BULLETIN DE L'ACADEMIE
POLONAISE DES SCIENCES
Serie des sciences math., astr.
et phys. Vol. XXII, No. 9, 1974

THEORETICAL PHYSICS

Extensions of Projective Representations to Unitary Group Representations

by

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Presented by J. Rzewuski on October 15, 1972

Summary. Sz-Nagy's extension theorem is applied to show that every projective representation of a group G can be obtained by projecting a (true) unitary representation onto a non-invariant cyclic subspace.

Some important problems of quantum theory lead to projective representations of groups. First of all the quantum theory itself may be considered as studying projective representations of the additive group of a classical phase space. Projective representations also arise in connection with the existence of. half-integer spin particles. As it is well known, the problem of classifying the projective representations (groups is closely related to the problem of group extensions and /or to the problem a structure of twisted group algebras and their representations (see [1] and references there). The aim of this note is to draw the reader's attention to the resting fact that every projective unitary representation of a group can be obtained projecting some true. unitary group representation onto non-invariant cyclic Impact. Let G be a group. A function $w : G \times G \longrightarrow T$ (T is the group of complex numbers of unit modulus) such that

$$\omega(e, s) = \omega(s, e) = 1, \tag{1}$$

$$\omega(s, tu)\omega(t, u) = \omega(st, u)\omega(s, t) \tag{2}$$

for all $s, t, u \in G$ is said to be a multiplier on G. It follows from (1) and (2) that $\omega(s, s^{-1}) = \omega(s^{-1}, s)$.

A multiplier ω such that for all $s \in G$

$$\omega(s, s^{-1} = \omega(s^{-1}, s) = 1 \tag{3}$$

is said to be symmetric. Every multiplier is in fact equivalent to a symmetric one. According to the generally accepted notation the group of all symmetric multipliers denoted by $Z_s^2(G_d, T)$, where G_d stands for G endowed with the discrete topology. For every $\omega \in Z_s^2(G_d, T)$ let G^ω be the central extension of G by T defined by

$$(s, \alpha)(t, \beta) = (st, a\beta\omega(s, t))$$

, for all $(s, a) \in G \times T$. If $s \mapsto U(s)$ is a projective unitary representation of G with a multiplier ω, then $(s, \alpha) \mapsto \alpha U(s)$ is a (true) unitary representation of G^ω. Since

$$U(e) = I,$$

$$U(s^{-1}) = U(s)^{-1},$$

$$U(s)U(t) = \omega(s, t)U(st),$$

it follows that for every finite sequence $s_1, \ldots, s_n \in G$ and $\alpha_1, \ldots, \alpha_n \in C$ the function $\mu(s) = (\Phi, U(s)\Phi)$ satisfies

$$\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j \mu(s_i^{-1} s_j)\omega(s_i^{-1}, s_j) \geq 0. \tag{4}$$

A function μ satisfying (4) is said to be ω–positive-definite. More generally, we may talk about operator-valued ω–positive definite functions, with values in some $L(\mathcal{H})$, and the definition above applies *mutatis mutandis.* It follows from the definition itself that μ is ω-positive-definite on G if and only if $(s, \alpha) \mapsto \alpha\mu(s)$ is a positive–definite function on G^ω. From this remark and the well–known properties of positive–definite functions (see e.g. [2]Th. 32. 4.) one gets

$$\mu(s^{-1}) = \mu(s),$$

$$|\mu(s) \leq \mu(e),$$

for every symmetric multiplier ω and every ω–positive–definite function μ. We remark that if ω is a multiplier for G then $\bar{\omega}$ is also a multiplier and if μ is ω–positive–deffinite then $\bar{\mu}$ $\bar{\omega}$–positive–definite. If ω and ω' are two multipliers then their product is also a multiplier and if μ and μ' are ω– and ω'–positive–definite respectively, then $\mu\mu'$ is $\omega\omega'$–positive definite. In particular, if μ and μ' are ω–positive-definite, then $\mu\bar{\mu}'$ is a positive–definite function on G.

THEOREM: *Let $s \mapsto V(s)$ be a projective unitary ,representation of G on a Hilbert space $\mathcal{H}_0$ with a symmetric multiplier ω.. Let μ be an ω–positive–definite function on G with $\mu(e) = l$. There exists a Hilbert space $\mathcal{H}_\mu$ that contains $\mathcal{H}_0$ as a subspace and a unitary representation $s \mapsto U_\mu(s)$ of G on $\mathcal{H}_\mu$ such that* *(i)* $P_\mu U_\mu(s) = \bar{\mu}(s)V(s)\, on\, \mathcal{H}_0$,

(ii) $\mathcal{H}_0$ *is cyclic for* U_μ, *where* P_μ *is the orthogonal projection from* $\mathcal{H}_\mu$ *onto* $\mathcal{H}_0$. $\mathcal{H}_\mu$ *and* U_μ *are uniquely determined by the conditions* (*i*) *and* (*ii*). *If* μ *and* V *are continuous, then* $U\mu$ *is also continuous.*

Proof. Since $s \mapsto V(s)$ is an ω–positive– definite operator valued function, it follows that $s \mapsto \bar{\mu}(s)V(s)$ is positive definite. The statement follows directly from Sz-Nagy's theorem on unitary dilations of positive–definite ([3]Th. 7.1.).

Remark 1. Let Φ be any unit vector in $\mathcal{H}_0$. Then the function $\mu_0 : s \mapsto (Phi, V(s)\Phi)$ is ω–positive-definite. Thus, for each $\Phi \in \mathcal{H}_0$ we have a unitary representation U_Φ uniquely characterized by

$$P_\Phi U_\Phi(s) = (V(s)\Phi, \Phi)V(s),$$

and cyclicity of $\mathcal{H}_0$.

Remark 2. Let G be locally compact Abelian, μ and V continuous. By the Sz-Nagy Theorem (see e.g. [4], p. 377) we have

$$U_\mu(s) = \int_{\hat{G}} \chi(s) dF_\mu(\chi),$$

where $\hat{G}$ is the dual of G. Let $E_\mu(\chi) = P_\mu F_\mu(\chi)\, \mathrm{on}\, \mathcal{H}_0$. Then E_μ is a positive operator measure and we obtain

$$\bar{\mu}(s)V(s) = \int_{\hat{G}} \chi(s) dE_\mu(\chi),$$

which is a kind of spectral decomposition for a projective representation. The measure E_μ uniquely determined by (μ, V).

Remark 3. If G is connected locally compact and μ is continuous then V is irreducible if and only if the pair (P_μ, U_μ) is irreducible.

Remark 4. Clearly, the family of all extensions is parametrized by μ–s only and does not depend on the choice of multiplier in a given equivalence class: with $\omega'(s,t) = a(a)a(t)\bar{a}(st)\omega(s,t)$, the function $\mu'(s) = a(s)\mu(s)$ is ω'–positive and (V, μ) and (V', μ') determine the same U.

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