Galilei general relativistic quantum mechanics revisited

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After this paper several results,
including the quantum vector fields and the Schrödinger equation,
have been improved in subsequent papers.

Abstract

We review the recent advances in the generally covariant and geometrically intrinsic formulation of Galilei relativistic quantum mechanics. The main concepts used are Galilei-Newton space-time, Newtonian gravity and electromagnetism, space-time connection and cosymplectic form, quantum line bundle and quantum connection, Schrödinger equation and Hilbert bundle, quantisable functions and quantum operators. The paper contains a number of improvements and simplifications with respect to the already published or announced results [31, 33].

Contents

Introduction

Quantum theory is one of the most successful and at the same time most mysterious physical theories. As Feynman puts it, "nobody understands quantum theory". There are several reasons for these difficulties. One of them being the fundamental statistical character of the theory which caused several physicists to doubt its completeness. Another reason is in the philosophical problems of the quantum measurement theory, whose repeated terms "observation" or "measurement" are never defined [5, 6]. These problems are extensively discussed and a solution is proposed in a so called "Event Enhanced Quantum Theory" developed in a series of papers by Ph. Blanchard and one of us (A. J.) (see [8] and references there). Finally quantum theory seems to contradict relativity and this fact was discussed by several authors (see e.g. [2, 65]). We feel that it is indeed necessary to formulate quantum theory in an intrinsic geometrical way, as such a formulation can throw some light on the meaning of the theory in general and of the "quantisation" procedure in particular. As the Planck constant can be expressed, via the fine structure constant, in terms of the light speed constant c and electric charge e, there is a hope that a unified geometrical theory of gravitation and electromagnetism will be able to derive quantum theory as an effective theory of a complex and chaotic "classical" space-time structure. Thus the Einstein dream may yet come true.

Time will show whether it will be so or not, but anyhow an intrinsically geometrical formulation of the quantum theory is a necessary initial step which can show us ways toward further developments. The present paper discusses the mathematical aspects of the approach to Galilei general relativistic quantum mechanics developed originally in [31, 33]. The fact that we restrict our discussion to a Galileian (sometimes called also Newtonian) general relativity is not an essential restriction of our approach. As it has been discussed by many authors (cf. [19, 20, 21, 28, 29, 50, 66, 67]), by using Fock-Schwinger "proper time" formalism, Einstein relativistic quantum mechanics can be represented as Galilei relativistic one with an extra "proper-time" parameter. On the other hand it is also possible to apply most of the mathematical ideas presented below directly to the Einstein relativistic formulation, and this has been done in [36, 37, 38].

The paper is addressed to a mathematically oriented reader. Thus, we concentrate on mathematical concepts without trying to justify them. The starting concept is that of a Galileian general relativity, a concept that was studied originally by Trautman [74, 75] and then analysed by several other authors [12, 13, 14, 15, 16, 17, 18, 26, 44, 45, 46, 47, 49, 52, 53, 57, 64, 70, 76]. We take a somewhat different approach than these authors as we are able to show that the constraints on the curvature tensor postulated by them are consequences of the closure of the cosymplectic form. We work consequently with jet spaces which allow us to develope a fully covariant approach to quantisation without any need to apply the arbitrary polarisation method of geometrical quantisation (see [72, 79]). Thus although our method borrows from the Souriau's idea of pre–quantisation ([73]) we achieve the goal of getting rid of extra degrees of freedom via the concept of a universal quantum connection and a projection procedure. Our method produces, in fact, a one-parameter family of theories out of which exactly one happens to be projectable. It would be interesting to investigate a possible physical meaning of the other theories as well. It would also be interesting to apply the above methods to the deformation-quantisation scheme of Flato and al. [4]. Although we are considering here only a scalar case (that is spin zero), our method can be easily generalised to include generally covariant Pauli equation for spin one-half particle [10].

The reader will be perhaps puzzled by our half-spaces of units of time, length and mass, as they usually do not appear in the textbooks. This is however the only rigorous way that we know to include "physical dimensions" in a covariant formulation of the theory. Their explicit introduction suggests also generalisations. Physicists often dreamed about some "ultimate theory" in which there will be no dimensional coupling constants. To work towards a realisation of such a dream we could replace our half-spaces by line bundles over space-time without an a priori selected connection. We hope to return to this problem in the future.

In Section 1.1 we introduce the arena for our play that is the Galileian space-time for one massive particle. It is a fibred manifold over one dimensional affine space of "absolute time". Each fibre has to be thought of as "space at a given time" equipped with a Riemannian metric. There are several possible variations of this formalism. Replacing space with a Cartesian product of n spaces we can describe classical and quantum mechanics of a system of n particles (as in [33] Chapters I.7 and I.6). On the other hand, thinking of the base space as a "proper time" and the fibers as four dimensional pseudo Riemannian manifolds we can describe Einstein relativistic systems along the lines of [19]. Phase space is defined as the space of 1–jets of sections, and it is odd-dimensional. We discuss space-time connections, construct a cosymplectic two-form Ω and find necessary and sufficient conditions for Ω to be closed in terms of the connection. Electromagnetic field acting on the particle is introduced as usual in symplectic mechanics ([1]), that is by adding the electromagnetic 2—form to Ω. Then, we show that it can be recovered by adding a term to the gravitational connection. Autoparallel motions of this connection contain then the correct electromagnetic force. In Section 1.9 we discuss in some detail classical particle mechanics and the concepts of Poincare-Cartan form, Lagrangian, momentum and Hamiltonian.

Section 2 deals with generally covariant quantisation. The quantum bundle is defined as a Hermitian complex line bundle over space-time. This is then prolonged to a line bundle over the phase space. The quantum connection is defined as a "universal" connection on the prolonged quantum bundle whose curvature 2–form is proportional to Ω. Theorem 2.5 gives necessary and sufficient conditions for existence of a quantum structure. Section 2.3 deals with quantum dynamics. We show that the Schrödinger equation can be formulated in terms of the quantum connection even if the quantum connection lives on the prolonged quantum bundle over the phase space. A unique combination of first and second order derivatives of the quantum section makes it possible for the velocity variables to drop out of the equation (Proposition 2.6). In Section 2.4 we introduce the concept of a quantisable function. Quantisable functions are functions over the classical phase space (but it should be kept in mind that our phase space is odd dimensional and includes time as one of its coordinates). By a functorial procedure we associate with each quantisable function a differential operator acting on sections of the quantum bundle over spacetime (Theorem 2.30). Eventually, Section 2.8 introduces the concept of the quantum Hilbert bundle i.e. a family of Hilbert spaces parametrised by time. Solutions of the Schrödinger equation can then be thought of as sections of this bundle. Then a covariant quantisation procedure associates with each "good" classical observable (represented by a quantisable function) a symmetric operator acting on the fibres of the Hilbert bundle.

We stress that in the flat case, our method recovers the standard Scrödinger equation and quantum operators associated with position coordinates, momentum and Hamiltonian.

We assume the following fundamental unit spaces, which are positive 1–dimensional semi–vector spaces over $I\!R^+$ [33] (here we adopt a minor change in the notation $\mathbb{T}$ with respect to the previous literature):

(1) the space $\mathbb{T}$ of *time intervals*,
(2) the space $\mathbb{L}$ of *lengths*,
(3) the space $\mathbb{M}$ of *masses*.

A *time unit of measurement* is defined to be an element of $\mathbb{T}$ or its dual $\mathbb{T}^*$

$$u_0 \in \mathbb{T}, \qquad u^0 \in \mathbb{T}^* .$$

Moreover, we refer to the *Planck constant*

$$\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M} .$$

We shall be concerned with smooth manifolds and maps.

Let $\boldsymbol{M}$ be a manifold and $\boldsymbol{F} \to \boldsymbol{B}$, $\boldsymbol{G} \to \boldsymbol{B}$ fibred manifolds.

