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DIMENSIONAL REDUCTION AND KALUZA-KLEIN THEORIES. SOME DIFFERENTIAL GEOMETRIC IDEAS

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ABSTRACT

We discuss geometrical aspects of the consistency problem in Kaluza-Klein theories.

1) The hypothesis that space-time may have more than four dimensions deserves the most serious study. This idea is usually connected with the papers by Kaluza [1] and Klein [2], as these authors were the first to implement in terms of physical concepts and mathematical equations the idea that nurtured the philosophers since long (see e.g. [3], [4, Ch.X]). The methods of Kaluza and Klein, who proposed to use a five-dimensional Riemannian metric for a unified description of Einstein's gravity and Maxwell electromagnetism in four dimensions, were further developed with the invention of non-Abelian gauge theories [5-8] and, later, supergravity [9-10]. A new motivation for studying physics in higher dimensions, and also the way it can produce observable phenomena in our four-dimensional world, came with string and superstring theories which, as a rule, need for a consistent quantum formulation more than four (e.g. 26 or 10) space-time dimensions[1)]

1) We refer the reader to the review article [11] for more information, references, and recent applications of the Kaluza-Klein ideas.

2) Accepting the hypothesis of more than four space-time dimensions one has to, consequently, take the extra dimensions seriously; otherwise the theory has no more predictive power than an ad hoc built proviso. Now, what it does mean "to take the extra dimensions seriously"? It means that one has to find a convincing answer to the following question:"why the number of dimensions in the world we see under normal conditions seems to be four, and which (physical) conditions are necessary for the extra dimensions to show up"? The physical mechanism (and its mathematical description) by which the number of "effective" (whatever this term may mean) dimensions is reduced is known under the name *dimensional reduction*. Now, of course, there are many ways of, formally, reducing the number of dimensions, for instance, one can simply forget about some of them or put the corresponding coordinates to some fixed values. However, if the extra dimensions are to be taken seriously, then the dimensional reduction must be, at least, *consistent*. In the following we will make the word "consistent" into a precise technical term, but first let us look at few examples.

3) Consider an N-body problem of Newton's gravitation. Here the number of dimensions can be consistently reduced from three to two. Indeed, once the initial conditions (positions and velocities) are restricted to a plane, $z=0$, then the motion will be planar forever. This of course does not explain *why*, for instance, our planetary system hppens to be nearly planar; nevertheless it tells us that it is *consistent* to freeze one of the three space dimensions. The resulting two-dimensional theory of gravitation is *consistent* with the original three-dimensional one. It is also worthwhile to observe that one could, in principle, rediscover the third dimension by contemplating the force law of the effective planar theory; indeed, the effective planar force results from the potential $1/r$, which is a solution of the Laplace equation in *three dimensions* but is rather unnatural in two.
Similar considerations apply if the bodies are charged and act on themselves by, say instantaneous, electromagnetic forces. But there is no consistent reduction to two dimensions if magnetic poles are also allowed. Indeed, for a generic initial planar configuration (exceptional

configurations may be, however, possible), the Biot-Savart law will make the motion necessarily three dimensional.

4) The above examples, although transmitting quite well the idea of consistent versus inconsistent reduction of degrees of freedom, should be taken only as a hint as to the situation occurring in a field theory[2]. The best example of an *inconsistent* dimensional reduction in field theory is given by the "Ansatz" made by Kaluza and Klein:the reduction of a five-dimensional gravity to four-dimensional gravity and electromagnetism is inconsistent since generic solutions of four-dimensional Einstein-Maxwell equations cannot be interpreted as special solutions of Einstein's gravity in five dimensions. However, the situation changes if the content of the effective four-dimensional theory is enlarged just by one field - the Jordan-Thiry scalar field carrying the information about space-time dependence of the radius of the internal space. We shall discuss this example later on in some more details.

5) We will now make the term "consistent reduction" formally precise. To this end it is instructive to consider the mechanism of dimensional reduction as a particular case of a more general mathematical scheme. Let A be a function on a manifold F (with F possibly infinite dimensional Hilbert or Banach manifold). Let F' be a submanifold of F. We will say that "the reduction from F to F' is consistent with respect to A" if every critical point of the function A restricted to F' is also a critical point of A in F. In other words: if for every point $\phi \in F'$ such that $(dA)(X)=0$ for all vectors X tangent to F' at ϕ, we automatically have $(dA)(X)=0$ for all X tangent to F at ϕ. The most evident example of a submanifold F' which is not consistent with A is to take for F one of the constant-value submanifolds for A : $F'_c=\{\phi \in F: A[\phi]=c\}$.Then, clearly, every point of F'_c is a critical point for A restricted to F'_c but, in general, not a critical point of A in F.

