BULLETIN DE L'ACADÉMIE POLONAISE DES SCIENCES
Série des sciences math., astr. et phys. — Vol. XX, No. 8, 1972

A Note on Scale Transformations

by

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Presented by J. RZEWUSKI on March 9, 1972

Summary. We define automorphisms of the algebra of quasilocal observables of the free scalar field of zero mass that correspond to scale transformations in $\nu+1$ space-time dimensions. It is shown that these automorphisms are unitarily implemented in the representations constructed by Streater and Wilde [3] for $\nu=1$. For $\nu=3$ we consider a family of vacuum representations and show that the only vacuum representation that exhibits dilatation symmetry is the Fock one.

1. Introduction. In this paper we make some elementary comments on automorphisms of observables corresponding to scale transformations. We consider free scalar field of zero mass in $\nu+1$ space-time dimensions. Classically such a field is described by solutions of the wave equation which is known to be invariant not only with respect to the Poincaré group but also with respect to the conformal group. This group acts on the Minkovski space in a locally causal way [1]. Its subgroup consisting of the restricted Poincaré group and the positive dilatations is known to be the group of all automorphisms of the causal structure of the Minkovski space [2]. We deal with this subgroup only. In Sec. 3 we show that the scale transformations are unitarily implemented in the representations constructed by Streater and Wilde [3] for $\nu=1$. We also consider a family of vacuum representations which have been discussed by Streater [4] and Doplicher [5]. The scale transformations are unitarily implemented in none of these representations except the Fock one.

2. The free scalar field of zero mass. a) *Notation.* Let $M^{\nu+1}$ be a $\nu+1$ dimensional Minkovski space with $(x_0, \bar{x})=x \in M^{\nu+1}$ and $xy=x_0 y_0-\bar{x}\bar{y}$. We write $(a, \wedge)$ for an element of $\mathcal{P}_+^\uparrow$-the restricted Poincaré group in $M^{\nu+1}$. For every function φ defined on R^ν, $\tilde{\varphi}(\bar{p})$ stands for the Fourier transform of φ

$$\tilde{\varphi}(p)=(2\pi)^{-\nu/2}\int \varphi(\bar{x})\, e^{i\bar{p}\bar{x}}\, d^\nu x.$$

Everywhere in this paper $p_0=|\bar{p}|=(\bar{p}\bar{p})^{1/2}$.

b) *The wave equation.* The wave equation

$$\Box\, \xi(x)=0$$

is invariant under $\mathcal{P}_+^\uparrow$. Let $\Delta(x)$ be the Jordan—Pauli invariant D-function

$$\Delta(x)=-i\,(2\pi)^{-\nu/2}\int \varepsilon(p_0)\,\delta(p^2)\, e^{ipx}\, d^{\nu+1} x.$$

The solution of the Cauchy problem at $x_0=0$ is given by

$$\xi(x)=\{\Delta(x-y), \xi(y)\}_{x_0=0},$$

where

$$\{\xi_1, \xi_2\}=\int \dot{\xi}_1(\bar{x}, 0)\,\xi_2(\bar{x}, 0)-\xi_1(\bar{x}, 0)\,\dot{\xi}_2(\bar{x}, 0)\,d^\nu x.$$

The set of real solutions $\xi(x)$ with $\xi(\bar{x}, 0), \dot{\xi}(\bar{x}, 0)\in\mathcal{D}(R^\nu)$ and $\int \dot{\xi}(x_1, 0)\,dx_1=0$ for $\nu=1$ is denoted by $\mathcal{M}^\nu$. For every open double cone $\mathcal{V}$ in $M^{\nu+1}$ we denote by $\mathcal{M}^\nu(\mathcal{V})$ the set of all $\xi\in\mathcal{M}^\nu$ such that ξ vanishes on $\mathcal{V}'=\{x: x\times\mathcal{V}\}$. We remark that the symplectic form $\{\xi_1, \xi_2\}$ on $\mathcal{M}^\nu$ is Poincaré inveariant and $\{\xi_1, \xi_2\}=0$ for $\xi_i\in\mathcal{M}^\nu(\mathcal{V}_i)$ with $\mathcal{V}_1\times\mathcal{V}_2$.