Then, $\pi_{\boldsymbol{M}} : T\boldsymbol{M} \to \boldsymbol{M}$ denotes the tangent bundle of $\boldsymbol{M}$ and $V\boldsymbol{F} \subset T\boldsymbol{F}$ the vertical subbundle.

Moreover, $\mathcal{F}(\boldsymbol{M}, \mathbb{S})$ denotes the sheaf of local maps $f : \boldsymbol{M} \to \mathbb{S}$ with values in a set $\mathbb{S}$; in particular, $\mathcal{F}(\boldsymbol{M}) \equiv \mathcal{F}(\boldsymbol{M}, I\!R)$ is the sheaf of local real valued functions.

Furthermore, $\mathcal{S}(\boldsymbol{F})$ denotes the sheaf of local sections $s : \boldsymbol{B} \to \boldsymbol{F}$; in particular, $\mathcal{T}(\boldsymbol{M}) \equiv \mathcal{S}(T\boldsymbol{M})$ is the sheaf of local vector fields $X : \boldsymbol{M} \to T\boldsymbol{M}$ and $\mathcal{T}^*(\boldsymbol{M}) \equiv \mathcal{S}(T^*\boldsymbol{M})$ is the sheaf of local forms $\alpha : \boldsymbol{M} \to T^*\boldsymbol{M}$.

Additionally, $\mathcal{M}(\boldsymbol{F}, \boldsymbol{G})$ denotes the sheaf of local fibred morphisms $f : \boldsymbol{F} \to \boldsymbol{G}$ over the base space $\boldsymbol{B}$.

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1 Classical spacetime

First, we present a model of classical curved spacetime fibred over absolute time and equipped with a vertical metric a gravitational connection and an electromagnetic form, fulfilling certain relations. This structure is encoded in a cosymplectic form on the phase space. We analyse the Lagrangian setting of the mechanics of a classical particle as well.

1.1 Spacetime

We start with the fibred and metric structure of spacetime. Later we shall add the connection structure.

Assumption C.1. We assume *spacetime* to be a 4-dimensional oriented fibred manifold

(1.1) $$t : \boldsymbol{E} \to \boldsymbol{T}$$

over a 1-dimensional oriented affine space $\boldsymbol{T}$ (*time*), associated with the vector space $\mathbb{T} \otimes I\!R$, equipped with a Riemannian metric on the fibres, i.e. with a *vertical Riemannian metric*,

(1.2) $$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E}) \,. \square$$

Thus, for each unit of measurement of lengths $l \in \mathbb{L}$, the metric g yields a standard real valued metric on the fibres.

We choose an orientation of $\boldsymbol{E}$.

We shall refer to *spacetime charts* (x^0, x^i), which are adapted to the fibring, to a time unit of measurement u_0 and to the chosen orientation of $\boldsymbol{E}$.

The index 0 will refer to the base space, Latin indices $i, j, \dots = 1, 2, 3$ will refer to the fibres, while Greek indices $\lambda, \mu, \dots = 0, 1, 2, 3$ will refer both to the base space and the fibres.

We shall denote the induced local bases of $\mathcal{T}(\boldsymbol{E})$ and $\mathcal{T}^*(\boldsymbol{E})$ by (∂_λ) and (d^λ), respectively. Moreover, the chart induced on $T\boldsymbol{E}$ will be denoted by $(x^\lambda, \dot{x}^\lambda)$.

We can regard $u^0 \in \mathbb{T}^* \subset \mathbb{T}^* \otimes I\!R$ as a form $u^0 : \boldsymbol{T} \to T^*\boldsymbol{T}$ and, by pullback with respect to $t : \boldsymbol{E} \to \boldsymbol{T}$, as the form

$$u^0 = d^0 : \boldsymbol{E} \to T^*\boldsymbol{E} \,.$$

We have the *time-form*

(1.3) $$dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E} \,,$$

with coordinate expression

$$dt = u_0 \otimes d^0 \,.$$

From now on, the check ' $^\vee$ ' will denote vertical restriction of forms.

The coordinate expression of g is

$$g = g_{ij}\, \check{d}^i \otimes \check{d}^j \,, \qquad g_{ij} \in \mathcal{F}(\boldsymbol{E}, \mathbb{L}^2 \otimes I\!\!R)\,.$$

We denote the contravariant metric by

$$\bar{g} : \boldsymbol{E} \to \mathbb{L}^{*2} \otimes (V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E})\,;$$

its coordinate expression is

$$\bar{g} = g^{ij} \partial_i \otimes \partial_j \,, \qquad g^{ij} \in \mathcal{F}(\boldsymbol{E}, \mathbb{L}^{*2} \otimes I\!\!R)\,.$$

The metric yields the linear fibred isomorphisms over $\boldsymbol{E}$

$$g^\flat : V\boldsymbol{E} \to \mathbb{L}^2 \otimes V^*\boldsymbol{E}\,, \qquad g^\sharp : V^*\boldsymbol{E} \to \mathbb{L}^{*2} \otimes V\boldsymbol{E}\,.$$

Given a mass $m \in \mathbb{M}$, it is convenient to introduce the "normalised" metric

$$\text{(1.4)} \quad G \equiv \frac{m}{\hbar} g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E})\,, \qquad \bar{G} \equiv \frac{\hbar}{m} \bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes (V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E})$$

with coordinate expression

$$G = G_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j \,, \qquad G_{ij} \equiv \frac{m}{\hbar} g_{ij} u^0 \in \mathcal{F}(\boldsymbol{E})\,,$$
$$\bar{G} = G^{ij}\, u^0 \otimes \partial_i \otimes \partial_j \,, \qquad G^{ij} \equiv \frac{\hbar}{m} g^{ij} u_0 \in \mathcal{F}(\boldsymbol{E})\,.$$

We stress the fact that the normalised metric and all objects which will be derived from it incorporate the mass and the Planck constant.

The metric g and the spacetime orientation yield the *space-like volume form*

$$\text{(1.5)} \qquad \eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3 V^*\boldsymbol{E}\,,$$

with coordinate expression

$$\eta = \sqrt{|g|}\, \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3 \,,$$

where

$$|g| \equiv \det(g_{ij})\,.$$

Let $\alpha : \boldsymbol{E} \to \Lambda^3 V^*\boldsymbol{E}$ be a vertical 3–form and $\tilde{\alpha} : \boldsymbol{E} \to \Lambda^3 T^*\boldsymbol{E}$ any extension of α. Then, we observe that the wedge product $dt \wedge \tilde{\alpha} : \boldsymbol{E} \to \mathbb{T} \otimes \Lambda^4 T^*\boldsymbol{E}$ does not depend on the choice of the extension. Accordingly, we shall write $dt \wedge \alpha \equiv dt \wedge \tilde{\alpha}$.

Then, we obtain the *spacetime volume form*

$$\upsilon := dt \wedge \eta : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^3) \otimes \Lambda^4 T^* \boldsymbol{E} \,, \tag{1.6}$$

with coordinate expressions

$$\upsilon = \sqrt{|g|}\, u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 \,.$$

An (observer independent) *motion* is defined to be a section

$$s : \boldsymbol{T} \to \boldsymbol{E} \,. \tag{1.7}$$

1.2 Phase space

Now, we introduce the phase space as the first jet space of spacetime. The choice of such a phase space, instead of the tangent or cotangent space of spacetime, is an important feature of our model aimed at fulfilling the covariance of the theory.

The space of jets is equipped with the contact structure, which will play an important role throughout the paper.

In the framework of jets we can easily introduce the notion of observer, which provides a splitting of the tangent space of spacetime.

