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A geometrical approach to general relativistic Galilei quantum mechanics

Arkadiusz Jadczyk† and Marco Modugno‡

† Institute of Theoretical Physics, pl. Maxa Borna 9, 50–204 Wrocław, Poland
‡ Department of Applied Mathematics, 'G. Sansone', Via S. Marta 3, 50139 Florence, Italy

1. Introduction

First we present a geometrical model for the Galilei general relativistic classical space–time.

The classical space–time is a fibred manifold $t : \mathbf{E} \to \mathbf{T}$ or dimension 4 over a one dimensional oriented affine space $\mathbf{T}$. A motion is defined to be a section $s : \mathbf{T} \to \mathbf{E}$. The first jet bundle $J_1\mathbf{E} \to \mathbf{E}$ can be interpreted both as the space of velocities $j_1 s : \mathbf{T} \to J_1\mathbf{E}$ and of observes $o : \mathbf{E} \to J_1\mathbf{E}$.

Space–time is equipped with a positive definite vertical metric $g : \mathbf{E} \to S^2V^*\mathbf{E}$, which yields a space–time volume form.

A space–time connection is defined to be a dt–preserving linear connection $K : T\mathbf{E} \to T^*\mathbf{E} \otimes TT\mathbf{E}$ on the manifold $\mathbf{E}$, or equivalently, an affine torsion free connection $\Gamma : J_1\mathbf{E} \to T^*\mathbf{E} \otimes TJ_1\mathbf{E}$ on the bundle $J_1\mathbf{E} \to \mathbf{E}$. A space–time connection Γ yields a second order connection $\gamma : J_1\mathbf{E} \to T^*\mathbf{T} \otimes TJ_1\mathbf{E}$ and a contact 2–form $\Omega : J_1\mathbf{E} \to \Lambda^2 T^* J_1\mathbf{E}$, which characterize Γ itself.

We assume space–time to be equipped with a space–time "gravitational" connection $\Gamma^\natural$ and an "electromagnetic" field $F : \mathbf{E} \to \Lambda^2\Gamma^*\mathbf{E}$. Given a mass m and a charge q, there is a unique consistent way to deform the gravitational space–time connection $\Gamma^\natural$ and the related second order connection $\gamma^\natural$ and contact 2–form $\Omega^\natural$ into corresponding dynamical objects Γ, γ and Ω, through the electromagnetic field F.

We introduce the gravitational and electromagnetic field equations in the following way.

First, we assume the dynamical contact 2–form Ω to be closed. This equation implies the gravitational connection $\Gamma^\natural$ to be metrical, yields the standard algebraic Riemannian identities for the curvature of $\Gamma^\natural$ and the first Maxwell equation for the electromagnetic field F.

Moreover, given an incoherent charged fluid, we define a time–like energy tensor and assume the dynamical space–time connection to fulfill the Einstein equation. This equation yields the law of gravitation and the second Maxwell equation.

The only observer independent approach to classical mechanics can be achieved in terms of the second order dynamical connection γ. The Hamiltonian and Lagrangian approaches depend on the choice of a frame of reference.

The contact 3–form Ω is degenerate and yields a Hamiltonian formalism on the

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odd dimensional bundle J_1E. This result is more interesting for quantum than for classical mechanics.

The Galilei special relativistic classical fields and mechanics are obtained as a particular case of the previous model.

2. Galilei general relativistic quantum mechanics

The quantum mechanical model is introduced in the following way.

The quantum bundle is defined to be a Hermitean complex one dimensional bundle $\pi : \mathbf{Q} \to \mathbf{E}$. A quantum wave function is defined to be a section $\Psi : \mathbf{E} \to \mathbf{Q}$.

Our main assumption is the following. We assume the pullback $\mathbf{Q}^\uparrow \to J_1\mathbf{E}$ (of the quantum bundle $\mathbf{Q} \to \mathbf{E}$ over the bundle of classical observers $J_1\mathbf{E} \to \mathbf{E}$) to be equipped with a Hermitean linear complex connection $\kappa : \mathbf{Q}^\uparrow \to T^* J_1\mathbf{E} \otimes T\mathbf{Q}^\uparrow$, which is a "universal connection" and such that its curvature is $i\frac{m}{\hbar}\Omega$.

Thus, the quantum connection κ depends on the observers, but we are looking for objects naturally associated with κ and observer independent. On the other hand, we notice that κ yields, in several contexts, pairs of observer depend objects (with the same source an target) and that there is a unique coupling of them, which is observer independent. This guideline leads us to discover the basic intrinsic quantum objects.

By this geometrical criterion, we find the intrinsic four-dimensional tangent valued quantum momentum $\wp_\Psi$ associated with a section Ψ.

Analogously, we find the intrinsic quantum lagrangians on the quantum bundle, or, equivalently, on its principal bundle. The derived Euler–Lagrange equation turns out to be the Schroedinger equation, which includes the correction terms related to the curved classical metric, gravity and electromagnetism. The momentum obtained by the Lagrangian formalism coincides with the aforesaid quantum momentum.

We give a further geometrical interpretation of the Schroedinger operator as the sum of the time-like covariant differential of the quantum section Ψ and the divergence $\delta\wp_\Psi$ of its quantum momentum $\wp_\Psi$. We observe that both terms are observer dependent, but their difference is observer independent.

A similar technique is used to obtain the probability current.

Finally, we classify all projectable vector fields $X : \mathbf{Q}^\uparrow \to T\mathbf{Q}^\uparrow$, which preserve the basic structures (Hermitean complex and quantum connection); namely, we prove that they are of the type $X_\phi = \kappa(\phi^\#) + i\phi$, where $\phi : J_1\mathbf{E} \to \mathbb{R}$ is a (classical) function and # denotes the classical Hamiltonian isomorphism. The map $\phi \to X_\phi$ turns out to be a Lie algebra isomorphisms. In particular, we recover the standard time, position, momentum and energy operators.

The Hilbert treatment of the above operators is obtained by introducing a Hilbert bundle over time and by interpreting the Schroedinger operator as a connection on this infinite dimensional bundle.

3. Comments

Thus, the above formulation of classical and quantum mechanics seems to have several original aspect. In particular, it is fully geometrical, general relativistic (in the Galilei sense. i.e. with absolute time, but with a curved space–time) and explicitly covariant (i.e. intrinsic).

Moreover, a fundamental role is played by the fibring over time, connections and jet spaces; conversely, the Hamiltonian and Lagrangian formalisms have a limited application.

We expect that the above method can be extended to particles with spin and to the Einstein general relativistic framework.

References

Further details can be found in the comprehensive forthcoming paper:
Jadczyk A and Modugno M *A geometrical approach to the Schrödinger equation on a curved space–time*