2) Indeed, mechanics can be considered as a one-dimensional field theory, so the reduction of some space dimensions of a mechanical system is analogous to reduction of the number of field components rather than field's argumensts

In physical applications we can take for F a manifold of field configurations of some physical system, and for A the action functional, usually constructed from some local Lagrangian density L. Then a reduction of field configurations from F to F' can be interpreted as freezing of some degrees of freedom of the system to their constant values. Effectively one obtains a reduced (or constrained) physical system with the action functional A' which is induced from A by simply restricting its domain to F'. Such a freezing of some degrees of freedom and the corresponding reduction of the system from (F,A) to (F',A') is then said to be "consistent", or "consistent" with the dynamics" if every solution of the field equations of the reduced (or "constrained") system is also a solution of the field equations of the original system (cf. [12,13]). Since the solutions of field equations of a classical field theory are nothing but critical points of the actionfunctional, it is clear that the two concepts of consistency coincide.

6) Let us now clarify some of the above points. In classical mechanics the action functional is a function on the space of all possible trajectories, and the trajectories which are the critical points of this function are candidates for the classically observed motions - they satisfy the Euler-Lagrange equations. Similarly in field theory, the action functional is a function on the space of all possible space-time field configurations, and the configurations which are critical points of this function are those satisfying classical field equations.
Let F be a manifold, thought of as the manifold of (continuous, or smooth) trajectories or field configurations of some physical system S living in a space-time manifold M, and let $A: F \to \mathbb{R}$ be a function on F, thought of as representing the action functional of the system. Typically F will consists of smooth functions $\phi : M \to \mathbb{R}$, and will be of the form

$$A[\phi] = \int_M L(\phi, \partial\phi) \quad ,$$

where $L(\phi,\partial\phi)$ is a Lagrangian density. (Of course, for the above action function to have a meaning one has to take M compact, or ϕ of compact support, or to take some other appropriate measures; we will not enter into a discussion of these problems here). For instance in a standard Kaluza-Klein theory one starts with F being the space of all pseudo-Riemannian metrics g_{AB} on M (usually with signature (-+++...)), and A - the Einstein-Hilbert action functional

$$A[g] = \int_M (R[g] + \Lambda)\ |\det g_{AB}|^{\frac{1}{2}} d^m x,$$

where R[g] is the scalar curvature of g, Λ is a (cosmological) constant, and $m=\dim M$.

7) Dimensional reduction of a physical system S implies a particular choice of a submanifold of the manifold F of field configurations of S, a choice usually implemented by a certain "Ansatz" or, as we will see, by some constraint equations. While many reasonable submanifolds of F can be usually chosen, dimensional reduction (as implied by its very name) distinguishes a certain submanifold F' of F in such a way that F' is isomorphic to a manifold of all field configurations of some other physical system S' living on a manifold M', with $m'=\dim M'<m=\dim M$.

Example

Let us take for F the manifold of Riemannian metrics on a product manifold $M=M'\times\mathbb{R}$, and let F' be a submanifold of F consisting of all those metrics $g_{AB}(x,\xi)$, $x\in M$, $\alpha\in\mathbb{R}$, which are of the form

$$g_{AB}(x,\xi) = \begin{pmatrix} g_{\alpha\beta}(x)+\lambda(x)A_\alpha(x)A_\beta(x), & \lambda(x)A_\beta(x) \\ \lambda(x)A_\alpha(x) & , & \lambda(x) \end{pmatrix}$$

where $A,B=1,2,\dots,m$ and $\alpha,\beta=1,2,\dots,m'=m-1$; $g_{\alpha\beta}(x)$, $A_\alpha(x)$ and $\lambda(x)$ are, respectively, a Riemannian metric, a one-form, and a scalar field on M'. This is the Ansatz for an $m'+1$ dimensional Kaluza-Klein theory with the real line $\mathbb{R}$ as internal space, and with a Jordan-Thiry field $\lambda(x)$.
A still smaller submanifold F_1' can be distinguished as consisting of

g_{AB}'s of the above form, but with the Jordan-Thierry field frozen to a prescribed fixed value: $\lambda(x)\equiv l$ (usually one puts $l=1$). We will see that the first reduction, from F to F', is *consistent* with the Einstein -Hilbert action functional, while the second one, which historically was the first, is not.

Remark

It should be observed that the above choice of F (or F_1') was possible owing to the assumed extra structure carried by the manifold M; in this case by the product structure $M=M'\times\mathbb{R}$. For a physical understanding of the Kaluza-Klein mechanism one has to explain this extra structure too. Usually it is assumed that it results from some kind of a "spontaneous compactification of extra-dimensions" , the term suggesting an analogy to the mechanisms of a "spontaneous magnetisation" etc.

8) In [14] a general geometrical scheme of the so called "G-invariant dimensional reduction" was developed. One starts there with a manifold M on which a compact Lie group G acts (say, from the right) by a simple (all orbits of the same type) action. One takes then for the field configuration manifold F the space of all Riemannian metrics g_{AB} on M and for F' the submanifold of all G-invariant metrics. One proves then that F' is isomorphic to the manifold of field configurations on $M'=M/G$ (the manifold of orbits), where a field configuration on M' is a triplet consisting of: a Riemannian metric $g_{\alpha\beta}$ on M', a principal connection (gauge field) $A^{\hat{a}}_{\mu}$ in a certain principal bundle P over M', and a multiplet of scalar fields $\lambda_{\alpha\beta}$ (a Jordan-Thiry multiplet) on M'. One also finds that the Einstein-Hilbert action A of g_{AB}, when reduced to F', gives a reasonable effective action for the fields $g_{\alpha\beta}$, $A^{\hat{a}}_{\mu}$ and $\lambda_{\alpha\beta}$ on M'. The effective gauge group, resulting from this kind of dimensional reduction, i.e. the structure group of the principal bundle P, is $N(H)/H$, where H is the typical isotropy group of the action of G on M and $N(H)$ is the normalizer of H in G.