1.1. Definition. The *phase space* is defined to be the first jet space of sections $\boldsymbol{T} \to \boldsymbol{E}$

$$\pi^1_0 : J\boldsymbol{E} \equiv J_1\boldsymbol{E} \to \boldsymbol{E} \,. \square \tag{1.8}$$

We can naturally identify $J\boldsymbol{E}$ with the subbundle over $\boldsymbol{E}$

$$J\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E} \,,$$

which projects on $1 \in \mathbb{T}^* \otimes \mathbb{T}$. It follows that $J\boldsymbol{E} \to \boldsymbol{E}$ is an affine bundle associated with the vector bundle $\mathbb{T}^* \otimes V\boldsymbol{E}$.

We have the natural fibred isomorphism over $J\boldsymbol{E}$

$$V_{\boldsymbol{E}} J\boldsymbol{E} \simeq J\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\mathbb{T}^* \otimes V\boldsymbol{E}) \,.$$

A spacetime chart (x^λ) induces on $J\boldsymbol{E}$ the chart (x^0, x^i, x^i_0).

We shall be involved with the natural complementary *contact maps*

$$д : J\boldsymbol{E} \times \mathbb{T} \to T\boldsymbol{E} \,, \qquad \theta : J\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to V\boldsymbol{E} \,, \tag{1.9}$$

with coordinate expressions

$$д = u^0 \otimes д_0 = u^0 \otimes (\partial_0 + x^i_0\, \partial_i) \,, \qquad \theta = \theta^i \otimes \partial_i = (d^i - x^i_0\, d^0) \otimes \partial_i \,.$$

The (observer independent) *velocity* of a motion s is defined to be the jet prolongation

$$js \equiv j_1 s : \boldsymbol{T} \to J\boldsymbol{E} . \tag{1.10}$$

1.2. Definition. An *observer* is defined to be a section

$$o : \boldsymbol{E} \to J\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E} . \square \tag{1.11}$$

The coordinate expression of o is of the type

$$o = u^0 \otimes (\partial_0 + o^i_0\, \partial_i) , \qquad o^i_0 \in \mathcal{F}(\boldsymbol{E}) .$$

A spacetime chart is said to be *adapted* to an observer o if $o^i_0 = 0$. There are many spacetime charts adapted to an observer; conversely, each spacetime chart yields the observer

$$o = u^0 \otimes \partial_0 ,$$

which is the only one such that $o^i_0 = 0$.

An observer o can be regarded as a (non linear) connection on the fibred manifold $\boldsymbol{E} \to \boldsymbol{T}$, which can be expressed, equivalently, by the tangent valued form, or by the vertical valued form on $\boldsymbol{E}$

$$o : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} , \qquad \nu[o] : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E} , \tag{1.12}$$

respectively, with coordinate expressions

$$o = u^0 \otimes (\partial_0 + o^i_0\, \partial_i) , \qquad \nu[o] = (d^i - o^i_0\, d^0) \otimes \partial_i .$$

Hence, the observer o yields the linear splitting over $\boldsymbol{E}$

$$T\boldsymbol{E} \simeq (\boldsymbol{E} \times \mathbb{T} \otimes I\!\!R) \underset{\boldsymbol{E}}{\oplus} V\boldsymbol{E} ,$$

given by

$$X = \big\langle o, dt(X) \big\rangle + \Big(X - \big\langle o, dt(X) \big\rangle \Big)$$

i.e. , in coordinates, by

$$X = X^0\, (\partial_0 + o^i_0\, \partial_i) + (X^i - o^i_0\, X^0)\, \partial_i .$$

Then, the covariant differential operator associated with o can be regarded as the affine fibred morphism over $\boldsymbol{E}$

$$\nabla[o] : J\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E} : j \mapsto j - o(\pi^1_0(j)) , \tag{1.13}$$

with coordinate expression

$$\nabla[o] = u^0 \otimes (x^i_0 - o^i_0)\,\partial_i\,.$$

So, if s is a motion, then the covariant differential of s with respect to o turns out to be the section

$$\nabla[o]s := \nabla[o] \circ js : \boldsymbol{T} \to \mathbb{T}^* \otimes V\boldsymbol{E}\,,$$

with coordinate expression

$$\nabla[o]s = (\partial_0 s^i - o^i_0 \circ s)u^0 \otimes (\partial_i \circ s)\,,$$

which will be interpreted as the *velocity of s observed by o.*

We define the *kinetic momentum* and the *kinetic energy* associated with o to be maps

$$\check{\mathcal{Q}}[o] \;\equiv\; G^\flat \circ \nabla[o] : J\boldsymbol{E} \to V^*\boldsymbol{E}\,, \tag{1.14}$$

$$\mathcal{K}[o] \;\equiv\; \tfrac{1}{2}\, G \circ (\nabla[o],\, \nabla[o]) : J\boldsymbol{E} \to \mathbb{T}^* \otimes I\!\!R\,. \tag{1.15}$$

We observe that the kinetic momentum is the vertical restriction of the 1–form on $\boldsymbol{E}$

$$\mathcal{Q}[o] \;\equiv\; \theta \lrcorner \check{\mathcal{Q}} : J\boldsymbol{E} \to T^*\boldsymbol{E}\,;$$

moreover, by pullback with respect to $t : \boldsymbol{E} \to \boldsymbol{T}$, the kinetic energy can be regarded as a 1–form on $\boldsymbol{E}$

$$\mathcal{K}[o] : J\boldsymbol{E} \to T^*\boldsymbol{E}\,.$$

In a chart adapted to o we have the following coordinate expressions

$$\check{\mathcal{Q}}[o] = G^0_{ij}\, x^j_0\, \check{d}^i\,, \qquad \mathcal{Q}[o] = G^0_{ij}\, x^j_0\, (d^i - x^i_0\, d^0)\,,$$
$$\mathcal{K}[o] = \tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0\, u^0 \;\equiv\; \mathcal{K}_0\, u^0 \;\simeq\; \mathcal{K}_0\, d^0\,.$$

If o, o' are two observers, then we can write

$$o' = o + V\,,$$

where the section $V : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ can interpreted as the *velocity of o' observed by o.*

1.3 Distinguished connections

Next, we consider the vertical Riemannian metric generated by the vertical metric. Moreover, we analyse certain distinguished connections that can be defined on spacetime and on the phase space and discuss their mutual relations. Later we shall make some assumptions on them.

First of all, we observe that the vertical Riemannian metric g yields the vertical Riemannian connection, i.e. the Riemannian connection on the fibres of $\boldsymbol{E} \to \boldsymbol{T}$, which can be expressed, equivalently, by a tangent valued form, or by a vertical valued form on $T\boldsymbol{E}$

$$\varkappa : V\boldsymbol{E} \to V^*\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} TV\boldsymbol{E}\,, \qquad \nu[\varkappa] : V\boldsymbol{E} \to V^*V\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} V\boldsymbol{E}\,, \tag{1.16}$$

respectively, with coordinate expressions

$$\varkappa = \check{d}^i \otimes (\partial_i + \varkappa_i{}^j{}_h\, \dot{x}^h\, \dot{\partial}_j)\,, \qquad \nu[\varkappa] = (\dot{\check{d}}^i - \varkappa_j{}^i{}_h\, \dot{x}^h\, \check{d}^j) \otimes \partial_i\,,$$

where

$$\varkappa_h{}^i{}_k = -\tfrac{1}{2}\, g^{ij}\, (\partial_h g_{jk} + \partial_k g_{jh} - \partial_j g_{hk}) \in \mathcal{F}(\boldsymbol{E})\,,$$

and $(\partial_i, \dot{\partial}_i)$ and $(\check{d}^i, \dot{\check{d}}^i)$ are the induced local bases of $\mathcal{T}(V\boldsymbol{E})$ and $\mathcal{T}^*(V\boldsymbol{E})$, respectively.

Next, we analyse some distinguished types of connections on spacetime.

