_0^t \lambda(\phi(s,\, x))ds)$$

Because

$$P_x[T_2 > t] \;=\; P_x[T_1 > t] \;+\; E_x[1_{\{T_1\le t\}} \exp(-\Lambda(t - T_1,\, X_1)]$$

and, on the other hand,

$$P_x[T_2 > t] \;=\; P_x[T_2 > t \wedge T_1 \le t] \;+\; P_x[T_2 > t \wedge T_1 > t]$$

so the probability that exactly one jump happens up to time t is given by

$$P_x[T_2 > t \wedge T_1 \le t] \;=\; \int\limits_0^t 1_{\{u\le t\}} dF_{T_1}(u) \int_E \exp(-\Lambda(t-u,\, y))dF_{X_1|T_1}(y|u) \;=$$

$$\int_0^t \int_E \lambda(\phi(u,\, x)) \exp(-\Lambda(u,\, x)) \exp(-\Lambda(t-u,\, y))Q(dy,\, \phi(u,\, x))du$$

Now let us define a random variable $T_\infty = \lim_{n\to\infty} T_n$. For every $t < T_\infty$ we construct the process $\mathbf{x}_t$ by putting

$$\mathbf{x}_t(\omega) \;=\; \phi(t - T_k(\omega),\, X_k(\omega)) \quad if \quad T_k(\omega) \le t < T_{k+1}(\omega)$$

In general we can have the process with the lifetime. We show that in our case, due to the boundness of the jumping rate, $T_\infty = \infty$ a.s. Let $C = \sup_{x\in E} \lambda(x)$. Then for every $t > 0$

$$\sup_{x\in E}(1 - \exp(-\Lambda(t,\, x)) \leq 1 \;-\; e^{-Ct}$$

Let us fix t and denote $C_1 \;=\; 1 \;-\; e^{-Ct}$, which is strictly less then 1. Then

$$P_x[T_{n+1} \leq t] \;=\; E[1_{\{T_n \leq t\}}(1 - \exp(-\Lambda(t - T_n,\, X_n))] \leq C_1 P_x[T_n \leq t] \leq C_1^{n+1}$$

by induction. It implies that

$$P_x[\bigcap_{n=0}^{\infty}\{T_n \leq t\}] \;=\; \lim_{n\to\infty} P_x[T_n \leq t] \;=\; 0$$

It follows that $\mathbf{x}_t$ is defined for all $t \in \mathbf{R}_+$ and is a *cadlag* process i.e. possesses right continuous with left limits paths.

To end the construction of ingredients needed for a Markov process let us introduce a natural filtration on Ω_x given by $\mathcal{F}_t^0 = \sigma\{\mathbf{X}_s,\; s \leq t\}$ and take $\mathcal{F}_\infty^0 = \vee_t \mathcal{F}_t^0$. Let $\mathcal{F}_t$ and $\mathcal{F}_\infty$ denote the P_x-completion of $\mathcal{F}_t^0$ and $\mathcal{F}_\infty^0$ respectively. Because, after a jump, the process evolves deterministically, so the filtration $(\mathcal{F}_t)_{0\leq t\leq\infty}$ is right continuous. Thus we arrive at:

Theorem 1.

a) The filtered probability space $(\Omega,\, \mathcal{F}_\infty,\, P_x,\, \mathcal{F}_t)$ satisfies the usual hypothesis for every $x \in E$.

b) $\mathbf{x}_t$ is an adapted and *cadlag* process.

c) $(\Omega,\, \mathcal{F}_\infty,\, P_x,\, \mathcal{F}_t,\, \mathbf{x}_t)$ is a strong Markov process with infinite lifetime.

Proof: only the Markov property need to be checked. It follows from two basic properties of $\mathbf{x}_t$. The distribution of T_1 depends only on the current state $\mathbf{x}_t$ and, after a jump, process starts again. For more details see[26].

Now we show another important property of the process $\mathbf{x}_t$, namely the quasi-left-continuity. Let us define a random set $\triangle = \{(t,\omega) : \mathbf{x}_{t^-} \neq \mathbf{x}_t\}$, where $\mathbf{x}_{t^-}$ is the left limit of $\mathbf{x}_t$. Then

$$\nu(\omega;\, dt,\, dx) \;=\; \sum_s 1_{\triangle}(s,\, \omega)\delta_{(s,\mathbf{x}_s(\omega))}(dt,\, dx),$$

where $\delta_{(s,x)}$ is the Dirac measure on $\mathbf{R}_+ \times E$ concentrated in $(s,\ x)$, is an integer-valued random measure. It leeds to a simple point process $\tilde{N}_t$ given by

$$\tilde{N}_t \;=\; \nu([0,\, t] \times E) \;=\; \sum_{n=1}^{\infty} 1_{\{T_n \leq t\}}$$