The five-dimensional Kaluza-Klein theory with a Jordan-Thiry field is a particular example of the above general scheme; in this case the group $U(1)$ acts on M without fix-points so that $G=U(1)$, $H=\{e\}$, $N(H)/H=G=$

$=U(1)$, and $P=M$ is an $U(1)$ principal bundle over $M'=M/U(1)$. The general G-invariant dimensional reduction scheme was shown to be consistent (for the Einstein-Hilbert action) by the explicit comparing of the effective and original field equations [15]. These calculations also show that, apart of some exceptional situations, the consistency also holds for a noncompact semisimple G and pseudoriemannian metrics. Any ad hoc introduced change in the field content (e.g. neglecting some of the Jordan-Thiry scalars λ, or "improving" the effective Lagrangian by neglecting a determinant of λ factor, lead in general to a dimensionally reduced theory which is inconsistent with the original one.

9) The consistency of G-invariant dimensional reduction scheme, at least for a compact G, can be also deduced from an interesting paper [16] by R.S. Palais, entitled "The Principle of Symmetric Criticality"[2]. The Author proves there the following remarkable result

> *Theorem.* (Principle of Symmetric Criticality) Let G be a compact Lie group, M a smooth G-manifold, $F\to M$ a smooth G-fiber bundle over M, and let $\mathcal{F}$ be the Banach manifold of sections of F with the natural action of G on $\mathcal{F}$ by $(g\phi)(x)=g\phi(xg)$. Let $A:\mathcal{F}\to\mathbb{R}$ be a smooth G-invariant function on $\mathcal{F}$. Then the set $\mathcal{F}'$ of G-invariant sections of F (i.e. those satisfying $(g\phi)(x)=\phi(x)$ for all $x\in M$) is a smooth submanifold of $\mathcal{F}$, and the reduction from $\mathcal{F}$ to $\mathcal{F}'$ is consistent with A i.e. every $\phi\in\mathcal{F}'$ which is critical point of $A'=A|_{\mathcal{F}'}$ is automatically a critical point of A in $\mathcal{F}$.

The Principle applies therefore to the standard Kaluza-Klein theory with a compact group G of isometries as discussed in Ref. [14]; indeed pseudoriemannian metrics of signature (p,q) can be identified with sections of the bundle associated to the frame bundle of M via the natural action of GL(m) on the coset space GL(m)/O(p,q). The principle applies also to Kaluza-Klein theories enlarged by other matter fields, as discussed in Ref. [18].

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2) for an earlier, more intuitive version of this principle see also the Coleman's paper [17].

10) A proof of the above principle can be based on the following considerations. Let G be a compact Lie group acting on a manifold F, and let F' be the submanifold of F consisting of the fix-points of this action. Now, if $\phi \in F'$ the group G leaves the point ϕ invariant but acts, in general, on a neighbour points. Therefore we have the isotropy representation of G on the tangent space $T_\phi F$. Denote by X_h, $h \in \mathrm{Lie}(G)$, the fundamental vector fields for the action of G on F, evidently one has $X_h(\phi)=0$ for $\phi \in F'$. Denote by $\partial X_h(\phi)$ the linear operators of the isotropy representation of G at ϕ. Let $A : F \to \mathbb{R}$ be a function on F which is G-invariant. If then follows from G-invariance of A that, for $\phi \in F'$, dA vanishes on the range of $\partial X_h(\phi)$ for all $h \in \mathrm{Lie}\,(G)$. Now, since the group G is compact, one can always make $T_\phi F$ into a Hilbert space so that the isotropy representation is unitary and the generators $\partial X_h(\phi)$ - selfadjoint. Now, for selfadjoint operators their ranges and kernels span the entire space. Thus to prove that ϕ is a critical point of A at $\phi \in F'$ it is enough to prove that dA vanishes on the kernel of $\partial X_h(\phi)$. Therefore to prove that criticality of ϕ for $A_{|F'}$ implies criticality for A in F it is enough to prove that $\Pi\{\mathrm{Ker}\,\partial X_h(\phi) : h \in \mathrm{Lie}\,G\} \subset T_\phi F'$. In Ref. [16] this fact is proven by first endowing F with a G-invariant Riemannian metric and then by observing that given a vector tangent at $\phi \in F'$ which is invariant under the isotropy representation, it generates a point-wise invariant geodesic (thus being contained in F') and so it is tangent to F'. It would be interesting to know if the Principle of Symmetric Criticality can be generalized to functionals invariant under supersymmetries.

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