A torsion free linear connection K on the bundle $T\boldsymbol{E} \to \boldsymbol{E}$ can be expressed, equivalently, by a tangent valued form, or by a vertical valued form on $T\boldsymbol{E}$

$$K : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E}\,, \qquad \nu[K] : T\boldsymbol{E} \to T^*T\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} T\boldsymbol{E}\,, \tag{1.17}$$

respectively, with coordinate expressions

$$K = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^\mu{}_\nu\, \dot{x}^\nu\, \dot{\partial}_\mu)\,, \qquad \nu[K] = (\dot{d}^\mu - K_\lambda{}^\mu{}_\nu\, \dot{x}^\nu\, d^\lambda) \otimes \partial_\mu\,,$$

where

$$K_\mu{}^\lambda{}_\nu = K_\nu{}^\lambda{}_\mu \in \mathcal{F}(\boldsymbol{E})$$

and $(\partial_\lambda, \dot{\partial}_\lambda)$ and $(d^\lambda, \dot{d}^\lambda)$ are the induced local bases of $\mathcal{T}(T\boldsymbol{E})$ and $\mathcal{T}^*(T\boldsymbol{E})$, respectively.

A torsion free linear connection K is said to be *time–preserving* if

$$\nabla[K](dt) = 0\,; \tag{1.18}$$

in coordinates it reads

$$K_\mu{}^0{}_\nu = 0\,.$$

1.3. Definition. A *spacetime connection* is defined to be a time–preserving torsion free linear connection K on the bundle $T\boldsymbol{E} \to \boldsymbol{E}$. □

We observe that a spacetime connection K yields, by restriction, a linear connection on the bundle $V\boldsymbol{E} \to \boldsymbol{E}$

$$K' : V\boldsymbol{E} \to T^*\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} TV\boldsymbol{E}\,, \tag{1.19}$$

whose coordinate expression is

$$K' = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i{}_j\, \dot{x}^j\, \dot{\partial}_i)\,.$$

Moreover, a spacetime connection K yields, by further restriction, a linear connection on the fibres of $\boldsymbol{E} \to \boldsymbol{T}$

$$\check{K} : V\boldsymbol{E} \to V^*\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} VV\boldsymbol{E}\,, \tag{1.20}$$

whose coordinate expression is

$$\check{K} = d^h \otimes (\partial_h + K_h{}^i{}_j\, \dot{x}^j\, \dot{\partial}_i)\,.$$

A spacetime connection is said to be *metric* if

$$\nabla[K']G = 0\,. \tag{1.21}$$

This condition reads in coordinates as

$$K_h{}^i{}_k = -\tfrac{1}{2}\, G^{ij}(\partial_h G_{jk} + \partial_k G_{jh} - \partial_j G_{hk})\,, \qquad K_{0ij} + K_{0ji} = -\,\partial_0 G_{ij}\,.$$

Thus, in particular, for a metric spacetime connection K, we obtain

$$\check{K} = \varkappa\,. \tag{1.22}$$

Then, let us discuss a distinguished type of connections on the phase space and their relation with spacetime connections.

A torsion free affine connection Γ on the affine bundle $J\boldsymbol{E} \to \boldsymbol{E}$ can be expressed, equivalently, by a tangent valued form, or by a vertical valued form

$$\Gamma : J\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J\boldsymbol{E}}{\otimes} TJ\boldsymbol{E}\,, \qquad \nu[\Gamma] : J\boldsymbol{E} \to \mathbb{T}^* \otimes (T^*J\boldsymbol{E} \underset{J\boldsymbol{E}}{\otimes} V\boldsymbol{E})\,, \tag{1.23}$$

respectively, with coordinate expressions

$$\Gamma = d^\lambda \otimes (\partial_\lambda + (\Gamma_{\lambda 0 0}^{\ i} + \Gamma_{\lambda 0 j}^{\ i}\, x_0^j)\, \partial_i^0)\,,$$
$$\nu[\Gamma] = u^0 \otimes (d_0^i - (\Gamma_{\lambda 0 0}^{\ i} + \Gamma_{\lambda 0 j}^{\ i}\, x_0^j)\, d^\lambda) \otimes \partial_i\,,$$

where

$$\Gamma_{\lambda 0 \mu}^{\ i} = \Gamma_{\mu 0 \nu}^{\ i} \in \mathcal{F}(J\boldsymbol{E})\,.$$

We remark that the torsion of a connection on the phase space can be defined via the contact form θ and the Frölicher–Nijenhuis bracket [33, 61].

1.4. Definition. A *phase connection* is defined to be a torsion free affine connection Γ of the bundle $J\boldsymbol{E} \to \boldsymbol{E}$. □

In view of the correspondence between spacetime and phase connections, we observe that a linear connection $\nu[K]$ on $T\boldsymbol{E} \to \boldsymbol{E}$ induces a linear connection $\nu[\tilde{K}] : T(\mathbb{T}^* \otimes T\boldsymbol{E}) \to \mathbb{T}^* \otimes T\boldsymbol{E}$ on the vector bundle $\mathbb{T}^* \otimes T\boldsymbol{E} \to \boldsymbol{E}$, with coordinate expression

$$\nu[\tilde{K}] = u^0 \otimes (\dot{d}_0^\mu - K_\lambda{}^\mu{}_\nu \, \dot{x}_0^\nu \, d^\lambda) \otimes \partial_\mu \,,$$

where $(x^\lambda, \dot{x}_0^\lambda)$ denotes the induced chart on $\mathbb{T}^* \otimes T\boldsymbol{E}$.

1.5. Theorem. *Given a spacetime connection K, we obtain ([33, 37, 39]) a phase connection $\Gamma[K]$ characterised by*

$$\nu[\Gamma[K]] = \theta \circ \nu[\tilde{K}] \circ T\text{д} \,,$$

whose coordinate expression is

$$\Gamma[K]_{\lambda 0 \mu}^{\ i} = K_\lambda{}^i{}_\mu \,.$$

Thus, the map

$$K \mapsto \Gamma[K] \tag{1.24}$$

is a bijection between spacetime and phase connections. □

If Γ is a phase connection and o an observer, then we obtain the covariant differential

$$\nabla[\Gamma]o : \boldsymbol{E} \to \mathbb{T}^* \otimes (T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}) \,, \tag{1.25}$$

with coordinate expression in adapted coordinates

$$\nabla[\Gamma]o = -\Gamma_{\lambda 0 0}^{\ i} \, u^0 \otimes d^\lambda \otimes \partial_i \,.$$

Moreover, the metric G and the inclusion $\nu[o]^* : V^*\boldsymbol{E} \subset T^*\boldsymbol{E}$ yield the tensor

$$G^\flat(\tilde{\nabla}[\Gamma]o) : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E} \,,$$

whose expression in adapted coordinates is

$$G^\flat(\tilde{\nabla}[\Gamma]o) = - G_{jh} \, \Gamma_{\lambda 0 0}^{\ h} \, d^\lambda \otimes d^j \,.$$

We can split $G^\flat(\tilde{\nabla}[\Gamma]o)$ into its symmetric and antisymmetric components

$$G^\flat(\tilde{\nabla}[\Gamma]o) = \tfrac{1}{2} \, (\Sigma[\Gamma, o] + \Phi[\Gamma, o]) \,,$$

where

$$\Sigma[\Gamma, o] : \boldsymbol{E} \to S^2 T^* \boldsymbol{E}\,, \qquad \Phi[\Gamma, o] : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E}\,, \tag{1.26}$$

have coordinate expressions

$$\Sigma[\Gamma, o] = 2\,\Sigma_{0j}\, d^0 \vee d^j + \Sigma_{ij}\, d^i \vee d^j\,, \qquad \Phi[\Gamma, o] = 2\,\Phi_{0j}\, d^0 \wedge d^j + \Phi_{ij}\, d^i \wedge d^j\,,$$

with

$$\begin{aligned} \Sigma_{ij} &= -(G_{jh}\Gamma_{i}{}^{h}_{00} + G_{ih}\Gamma_{j}{}^{h}_{00})\,,\\ \Sigma_{0j} &= -G_{jh}\Gamma_{0}{}^{h}_{00} = \Phi_{0j}\,,\\ \Phi_{ij} &= -(G_{jh}\Gamma_{i}{}^{h}_{00} - G_{ih}\Gamma_{j}{}^{h}_{00})\,. \end{aligned}$$

Furthermore, we are involved with the vertical restriction of $\Sigma[\Gamma, o]$

$$\check{\Sigma}[\Gamma, o] : \boldsymbol{E} \to S^2 V^* \boldsymbol{E}\,,$$

with coordinate expression

$$\check{\Sigma}[\Gamma, o] = \Sigma_{ij}\, \check{d}^i \vee \check{d}^j\,.$$

1.6. Remark. Let us consider a spacetime connection K. Then, the maps

$$K \quad\mapsto\quad (\check{K},\, \nabla[\Gamma]o) \quad\mapsto\quad (\check{K},\, \check{\Sigma}[\Gamma[K], o],\, \Phi[\Gamma[K], o]) \tag{1.27}$$

are bijections.