Because $T_\infty = \infty$ a.s. so $\tilde{N}_t$ is a.s. finite valued. It is also integrable because

$$E_x[\tilde{N}_t] \;=\; \sum_{n=1}^{\infty} nP_x[T_n \leq t] \leq \;\sum_{n=1}^{\infty} nC_1^n \;<\; \infty$$

Moreover it was shown in[26] that the compensator of $\tilde{N}_t$ is equal to $\int_0^t \lambda(\mathbf{x}_s)ds$ and so $M_t := \tilde{N}_t - \int_0^t \lambda(\mathbf{x}_s)ds$ is an $(P_x,\, \mathcal{F}_t)$-martingale. Using this fact it can be calculated that the dual predictible projection of ν is given by

$$\nu^p(\omega; dt,\, dx) \;=\; Q(dx,\, \mathbf{x}_t(\omega))\lambda(\mathbf{x}_t(\omega))dt$$

Thus $\nu^p(\omega;\, \{t\},\, E) \;=\; 0$ and so $\mathbf{x}_t$ is quasi-left-continuous[28]. Thus we proved that $\mathbf{x}_t$ is a Hunt process. Moreover $\mathbf{x}_t$ is a Feller process i.e. the transition kernel of $\mathbf{x}_t$ generates a strongly continuous semigroup of contractions on the space of all continuous functions on E, see[32,33].

IV. STOCHASTIC REPRESENTATION OF THE CLASSICAL SYSTEM

In this section we discuss some properties of the stochastic process associated with the measuring apparatus. Let C be a state space of the classical system i.e. $C = \{1,\, 2,\, ...,\, m\}$.

Let us define a $\{0,\, 1\}$-valued process p_t^α by

$$p_t^\alpha(\omega) \;=\; \delta^\alpha_{\pi(\mathbf{x}_t(\omega))},$$

where δ^α_β is the Kronecker delta and π denotes the canonical projection $\pi :\; E \to C$. By $P(t,\, x,\, \Gamma)$, $x \in E$ and $\Gamma \subset E$ we denote the transition kernel of the process $\mathbf{x}_t$. It was shown in [jakol] that $P(t,\, x,\, \Gamma)$ is associated with the dynamical semigroup T_t and so

$$T_t(P_x) \;=\; \int\limits_E P_y P(t,\, x,\, dy)$$

Here P_y is the one-dimensional projector corresponding to $y \in E$ i.e. $P_y = \; |y><y|$. We show that the average of p_t^α gives the probability of finding the total system at time t in a classical state α. Let $\overline{p}_t^\alpha \;=\; E[p_t^\alpha]$. Then

$$\overline{p}_t^\alpha \;=\; \int\limits_E \delta^\alpha_{\pi(y)} P(t,\, x,\, dy) \;=\; \int\limits_{\mathbf{C}P_\alpha} Tr(P_y) P(t,\, x,\, dy) \;=$$

$$Tr(\int\limits_{\mathbf{C}P_\alpha} P_y P(t,\, x,\, dy)) \;=\; Tr(T_t(P_x)_\alpha)$$

Now we derive a differential equation for $\overline{p}_t^\alpha$. Let us start with the following example.

Example 1. Let $C \;= \{1,\, 2\}$. Then a change of the process p_t^α, $\alpha = 1,\; 2$, is given by

$$dp_t^1 \;=\; -p^1_{t^-} d\tilde{N}_t \;+\; p^2_{t^-} d\tilde{N}_t, \quad p_0^1 \;=\; 1$$

$$dp_t^2 \;=\; -p^2_{t^-} d\tilde{N}_t \;+\; p^1_{t^-} d\tilde{N}_t, \quad p_0^2 \;=\; 0,$$

where $\tilde{N}_t$ is the counting process introduced in the previous section. Solving the above equations we get

$$p_t^1 \;=\; \frac{1 + (-1)^{\tilde{N}_t}}{2}, \quad p_t^2 \;=\; \frac{1 - (-1)^{\tilde{N}_t}}{2}$$