c) *The Fock space.* The single-particle states are described by rays in $H_1=L^2(R^\nu, d^\nu p/2p_0)$. The Fock space $\mathcal{H}_0$ is the direct sum of the H_n-the space of symmetric functions, square integrable with respect to the product measure

$$\frac{d^\nu p^1}{2(p^1)_0}\cdots\frac{d^\nu p^n}{2(p^n)_0}.$$

The subspace $D_0\subset\mathcal{H}_0$ consisting of those $\psi\in\mathcal{H}_0$ for which $\psi^{(n)}=0$ for n sufficiently large is dense in $\mathcal{H}_0$. For every $F\in L^2(R^\nu, d^\nu p)$ the annihilation and creation operators are defined in a standard way

$$(a(F)\psi)^{(n)}(\bar{p}_1, \dots, \bar{p}_n)=\sqrt{n+1}\int F(\bar{p})\,\psi^{(n+1)}(\bar{p}, \bar{p}_1, \dots, \bar{p}_n)\frac{d^\nu p}{\sqrt{2p_0}},$$

$$(a^*(F)\psi)^{(n)}(\bar{p}_1, \dots, \bar{p}_n)=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\sqrt{2|\bar{p}_j|}\;\psi^{(n-1)}(\bar{p}_1, \dots, \bar{p}_j, \dots, \bar{p}_n).$$

The operators a and a^* satisfy the canonical commutation relations

$$[a(F), a^*(F')]=\int F(\bar{p})\,F'(\bar{p})\,d^\nu p$$

$$a^*(F)\subset a(F^*)^*$$

$$a(F)\psi_0=0 \quad \text{where} \quad \psi_0=\{1, 0, 0, \dots\}$$

The action of the Poincaré group is given by

$$U_0(a, \wedge)\,\psi^{(n)}(\bar{p}_1, \dots, \bar{p}_n)=\exp\left[i\sum_{j=1}^{n} ap_j\right]\psi^{(n)}(\overline{\wedge^{-1}p_1}, \dots, \overline{\wedge^{-1}P_n}).$$

d) *The field.* The field $\Phi(h)$ is defined on D_0 for every $h\in\mathcal{D}(R^{\nu+1})$ for $\nu>1$. For $\nu=1$ one has to restrict test functions by the condition $\tilde{h}(0)=0$, where $\tilde{h}(\bar{p})$ is defined as

$$\tilde{h}(\bar{p})=(2\pi)^{-\nu/2}\int e^{-ipx}\,h(x)\,d^{\nu+1}x.$$

The set of real functions defined above is denoted by $\mathcal{D}_\nu(R^{\nu+1})$ and with h in $\mathcal{D}_\nu$ the field is given in terms of a and a^* by

$$\Phi(h)=a\left((2p_0)^{-1/2}\,\tilde{h}(\bar{p})\right)+a^*\left((2p_0)^{-1/2}\,\tilde{h}^*(\bar{p})\right).$$

The commutation relations $[\Phi(x), \Phi(y)] = i\Delta(x-y)$ can also be expressed by means of Weyl operators $W_0(h) = \exp[i\Phi(h)]$. We get

$$W_0(h)\, W_0(h') = \exp[i/2\, \{h, h'\}]\, W_0(h+h'),$$

where

$$\{h, h'\} = \int h(x)\, \Delta(y-x)\, h'(y)\, d^{\nu+1}x\, d^{\nu+1}y.$$

Relativistic covariance of the field is expressed by

$$U_0(a, \wedge)\, W_0(h)\, U_0(a, \wedge)^* = W_0(h_{(a,\wedge)}).$$

Since $W_0(h) = W_0(h')$ if $\tilde{h} = \tilde{h}'$ it is convenient to map $\mathcal{D}(R^{\nu+1})$ into $\mathcal{M}^\nu$ in such a way that $\tilde{h} = \tilde{h}'$ implies $\xi_h = \xi_{h'}$. The following map has this property

$$\xi_h(x) = \int \Delta(y-x)\, h(y)\, d^{\nu+1}y.$$

Also

$$\{h, h'\} = \{\xi_h, \xi_{h'}\}$$

and

$$\operatorname{supp} h \subset \mathcal{V} \quad \text{implies} \quad \xi_h \in \mathcal{M}^\nu(\mathcal{V}).$$

For every $\xi \in \mathcal{M}^\nu$ let $f(\bar{x}) = \dot{\xi}(\bar{x}, 0)$ and $g(\bar{x}) = \xi(\bar{x}, 0)$. Then the field $\Phi(f, 0)$ and the canonical momentum $\pi(g, 0)$ at $x_0 = 0$ are defined by

$$\Phi(f, 0) = \frac{1}{\sqrt{2}}\left[a\left((p_0)^{-1/2} \tilde{f}(\bar{p})\right) + a^*\left((p_0)^{-1/2} \tilde{f}(\bar{p})^*\right)\right],$$

$$\pi(g, 0) = -i\frac{1}{\sqrt{2}}\left[a\left((p_0)^{1/2}\, \tilde{g}(\bar{p})\right) - a^*\left((p_0)^{1/2}\, \tilde{g}(\bar{p})^*\right)\right].$$