In other words, $\check{K}$ and $\nabla[\Gamma[K]]o$ carry independent information on K and characterise K itself. Moreover, $\check{\Sigma}[\Gamma[K], o]$ and $\Phi[\Gamma[K], o]$ carry independent information on $\nabla[\Gamma]o$ and characterise $\nabla[\Gamma]o$ itself. □

1.7. Proposition. Given a spacetime connection K and an observer o, the connection K is metric if and only if

$$\check{K} = \varkappa\,, \qquad \check{\Sigma}[\Gamma[K], o] = G^{\flat} L_o \bar{G}\,, \tag{1.28}$$

where $L_o \bar{G}$ denotes the Lie derivative of the contravariant metric $\bar{G}$, with respect to the scaled vector field $o : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$. □

Eventually, we analyse the second order connections of spacetime and discuss their relation with the phase connections.

1.8. Definition. A *second order connection* on $\boldsymbol{E}$ is defined as a section

$$\gamma : J\boldsymbol{E} \to J_2\boldsymbol{E}\,, \tag{1.29}$$

where $J_2\boldsymbol{E}$ is the space of second order jets of sections of $\boldsymbol{E} \to \boldsymbol{T}$. □

Therefore, by considering the inclusion $J_2\boldsymbol{E} \subset \mathbb{T}^* \otimes TJ\boldsymbol{E}$, each second order connection γ can be characterised as a (first order) connection on the fibred manifold $J\boldsymbol{E} \to \boldsymbol{T}$, which projects on the contact map д (see [58]). In other words, each second order connection γ can be regarded as a vector field

$$\gamma : J\boldsymbol{E} \to \mathbb{T}^* \otimes TJ\boldsymbol{E}\,,$$

such that

$$(\mathrm{id}_{\mathbb{T}^*} \otimes T\pi^1_0) \circ \gamma = \text{д}\,.$$

The coordinate expression of a second order connection γ is of the type

$$\gamma = u^0\,(\partial_0 + x^i_0\,\partial_i + \gamma_{00}{}^i\,\partial^0_i)\,, \qquad \gamma_{00}{}^i \in \mathcal{F}(J\boldsymbol{E})\,.$$

Given a second order connection γ and a motion s, the section

(1.30) $$\nabla[\gamma]js := j_2 s - \gamma \circ js : \boldsymbol{T} \to \mathbb{T}^{*2} \otimes V\boldsymbol{E}\,,$$

with coordinate expression

$$\nabla[\gamma]js = (\partial_{00}s^i - \gamma_{00}{}^i \circ js)\,u^0 \otimes u^0 \otimes (\partial_i \circ s)$$

will be called the (observer–independent) *acceleration* of s.

Now, for each phase connection Γ, the section [33],

(1.31) $$\gamma[\Gamma] := \text{д} \lrcorner \Gamma : J\boldsymbol{E} \to \mathbb{T}^* \otimes TJ\boldsymbol{E}$$

turns out to be a second order connection.

The coordinate expression of $\gamma[\Gamma]$ is

$$\gamma[\Gamma] = u^0 \otimes (\partial_0 + x^i_0\,\partial_i + \gamma_{00}{}^i\,\partial^0_i)\,, \qquad \gamma_{00}{}^i = \Gamma_{h0k}{}^i\,x^h_0 x^k_0 + 2\,\Gamma_{h00}{}^i\,x^h_0 + \Gamma_{000}{}^i\,.$$

By considering the algebraic structure of the bundle $J_2\boldsymbol{E} \to \boldsymbol{E}$, we can define the *homogeneous* second order connections; in coordinates, they are characterised by the fact that the coefficients $\gamma_{00}{}^i$ are second order polynomials in the coordinates x^i_0.

Now, given a phase connection Γ, the second order connection $\gamma[\Gamma]$ turns out to be homogeneous.

Even more, we have the following result.

1.9. Remark. The map

(1.32) $$\Gamma \mapsto \gamma[\Gamma] \equiv \text{д} \lrcorner \Gamma$$

turns out to be a bijection between phase connection and homogeneous second order connections. □

1.4 Phase 2–form

Now we analyse a distinguished 2–form that is induced on the phase space by a phase connection and the metric. Such a form encodes all structures of spacetime and plays a fundamental role throughout the paper.

Later we shall make assumptions on this form.

A phase connection Γ on $J\boldsymbol{E}$ and the vertical metric G yield the 2–form on $J\boldsymbol{E}$ [33],

$$\Omega[G,\Gamma] := \nu[\Gamma] \bar{\wedge} \theta : J\boldsymbol{E} \to \Lambda^2 T^* J\boldsymbol{E}\,, \tag{1.33}$$

with coordinate expression

$$\Omega[G,\Gamma] = G_{ij}\,(d^i_0 - (\Gamma_\lambda{}^i{}_{00} + \Gamma_\lambda{}^i{}_{0j}\, x^j_0)\, d^\lambda) \wedge \theta^j\,, \tag{1.34}$$

where the contracted wedge product is taken with respect to G.

1.10. Definition. A *phase 2–form* is defined to be a 2–form of the phase space of the type $\Omega[G,\Gamma]$, where Γ is a phase connection. □

A phase form will play a fundamental role throughout the paper; so, it is interesting to observe that the above form is the only natural 2–form which can be obtained from Γ and G, [34].

The form $\Omega[G,\Gamma]$ is non–degenerate, in fact

$$dt \wedge \Omega[G,\Gamma] \wedge \Omega[G,\Gamma] \wedge \Omega[G,\Gamma] : J\boldsymbol{E} \to \mathbb{T} \otimes \Lambda^7 T^* J\boldsymbol{E} \tag{1.35}$$

is a volume form on $J\boldsymbol{E}$.

Given a phase connection Γ and an observer o, we obtain

$$\Phi[\Gamma, o] = 2\, o^* \Omega[G,\Gamma] : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E}\,, \tag{1.36}$$

in fact, in adapted coordinates, we have

$$\Phi[\Gamma, o] = G_{ij}\,(d^i_0 - \Gamma_\lambda{}^i{}_{00}\, d^\lambda) \wedge d^j\,.$$

Later we shall be involved with a closed phase 2–form, thus with a *cosymplectic form* [54]. So, it is important to analyse such a closure.