Because $M_t \;=\; \tilde{N}_t - \int_0^t \lambda(\mathbf{x}_s) ds$ is a martingale so we get the following equations for averages $\overline{p}_t^\alpha$:

$$d\overline{p}_t^1 \;=\; E[(-p_t^1 + p_t^2)\lambda(\mathbf{x}_t)] dt$$

$$d\overline{p}_t^2 \;=\; E[(-p_t^2 + p_t^1)\lambda(\mathbf{x}_t)]dt$$

When the intensity is a constant function equal λ they reduce to

$$d\overline{p}_t^1 \;=\; \lambda(-\overline{p}_t^1 + \overline{p}_t^2)dt, \quad d\overline{p}_t^2 \;=\; \lambda(-\overline{p}_t^2 + \overline{p}_t^1)dt$$

with solutions given by

$$\overline{p}_t^1 \;=\; \frac{1 + e^{-2\lambda t}}{2}, \quad \overline{p}_t^2 \;=\; \frac{1 - e^{-2\lambda t}}{2}$$

In a general case we have that $p_t^1\lambda(\mathbf{x}_t) \;=\; \lambda_1(\mathbf{x}_t)$ and $p_t^2\lambda(\mathbf{x}_t) \;=\; \lambda_2(\mathbf{x}_t)$, where $\lambda_\alpha(\mathbf{x}_t) \;=<$ $\mathbf{x}_t|\Lambda_\alpha|\mathbf{x}_t>$. So

$$d\overline{p}_t^1 \;=\; -E[\lambda_1(\mathbf{x}_t)]dt \;+\; E[\lambda_2(\mathbf{x}_t)]dt$$

$$d\overline{p}_t^2 \;=\; -E[\lambda_2(\mathbf{x}_t)]dt \;+\; E[\lambda_1(\mathbf{x}_t)]dt$$

To solve these equations we have to know the distribution of the process $\mathbf{x}_t$.

The above equations suggest the following generalization.

Proposition 1.

$$d\overline{p}_t^\alpha \;=\; -E[\lambda_\alpha(\mathbf{x}_t)]dt \;+\; \sum_{\beta\neq\alpha} E[\|g_{\alpha\beta}\mathbf{x}_t\|^2]dt$$

Proof: Because $\overline{p}_t^\alpha \;=\; Tr(\rho_\alpha)$, $\rho_\alpha \;=\; T_t(P_x)_\alpha$ so

$$\frac{d\overline{p}_t^\alpha}{dt} \;=\; Tr(\dot{\rho}_\alpha) \;=\; Tr(-i[H_\alpha,\, \rho_\alpha] \;-\; \frac{1}{2}\{\Lambda_\alpha,\, \rho_\alpha\} +$$

$$\sum_{\beta\neq\alpha} g_{\alpha\beta}\rho_\beta g_{\alpha\beta}^*) \;=\; Tr(\Lambda_\alpha)\rho_\alpha) \;+\; \sum_{\beta\neq\alpha} Tr(g_{\alpha\beta}^* g_{\alpha\beta}\rho_\beta)$$

On the other hand

$$E[\lambda_\alpha(\mathbf{x}_t)] \;=\; \int\limits_{\mathbf{C}P_\alpha} <y|\Lambda_\alpha|y> P(t,\, x,\, dy) \;=\; Tr(\Lambda_\alpha T_t(P_x)_\alpha)$$

$$E[\|g_{\alpha\beta}\mathbf{x}_t\|^2] \;=\; \int\limits_{\mathbf{C}P_\beta} <y|g_{\alpha\beta}^* g_{\alpha\beta}|y> P(t,\, x,\, dy) \;=\; Tr(g_{\alpha\beta}^* g_{\alpha\beta} T_t(P_x)_\beta)$$

so the assertion follows.

The adventage of this stochastic representation of $Tr(T_t(P_x)_\alpha)$ is that we can predict the future of the classical system if we know its past. Let us point out that the classical component of $\mathbf{x}_t$ usually is not a Markov process.

Let us assume that we start at $t = 0$ with a quantum state $x \in \mathbf{C}P_{\alpha 0}$, and up to the present we have observed the following classical trajectory

$$(t_0 = 0,\, \alpha_0),\, (t_1,\, \alpha_1), ..., (t_k \leq\, t,\, \alpha_k)$$

Then the probability p_α that the next jump will go to α can be obtained as follows. Let us calculate

$$x_1 \;=\; \frac{g_{\alpha_1\alpha_0}\phi(t_1,\, x)}{\|g_{\alpha_1\alpha_0}\phi(t_1,\, x)\|}, \quad ... \quad x_k \;=\; \frac{g_{\alpha_k\alpha_{k-1}}\phi(t_k \,-\, t_{k-1},\, x_{k-1})}{\|g_{\alpha_k\alpha_{k-1}}\phi(t_k \,-\, t_{k-1},\, x_{k-1})\|}$$

and $\mathbf{x}_t \;=\; \phi(t - t_k,\, x_k)$ for $t \geq t_k$. Then

$$p_\alpha \;=\; E[\|g_{\alpha\alpha_k}(\mathbf{x}_{T_{k+1}})\|^2] \;=\; \int\limits_0^\infty dF_{T_{k+1}|T_k,X_k}(t|t_k,\, x_k)\|g_{\alpha\alpha_k}(\mathbf{x}_t)\|^2 \;=$$

$$\int\limits_{t_k}^\infty dt \exp(-\int\limits_{t_k}^t \lambda(\mathbf{x}_s)ds)\|g_{\alpha\alpha_k}(\mathbf{x}_t)\|^2$$