We have

$$\Phi(h) = \Phi(f, 0) - \pi(g, 0),$$
$$f(\bar{x}) = \dot{\xi}(\bar{x}, 0),$$
$$g(\bar{x}) = \xi(\bar{x}, 0).$$

e) *The algebras of observables.* For every $\xi \in \mathcal{M}^\nu$ let

$$W_0(\xi) = \exp\left[i\left(\Phi(f, 0) - \pi(g, 0)\right)\right].$$

The operators W_0 satisfy the Weyl commutation relations

$$W_0(\xi)\, W_0(\xi') = \exp[i/2\, \{\xi, \xi'\}]\, W_0(\xi+\xi').$$

For every open double cone $\mathcal{V}$ let $\mathfrak{A}(\mathcal{V})$ be the von Neumann algebra generated by $W_0(\xi)$ with $\xi \in \mathfrak{M}^\nu(\mathcal{V})$. Then the algebras $\mathfrak{A}(\mathcal{V})$ satisfy the following properties

(i) if $h \in \mathcal{D}_\nu(R^{\nu+1})$ and $\operatorname{supp} h \subset \mathcal{V}$ the $W_0(h) \in \mathfrak{A}(\mathcal{V})$,

(ii) $U_0(a, \wedge)\, \mathfrak{A}(\mathcal{V}) = \mathfrak{A}(\wedge \mathcal{V} + a)\, U_0(a, \wedge)$,

(iii) $\mathfrak{A}(\mathcal{V}) \subset \mathfrak{A}(\mathcal{V}')'$ for $\mathcal{V} \times \mathcal{V}'$,

(iv) $\mathfrak{A}(\mathcal{V}) \subset \mathfrak{A}(\mathcal{V}')'$ for $\underset{\mathcal{V}}{\overset{\mathcal{V}'}{\times}}$,

(v) $\mathfrak{A} = C^*\left(\bigcup_{\mathcal{V}} \mathfrak{A}(\mathcal{V})\right)$ is irreducible.

3. The scale transformations. The wave equation is invariant with respect to scale transformation:

$$\xi_\lambda(x)=\lambda^{-\frac{\nu-1}{2}}\,\xi(\lambda^{-1}x).$$

It is easy to see that $\mathcal{M}^\nu$ is invariant under these transformations and $\xi\in\mathcal{M}^\nu(\mathcal{V})$ implies $\xi_\lambda\in\mathcal{M}^\nu(\lambda\mathcal{V})$. Moreover, the fundamental symplectic form $\{\xi,\xi'\}$ is also invariant. Thus the map $W_0(\xi)\to W_0(\xi_\lambda)$ induces an automorphism $\mathcal{T}_\lambda$ of $\mathfrak{A}$. Clearly $\mathcal{T}_\lambda[\mathfrak{A}(\mathcal{V})]=\mathfrak{A}(\lambda\mathcal{V})$.

Let μ_0 be the generating functional for the Fock representation:

$$\mu_0(\xi)=\big(\psi_0,W_0(\xi)\,\psi_0\big)=\exp\left[-1/4\,\|\xi\|^2\right]$$

where

$$(\xi,\xi')=(F_\xi,F_{\xi'})=\int F_\xi(\bar{p})^*\,F_{\xi'}(\bar{p})\,d^\nu p$$

and the mapping $\mathcal{M}^\nu\ni\xi\to F_\xi\in L^2(R^\nu,d^\nu p)$ is given by

$$F_\xi(\bar{p})=(p_0)^{1/2}\,\tilde{g}(\bar{p})-i\,(p_0)^{-1/2}\tilde{f}(\bar{p}).$$

Since the operator $F_\xi\to F_{\xi_\lambda}$ is unitary it follows that μ_0 is invariant under dilatation and therefore the automorphisms $\mathcal{T}_\lambda$ are unitarily implemented on the Fock space.

Let now π be a representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}_\pi$. Let for each $\lambda>0$ a new representation π_λ be given by $\pi_\lambda(A)=\pi\big(\mathcal{T}_\lambda(A)\big)$. If $U_\pi(a,\wedge)$ is a representation of $\mathcal{P}_+^\uparrow$ implementing the Poincaré automorphisms on $\mathcal{H}_\pi$, then $\mathcal{P}_+^\uparrow$ is unitarily implemented on $\mathcal{H}_{\pi_\lambda}$ too. It is sufficient to define $U_{\pi_\lambda}(a,\wedge)=U_\pi(\lambda a,\wedge)$. The generators of translations are given by $(P_{\pi_\lambda})^\mu=\lambda\,(P_\pi)^\mu$ so, that the spectrum condition holds for π_λ provided it held for π (compare [6]). However, there are representations satisfying spectrum condition with dilatation symmetry being broken. Intuitively, some scale of length or field strength is distinguished in these representations. We give below examples.

a) *The Skyrme—Streater—Wilde model.* The model proposed by Skyrme [7] and linearized and put into a rigorous mathematical framework by Streater and Wilde [3] deals with the zero-mass free scalar field in two dimensions ($\nu=1$). We briefly describe the relevant points.