Let us consider the curvature tensor of a spacetime connection K

$$R[K] : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E}\,,$$

with coordinate expression

$$R[K] = (\partial_\lambda K_\mu{}^i{}_\nu + K_\lambda{}^j{}_\nu\, K_\mu{}^i{}_j)\, d^\lambda \wedge d^\mu \otimes \partial_i \otimes d^\nu\,.$$

1.11. Theorem. *Given a spacetime connection* K *and an observer* o*, the following conditions are equivalent:*

$$d\Omega[G, \Gamma[K]] = 0\,, \tag{1.37}$$

$$\nabla[K']g = 0\,, \qquad R[K]^i{}_\lambda{}^j{}_\mu = R[K]^j{}_\mu{}^i{}_\lambda\,, \tag{1.38}$$

$$\nabla[K']g = 0\,, \qquad d\Phi[\Gamma[K], o] = 0. \tag{1.39}$$

Thus, in a coordinate chart adapted to the observer, the phase 2–form $\Omega[G, \Gamma[K]]$ *is closed if and only if*

$$K_{ihj} = -\tfrac{1}{2}\left(\partial_i G_{hj} + \partial_j G_{hi} - \partial_h G_{ij}\right), \qquad K_{ij0} + K_{ji0} = -\partial_0 G_{ij}\,,$$
$$\partial_\nu \Phi_{\lambda\mu} + \partial_\mu \Phi_{\nu\lambda} + \partial_\lambda \Phi_{\mu\nu} = 0\,,$$

where the indices of K *have been lowered by means of* $G^\flat$*.* □

1.12. Proposition. Given a phase connection Γ, the second order connection $\gamma[\Gamma]$ fulfills

$$\gamma[\Gamma] \lrcorner \Omega[G, \Gamma] = 0\,. \square$$

A phase 2–form encodes a full information of the spacetime structure; in fact, it determines the second order connection, the phase connection, the spacetime connection and the metric.

1.13. Theorem. *Let* $\Omega[G, \Gamma]$ *be the phase 2–form generated by the metric* G *and the phase connection* Γ*.*

Then, there is a unique second order connection γ' *such that* $\gamma' \lrcorner \Omega[G, \Gamma] = 0$*; namely,* $\gamma' = \gamma[\Gamma]$*.*

Moreover, there is a unique phase connection Γ' *and a unique vertical metric* G' *such that* $\nu[\Gamma'] \bar{\wedge} \theta = \Omega[G, \Gamma]$*, where the contracted wedge product is taken with respect to* G'*; namely,* $\Gamma' = \Gamma$ *and* $G' = G$*.* □

1.5 Gravitational objects

Now we specify the gravitational connection and the derived objects.

Assumption C.2. Spacetime is assumed to be equipped with a spacetime connection

$$K^\natural : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E}\,, \tag{1.40}$$

which will be called *gravitational*. □

We denote the induced phase connection, second order connection and phase 2–form by

$$\Gamma^\natural \equiv \Gamma[K^\natural] : J\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J\boldsymbol{E}}{\otimes} TJ\boldsymbol{E}\,, \tag{1.41}$$

$$\gamma^\natural \equiv \gamma[\Gamma^\natural] : J\boldsymbol{E} \to \mathbb{T}^* \otimes TJ\boldsymbol{E}\,, \tag{1.42}$$

$$\Omega^\natural \equiv \Omega[G, \Gamma^\natural] : J\boldsymbol{E} \to \Lambda^2 T^* J\boldsymbol{E}\,. \tag{1.43}$$

Analogously, given an observer o, we set

$$\Sigma^\natural[o] \equiv \Sigma[\Gamma^\natural, o] : \boldsymbol{E} \to S^2 T^*\boldsymbol{E}\,,$$
$$\Phi^\natural[o] \equiv \Phi[\Gamma^\natural, o] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}\,.$$

Moreover, we postulate the following condition as a field equation for g and $K^\natural$.

Assumption C.3. We assume the phase 2–form to be closed

$$d\Omega^\natural = 0\,.\,\square \tag{1.44}$$

Thus, under assumptions C.1, C.2 and C.3, the form $\Omega^\natural$ turns out to be "cosymplectic" (see [54]). This form, on one hand, encodes the main information arising from the classical spacetime background and, on the other hand, will be the source of the quantisation procedure.

Summarising, the gravitational objects on the phase space $J\boldsymbol{E}$ fulfill the following mutual relations

$$\gamma^\natural = д \,\lrcorner\, \Gamma^\natural\,, \qquad \Omega^\natural = \nu[\Gamma^\natural] \bar{\wedge} \theta\,, \qquad \gamma^\natural \,\lrcorner\, \Omega^\natural = 0\,, \qquad d\Omega^\natural = 0\,. \tag{1.45}$$

We denote the local potential of $\Phi^\natural[o]$ (defined up to a gauge) as

$$A^\natural[o] : \boldsymbol{E} \to T^*\boldsymbol{E}\,, \tag{1.46}$$

according to $\Phi^\natural[o] = 2\,dA^\natural[o]$.

1.6 Electromagnetic field

So far, we have assumed on space-time the structure associated with the vertical Riemannian metric g and the gravitational connection $K^\natural$. Next, we introduce the electromagnetic field.

Assumption C.4. We assume the *electromagnetic field* to be a closed 2–form on $\boldsymbol{E}$

$$f : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}\,.\,\square$$

Given a charge $q \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$, it is convenient to introduce the "normalised" electromagnetic field

$$F \equiv \frac{q}{\hbar} f : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E} , \tag{1.47}$$

with coordinate expression

$$F = 2\, F_{0j}\, d^0 \wedge d^j + F_{ij}\, d^i \wedge d^j .$$

We stress the fact that the normalised electromagnetic field and all objects which will be derived from it incorporate the charge and the Planck constant.

We define the *"universal" electric* and *"universal" magnetic fields* to be the forms on $J\boldsymbol{E}$

$$E := -\text{д} \lrcorner F : J\boldsymbol{E} \to \mathbb{T}^* \otimes T^* \boldsymbol{E} , \tag{1.48}$$
$$B := F + 2\, dt \wedge E : J\boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E} , \tag{1.49}$$

with coordinate expression

$$\begin{aligned} E &= -u^0 \otimes \Big(- F_{0j} x^j_0\, d^0 + (F_{ij} x^i_0 + F_{0j})\, d^j \Big) , \\ B &= -2\, F_{ij} x^i_0\, d^0 \wedge d^j + F_{ij}\, d^i \wedge d^j . \end{aligned}$$

Thus, we can write

$$F = -\, 2\, dt \wedge E + B . \tag{1.50}$$

We call the above objects "universal" because the standard *electric field* $E[o]$ and *magnetic field* $B[o]$ observed by an observer o turn out to be the forms on $\boldsymbol{E}$ obtained by pullback with respect to o

$$E[o] = o^* E : \boldsymbol{E} \to \mathbb{T}^* \otimes T^* \boldsymbol{E} , \qquad B[o] = o^* B : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E} , \tag{1.51}$$

whose expressions in adapted coordinates are

$$E[o] = -F_{0j}\, d^0 \wedge d^j , \qquad B[o] = F_{ij}\, d^i \wedge d^j .$$

We stress that the universal electric field carries the full information of the electromagnetic field; in other words, if we know the electric field observed by every observer o, then, we know the electromagnetic field.

We denote the local potential of F (defined up to a gauge) as

$$A^{\mathfrak{e}} : \boldsymbol{E} \to T^* \boldsymbol{E} , \tag{1.52}$$

according to $F = 2\, dA^{\mathfrak{e}}$.

1.7 Total objects

Next, we show that the electromagnetic field can be naturally incorporated into the gravitational structures. Namely, we are looking for *total* objects obtained correcting the gravitational objects by an electromagnetic term, in such a way to preserve the mutual relations (1.45).

We start from the standard coupling of the electromagnetic field F with the gravitational phase 2–form $\Omega^\natural$ on $J\boldsymbol{E}$ [1].

Accordingly, we define the *total* phase 2–form to be

$$\Omega \;\equiv\; \Omega^\natural + \Omega^\mathfrak{e} : J\boldsymbol{E} \to \Lambda^2 T^* J\boldsymbol{E}\,, \tag{1.53}$$

where we have set

$$\Omega^\mathfrak{e} \;\equiv\; \tfrac{1}{2} F\,.$$

Here, the factor 1/2 is chosen in such a way to recover the standard formulas.