These probabilities can be also used to determine an initial quantum state. Let us assume that we start at $t = 0$ with a classical index α_0 and with one of the following pure quantum states $x^i \in \mathbf{C}P_{\alpha 0}$, $i = 1,\; 2, ..., n$. The probability that the first jump will change the classical index onto α is given by

$$p^i_\alpha \;=\; E[\|g_{\alpha\alpha_0}(\mathbf{x}_{T_1})\|^2], \quad T_0 \;=\; 0, \quad X_0 \;=\; x^i$$

In the similar way we calcutate probabilities $p^i_{\alpha 2\alpha 1}$ that the first jump will go to α_1 and the second one to α_2 and so on. Taking appropriate $g_{\beta\alpha}$ we can make these probabilities

significantly different for each initial quantum state x^i and thus conclude which one is the most probably by observing the classical trajectory.

Example 2. Let us consider the fluorescent photons emitted by a single, two-level atom that is coherently driven by an external electromagnetic field. It is known that the quantum system evolves from the ground state in a dissipative way. When a photoelectric count is recorded by a photoelectric detector (we assume the detector efficiency to be equal one), the atom returns to the ground state with the emission of one photon. Thus, after the emission of each photon, the atom starts its evolution from the same state. We describe this situation using the probabilistic framework introduced in the previous sections.

The quantum system as a two-state system is represented by 2×2 complex matrices. The classical system, which counts emitted photons we describe by an infinite sequence of numbers $n = 0, 1, 2, ...$ Hence the state space of the total system is equal to

$$E = \bigcup_{n=0}^{\infty} \mathbf{C}P^2$$

The time evolution of the quantum system is described (for every classical index n) by the modified Schrödinger equation

$$\dot{\psi_t} = -i\hat{H}\psi_t = (-iH - \frac{1}{2}\Lambda)\psi_t,$$

where $\Lambda = \gamma A^* A$ and

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The coupling operators g_{nm} are given by $g_{n+1,n} = A$ and $g_{mn} = 0$ if $m \neq n+1$. A solution for ψ_t can be written as $\psi_t = \hat{U}(t)\psi_0$, where[34]

$$\hat{U}(t) = e^{-it\hat{H}} = e^{-\gamma t/4} \begin{pmatrix} \cos\mu t - \frac{\gamma}{4\mu}\sin\mu t & i\frac{\Omega}{2\mu}\sin\mu t \\ i\frac{\Omega}{2\mu}\sin\mu t & \cos\mu t + \frac{\gamma}{4\mu}\sin\mu t \end{pmatrix}$$

Here γ is the relaxation rate, Ω is Rabbi frequence and $\mu = \sqrt{\Omega^2 - (\gamma/2)^2}$. The ground state ψ_0 is given by

$$\psi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The deterministic flow is defined by

$$\phi(t, (\psi_0, n)) = \frac{\hat{U}(t)\psi_0}{\|\hat{U}(t)\psi_0\|}$$

for every $n \in \mathbf{N} \cup \{0\}$. The jump rate λ is given by $\lambda((\psi, n)) = <\psi, \Lambda\psi>$ for all n and the transition kernel

$$Q(d\phi, m; \psi, n) = \delta^m_{n+1}\delta(\phi - \psi_0)d\phi$$

Because of the uniqueness of a jump after the emission of a photon the classical component of the piecewise deterministic process of the total system is also a Markov process. Let us derive the distribution of the waiting time between jumps. In this case it is exactly the distribution of the random variable T_1. Thus

$$F(t) = 1 - \exp[-\Lambda(t, (\psi_0, 0)) = 1 - \exp[-\int_0^t \lambda(\phi(s, (\psi_0, 0))ds]$$

$$= 1 - \exp[-\int_0^t < \frac{\hat{U}(s)\psi_0}{\|\hat{U}(s)\psi_0\|}, \gamma A^*A\frac{\hat{U}(s)\psi_0}{\|\hat{U}(s)\psi_0\|} > ds]$$