Let $\mathcal{N}(\mathcal{V})$ be the set of real solutions of $\Box\,\theta(x)=0$ with $\dot{\theta}\in\mathcal{M}(\mathcal{V})$. For every $\theta\in\mathcal{N}=\bigcup_{\mathcal{V}}\mathcal{N}(\mathcal{V})$ the transformations

$$W_0(\xi)\to e^{i\{\theta,\xi\}}\,W_0(\xi)$$

induce an automorphism γ of the algebra $\mathfrak{A}$. SW prove that the automorphisms γ lead to a family of inequivalent representations $\pi_{\alpha,\beta}$ of $\mathfrak{A}$ parametrized by the two parameters $\alpha=\theta(\infty,0)$ and $\beta=\theta(0,\infty)$. We show that the scale transformations are unitarily implemented in all these "charged" sectors. In fact, with $\xi\in\mathfrak{M}$, $\theta\in\mathcal{N}$, $\lambda>0$ we obtain $\gamma_\theta\circ\mathcal{T}_\lambda[W_0(\xi)]=\mathcal{T}_\lambda\circ\gamma_{\theta_{\lambda^{-1}}}[W_0(\xi)]$ and so $\gamma_\theta\circ\mathcal{T}_\lambda=\mathcal{T}_\lambda\circ\gamma_{\theta_{\lambda^{-1}}}$. However, since $\nu=1$, we have $\theta_\lambda(x)=\theta(\lambda x)$ and therefore $\theta_\lambda(\infty,0)=\theta(\infty,0)$ and $\theta_\lambda(0,\infty)=\theta(0,\infty)$. It follows that $\pi_{\alpha,\beta}\circ\mathcal{T}_\lambda\cong\pi_{\alpha,\beta}$ which completes the proof.

b) *Vacuum representations*. Streater [4] has shown that there exists a degeneracy of the vacuum for mass-zero scalar particles. In fact, let us define for each $s, t \in R$

$$W_{s,t}(f, g) = e^{i[s\tilde{f}(0)+t\tilde{g}(0)]}\, W_0(f, g).$$

Then $(f, g) \to W_{s,t}(f, g)$ is a representation of the CCR which is not equivalent to the Fock one unless $s=t=0$ (see e.g. [8], Th. 7.1). However, the Poincaré automorphisms are not unitarily implemented for $W_{s,t}$ for $t \neq 0$. In fact, for each time-translation a_0 we get $|\tilde{g}(0) - \tilde{g}_{a_0}(0)| = |a_0|\, \tilde{f}(0)$ and the statement follows by the non-continuity of the functional $f \to \tilde{f}(0)$ with respect to the norm we have introduced on $\mathcal{M}$. On the other hand, $\tilde{f}(0)$ is Poincaré invariant and for every bounded $\mathcal{V}$ one can find $\xi_{\mathcal{V}}$ such that

$$W_s(\xi) = W_0(s\xi_{\mathcal{V}})\, W_0(\xi)\, W_0(s\xi_{\mathcal{V}})^* \qquad \forall\, \xi \in \mathcal{M}(\mathcal{V}).$$

Therefore the map $W_0(\xi) \to W_s(\xi)$ extends to an unique automorphism γ_s of the algebra and defines new representation $\pi_s = \pi_0 \circ \gamma_s$ where π_0 is the Fock one (see [4]). Clearly γ_s commute with all $\mathcal{T}_g, g \in \mathcal{P}_+^\uparrow$ and therefore $\mathcal{T}_g$ is unitarily implemented in π_s by the same operators as in π_0. In particular, spectrum condition is satisfied in all π_s. All these representations coincide with the Fock one on a norm-dense subset of $\mathcal{M}^\nu$ and differ from the Fock one for $\nu > 1$ only. Modifying the arguments given by Doplicher [5] one can show that the π_s are the only vacuum representations with spectrum condition satisfied (after assuming some smoothness of $a(f, g)$). It is interesting to note that there is exactly one vacuum representation in which scale transformations are implemented. This is the Fock one. In all other π_s we have broken scale invariance. In fact, we have $\mathcal{T}_\lambda \circ \gamma_s = \gamma_{\lambda^{\frac{1-\nu}{2}} \cdot s} \circ \mathcal{T}_\lambda$.

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А. З. Ядчик, **Замечания о масштабе преобразований**

Содержание. В настоящей работе определены автоморфизмы алгебры квази-локальных обсерваблей для освободного скалярного поля нулевой массы, соответствующие масштабным преобразованиям в пространстве — времени $\nu+1$ измерений. Показано, что эти автоморфизмы имеют унитарное представление, построенное Стритером и Уайльдом для $\nu=1$. Для $\nu=3$ в работе рассматривается семейство вакуумных представлений. Показано, что единственным вакуумным представлением, проявляющим масштабную симметрию, является Фоковское представление.