Of course, we obtain

$$d\Omega = 0\,. \tag{1.54}$$

Moreover, we have

$$dt \wedge \Omega \wedge \Omega \wedge \Omega = dt \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural\,;$$

therefore, the electromagnetic field does not contribute to the total volume form on the phase space and the total phase 2–form is cosymplectic.

Furthermore, given an observer o, we obtain the closed total 2–form of $\boldsymbol{E}$

$$\Phi[o] \;\equiv\; \Phi^\natural[o] + \Phi^\mathfrak{e} := 2\,o^*\Omega\,, \tag{1.55}$$

where we have set

$$\Phi^\mathfrak{e} := 2\,o^*\Omega^\mathfrak{e} = 2\,\Omega^\mathfrak{e} = F\,.$$

Then, the local potential of $\Phi[o]$ (defined up to a gauge) turns out to be

$$A[o] = A^\natural[o] + A^\mathfrak{e} : \boldsymbol{E} \to T^*\boldsymbol{E}\,, \tag{1.56}$$

according to $\Phi[o] = 2\,d(A^\natural + A^\mathfrak{e})$.

Next, the total phase 2–form Ω yields a *total phase connection* [33, 34, 37] (see theorem 1.13).

1.14. Proposition. There is a unique phase connection Γ of the type

$$\Gamma = \Gamma^\natural + \Gamma^\mathfrak{e}\,, \tag{1.57}$$

where

$$\Gamma^{\mathfrak{e}} : J\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} (\mathbb{T}^* \otimes V\boldsymbol{E})$$

is a section, such that

$$\Omega = \nu[\Gamma] \,\bar{\wedge}\, \theta \,.$$

Namely, $\Gamma^{\mathfrak{e}}$ is the *electromagnetic soldering form*

$$\Gamma^{\mathfrak{e}} = -\tfrac{1}{2}\, G^{\sharp 2} \circ (F - 2\, dt \wedge E)\,, \tag{1.58}$$

with coordinate expression

$$\Gamma^{\mathfrak{e}} = G^{ij} \Big((F_{j0} + \tfrac{1}{2}\, F_{jh}\, x^h_0)\, d^0 + \tfrac{1}{2}\, F_{jh}\, d^h \Big) \otimes \partial^0_i \,,$$

where $G^{\sharp 2} : T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} T^*\boldsymbol{E} \to T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} V\boldsymbol{E}$ is the metric isomorphism on the second component after vertical restriction. □

Moreover, we have

$$-\,\Gamma^{\mathfrak{e}} \,\bar{\wedge}\, \theta = \Omega^{\mathfrak{e}} \,.$$

Analogously, the total phase 2–form Ω yields a *total second order connection* [33, 37] (see theorem 1.13).

1.15. Proposition. There is a unique second order connection γ of the type

$$\gamma = \gamma^{\natural} + \gamma^{\mathfrak{e}}\,, \tag{1.59}$$

where

$$\gamma^{\mathfrak{e}} : J\boldsymbol{E} \to \mathbb{T}^{*2} \otimes V\boldsymbol{E}$$

is a section, such that

$$\gamma \,\lrcorner\, \Omega = 0\,.$$

Namely, $\gamma^{\mathfrak{e}}$ turns out to be the *Lorentz force*

$$\gamma^{\mathfrak{e}} = G^{\sharp} \circ \check{E} : J\boldsymbol{E} \to \mathbb{T}^{*2} \otimes V\boldsymbol{E}\,, \tag{1.60}$$

with coordinate expression

$$\gamma^{\mathfrak{e}} = G^{ij}\, (F_{jh} x^h_0 + F_{j0})\, u^0 \otimes \partial^0_i \,. \; \square$$

Actually, we obtain

$$\gamma = \text{д} \lrcorner \Gamma \,.$$

Moreover, we have

$$\gamma^{\mathfrak{e}} = \text{д} \lrcorner \Gamma^{\mathfrak{e}} \,.$$

Eventually, the total phase 2–form Ω yields a *total spacetime connection* [33, 37] (see theorem 1.5).

1.16. Proposition. The spacetime connection K related to Γ, according to theorem 1.5 is given by

$$K = K^{\natural} + K^{\mathfrak{e}} \,, \tag{1.61}$$

where

$$K^{\mathfrak{e}} = G^{\sharp 2} \circ S_{13} \circ (dt \otimes F) : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E} \,, \tag{1.62}$$

with coordinate expression

$$K^{\mathfrak{e}} = G^{ij} \left(F_{j0}\, d^0 \otimes \partial_i \otimes d^0 + \tfrac{1}{2}\, F_{ih}\, d^0 \otimes \partial_i \otimes d^h + \tfrac{1}{2}\, F_{ih}\, d^h \otimes \partial_i \otimes d^0 \right),$$

where $G^{\sharp 2} : T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} T^*\boldsymbol{E} \to T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} V\boldsymbol{E}$ is the metric isomorphism on the second component after vertical restriction and S_{13} denotes the symmetrisation of the first and third indices. □

By summarising, the gravitational and electromagnetic fields induce the total metric spacetime connection K, the total phase connection Γ, the total second order connection γ and the total phase 2–form Ω.

1.8 Field equations

We can postulate [33] the interaction between the gravitational and electromagnetic objects and their mass and charge sources by means of a suitable version of the Einstein and Maxwell equations.

Actually, in our context, we can define the gravitational Ricci tensor and the divergence of the electromagnetic field. We can also define the contravariant energy momentum tensor and current associated with a charged moving continuum and write the continuity equations. However, because of the degeneracy of the four metric, we are unable to couple the above objects as in the Einstein's and Maxwell's general relativistic theories; we can only couple the gravitational and electromagnetic fields with the time components of the energy momentum tensor and current. So, we obtain equations which are general relativistically correct from a mathematical viewpoint, but which carry a weaker physical information with respect to the true Einstein's and Maxwell's theories. Thus, the present theory can be regarded as a rigorous mathematical model whose physical value stands in between the standard non relativistic theory and the true general relativistic theory. For further details see [33].

On the other hand the interaction between the gravitational and electromagnetic fields with their sources might be important for developing concrete models but does not play any direct role in our model for quantum mechanics, where we assume the classical gravitational and electromagnetic background as given.

1.9 Classical particle mechanics

We can formulate the mechanics of a classical charged particle in the given gravitational and electromagnetic fields $\Gamma^\natural$ and F in terms of the second order total connection γ [33].

Assumption C.5. The law of motion for a classical particle of mass m and charge q is assumed to be the generalised *Newton's equation* (see (1.30))

$$\nabla[\gamma] js = 0 \tag{1.63}$$

in the unknown motion $s : \boldsymbol{T} \to \boldsymbol{E}$. □

We can also write the above equation as (see Proposition 1.15))

$$\nabla[\gamma^\natural] js = \gamma^{\mathfrak{e}} \circ js \,,$$

i.e. , in coordinates, as (see (1.31))

$$\partial_{00} s^i - (\Gamma^\natural{}_h{}^i{}_k \circ s)\, \partial_0 s^h \partial_0 s^k - 2(\Gamma^\natural{}_h{}^i{}_0 \circ s)\, \partial_0 s^h - \Gamma^\natural{}_0{}^i{}_0 \circ s = F^i{}_0 \circ s + (F^i{}_h \circ s)\, \partial_0 s^h \,.$$

Thus, the solutions of the generalised Newton's equation are the motions whose velocity is autoparallel with respect to the total second order connection γ, or, equivalently, the motions whose acceleration with respect to the gravitational connection $\gamma^\natural$ equals the Lorentz force along the motion.

The above equation can be interpreted in the framework of a Lagrangian bicomplex [42, 63].