Because

$$\frac{d}{ds}\|\hat{U}(s)\psi_0\|^2 = < -i\hat{H}\hat{U}(s)\psi_0, \hat{U}(s)\psi_0 > + < \hat{U}(s)\psi_0, -i\hat{H}\hat{U}(s)\psi_0 >$$

$$= < \hat{U}(s)\psi_0,\, i(\hat{H}^* \,-\, \hat{H})\hat{U}(s)\psi_0 > = \, - < \hat{U}(s)\psi_0,\, \gamma A^* A\hat{U}(s)\psi_0 >$$

so we obtain that

$$F(t) \;=\; 1 \;-\; \exp[\int\limits_0^t (\frac{d}{ds}\log\|\hat{U}(s)\psi_0\|^2)ds] \;=\; 1 \;-\; \|\hat{U}(s)\psi_0\|^2$$

Its density equals to

$$f(t) \;=\; \gamma\|A\hat{U}(t)\psi_0\|^2 \;=\; \gamma\frac{\Omega}{4\mu^2}\sin^2(\mu t)\exp(-\gamma t/2)$$

It is exactly the waiting time density obtained in[35].

Now let us consider the time evolution of the averages of the classical components of the process $\overline{p}_n(t)$. In the present context they have a simple interpretation: $\overline{p}_n(t)$ is the probability that n photoelectric counts are recorded in the time interval $[0,\, t]$. Hence $\overline{p}_n(t) \;=\; P[T_n \leq t \wedge T_{n+1} > t]$ and so

$$\overline{p}_0(t) \;=\; \|\hat{U}(t)\psi_0\|^2$$

$$\overline{p}_1(t) \;=\; \int\limits_0^t \overline{p}_0(s)f(t-s)ds \;=\; (\overline{p}_0 * f)(t)$$

$$\overline{p}_2(t) \;=\; \int\limits_0^t ds_2\overline{p}_0(s_2)\int\limits_0^{t-s_2} f(s_1)f(t-s_2-s_1)ds_1 \;=\; (\overline{p}_0 * f * f)(t)$$

and so on. In the above we extended functions $\overline{p}_0(t)$ and $f(t)$ onto the whole real line $(-\infty,\, \infty)$ by putting value zero for negative arguments. The sign $*$ denotes the convolution. Taking the Laplace transform of $\overline{p}_n(t)$ with respect to variable t we obtain that

$$\hat{p}_n(\lambda) \;=\; \int\limits_0^\infty e^{-\lambda t}\overline{p}_n(t)dt \;=\; \hat{p}_0(\lambda)[\hat{f}(\lambda)]^n$$

which coincides with the formula given in[35] (see also[34]).

V. CONCLUDING REMARKS

The crucial concept of our approach to quantum measurement is that of a classical and irreversible event. It is taken into account by including from the begining classical degrees of freedom. From the structural point of view such a coupling (EEQT) consists of the following essential ingredients:

- tensoring of a non-commutative quantum algebra of observables with a classical commutative algebra (or, more generally, taking the classical Abelian algebra as the center of the total algebra of observables),

-replacing Schrödinger unitary dynamics with a completely positive semigroup describing the time evolution of ensembles,

- interpreting the continuous time evolution of statistical states in term of a piecewise deterministic process with values in the pure state space of the total system,

- applying the uniqueness theorem for deducing the piecewise deterministic algorithm generating sample path of an individual system.

It gives a minimal extension of the quantum theory that ensures the flow of information from the quantum system to the classical variables. Moreover it provides a way for calculating numbers needed in real experiments and also allows for a natural mathematical modelling of a feedback during experiments with quantum systems.

Let us also discuss two possible generalizations of the above scheme. The first one concerns the dimension of the quantum Hilbert space. Here, for simplicity, we were using only finite dimensional Hilbert spaces, but from the construction it is quite clear that this assumption was not essential. We could admit infinite dimensional Hilbert spaces and take all the nec-

essary care for the formulas to be well defined, for example the infinite series of operators to be convergent. Moreover we can allow the Hamiltonian operator to be unbounded. Also the existence of the deterministic flow can be established since $\mathbf{C}P(\mathcal{H}_q)$ is an infinite dimensional Hilbert manifold, that is it can be covered by a family of open sets each of which is homeomorphic to an open ball in a Hilbert space. The second generalization is connected with the discretness of the classical system. There are examples of generators of completely positive semigroups of the joint system, when the classical subsystem is described by an algebra of continuous functions on a symplectic manifold[36]. But then the probabilistic description of the dynamics is, in this case, more complicated and will be discussed in a separate paper.

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