We can prove [42, 63] that the pure differential structure of the fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ yields a commutative diagram which involves the de Rham sequence of forms on the manifold $J\boldsymbol{E}$, a certain de Rham "contact" subsequence and their quotient sequence.

Here, for short, we are just concerned with the square commutative subdiagram

$$\begin{array}{ccc} \mathcal{S}(T^* J\boldsymbol{E}) & \xrightarrow{d} & \mathcal{S}(\Lambda^2 T^* J\boldsymbol{E}) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle k} \\ \mathcal{F}(J_2\boldsymbol{E}, \mathbb{T}^* \otimes I\!\!R) & \xrightarrow{e} & \mathcal{M}(J_4\boldsymbol{E}, \mathbb{T}^* \otimes V^*\boldsymbol{E}) \end{array} \tag{1.64}$$

where d is the exterior differential, h and k are certain natural maps and ϵ is the standard Euler–Lagrange operator (for second order Lagrangians). Actually, we will be involved only with first order Lagrangians, but, in order to exhibit a commutative diagram we have been forced to introduce sheaves larger than the ones we really meet in our specific framework.

According to the standard scheme, a first order Lagrangian theory would start with a given Lagrangian $\mathcal{L} \in \mathcal{F}(J\boldsymbol{E}, \mathbb{T}^* \otimes I\!R)$ and derive from it the Poincaré–Cartan form $\Theta \in \mathcal{S}(T^*J\boldsymbol{E})$, the 2–form $\Omega \equiv d\Theta \in \mathcal{S}(\Lambda^2 T^* J\boldsymbol{E})$ and the Euler–Lagrange operator $\mathcal{E} \equiv \epsilon(\mathcal{L}) = k(\Omega) \in \mathcal{M}(J_2\boldsymbol{E}, \mathbb{T}^* \otimes V^*\boldsymbol{E})$. On the other hand, in our scheme it is natural to start from the total phase 2–form Ω because this is a global object exhibited by the spacetime structure. So, our approach will reverse some aspects of the standard scheme.

Thus, we start with the global total phase 2–form yielded by the spacetime structure

$$\Omega : J\boldsymbol{E} \to \Lambda^2 T^* J\boldsymbol{E}\,,$$

whose coordinate expression is

$$\Omega = G_{ij}(d^i_0 - (\Gamma_{\lambda}{}^{i}{}_{00} + \Gamma_{\lambda}{}^{i}{}_{0j}\, x^j_0)\, d^\lambda) \wedge \theta^j\,.$$

We can prove that $\mathcal{E} \equiv k(\Omega)$ turns out to be the global fibred morphism

$$\mathcal{E} = G^\flat(\nabla[\gamma]) : J_2\boldsymbol{E} \to \mathbb{T}^* \otimes V^*\boldsymbol{E}\,, \tag{1.65}$$

with coordinate expression

$$\mathcal{E} = (G_{ij}\, x^i_{00} - \Gamma_{hjk}\, x^h_0 x^k_0 - 2\Gamma_{hj0}\, x^h_0 - \Gamma_{0j0})\, u^0 \otimes \check{d}^j\,,$$

where $(x^\lambda, x^i_0, x^i_{00})$ is the induced chart of $J_2\boldsymbol{E}$ and the indices of Γ have been lowered by $G^\flat$.

Hence, the Euler–Lagrange type morphism $G^\sharp(\mathcal{E})$ derived from Ω is just the covariant differential $\nabla[\gamma]$ associated with the total second order connection γ which expresses the Newton's law of motion. We recall that, according to proposition 1.13, γ is the unique second order connection such that $\gamma \lrcorner \Omega = 0$.

Moreover, the phase 2–form Ω is closed, hence it can be locally derived from a potential $\Theta \in \mathcal{S}(T^*J\boldsymbol{E})$. On the other hand, we can see that Ω admits local potentials of the type

$$\Theta : J\boldsymbol{E} \to T^*\boldsymbol{E}\,; \tag{1.66}$$

the coordinate expressions of such special potentials Θ are of the type

$$\Theta = -(\frac{1}{2} G_{ij}\, x^i_0 x^j_0 - A_0)\, d^0 + (G_{ij} x^j_0 + A_i)\, d^i\,, \qquad A_\lambda \in \mathcal{F}(\boldsymbol{E})\,.$$

Each special local potential of the above type $\Theta \in \mathcal{M}(J\boldsymbol{E}, T^*\boldsymbol{E})$ will be called a *Poincaré–Cartan form* associated with Ω. A Poincaré–Cartan form Θ is defined up to a closed local form of $\boldsymbol{E}$

$$\alpha : \boldsymbol{E} \to T^*\boldsymbol{E}\,.$$

The restriction of h to $\mathcal{M}(J\boldsymbol{E}, T^*\boldsymbol{E})$ turns out to be

$$h = \text{д}\,\lrcorner : \mathcal{M}(J\boldsymbol{E}, T^*\boldsymbol{E}) \to \mathcal{F}(J\boldsymbol{E}, \mathbb{T}^* \otimes I\!R)\,.$$

We define the *Lagrangian* associated with a Poincaré–Cartan form Θ to be the local map

$$\mathcal{L} := \text{д}\,\lrcorner\,\Theta : J\boldsymbol{E} \to \mathbb{T}^* \otimes I\!R\,, \tag{1.67}$$

with coordinate expression

$$\mathcal{L} = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_i\, x^i_0 + A_0)\, u^0\,.$$

By considering the gauge of Poincaré–Cartan forms, a Lagrangian $\mathcal{L}$ turns out to be defined up to a map of the type

$$\text{д}\,\lrcorner\,\alpha : J\boldsymbol{E} \to \mathbb{T}^* \otimes I\!R\,,$$

where $\alpha \in \mathcal{S}(T^*\boldsymbol{E})$ is a closed form.

We observe that, by pullback with respect to $t : \boldsymbol{E} \to \boldsymbol{T}$, a Lagrangian $\mathcal{L}$ can be also regarded as a fibred morphism

$$\mathcal{L} : J\boldsymbol{E} \to T^*\boldsymbol{E}\,;$$

accordingly, we shall write

$$\mathcal{L} \;\equiv\; \mathcal{L}_0\, u^0 \;\simeq\; \mathcal{L}_0\, d^0\,.$$

Eventually, the Euler–Lagrange morphism associated with a Lagrangian $\mathcal{L}$ is just

$$\epsilon(\mathcal{L}) = G^\sharp(\nabla[\gamma]) : J_2\boldsymbol{E} \to \mathbb{T}^* \otimes V^*\boldsymbol{E}\,. \tag{1.68}$$

Of course, all Lagrangians $\mathcal{L}$ yield the same Euler–Lagrange operator $G^\sharp(\nabla[\gamma])$.

Thus, the commutative diagram (1.64) yields, in our specific case, the diagram

$$\begin{array}{ccc} \Theta & \xrightarrow{\;d\;} & \Omega \\ {\scriptstyle \text{д}\,\lrcorner}\big\downarrow & & \big\downarrow{\scriptstyle k} \\ \mathcal{L} & \xrightarrow{\;e\;} & G^\flat(\nabla[\gamma]) \end{array} \tag{1.69}$$

where the second column consists of global objects, while the first column consists of local objects defined up to a gauge determined by the closed forms α of $\boldsymbol{E}$.

We can say more about the relation between Lagrangians and Poincaré–Cartan forms.

We define, in the standard way, the *momentum* associated with a Lagrangian $\mathcal{L}$ as the map

$$\check{\mathcal{P}} := V_{\boldsymbol{E}}\mathcal{L} : J\boldsymbol{E} \to V^*\boldsymbol{E}\,; \tag{1.70}$$

the momentum turns out to be the vertical restriction of the 1–form on $\boldsymbol{E}$

$$\mathcal{P} \equiv \theta \lrcorner \check{\mathcal{P}} : J\boldsymbol{E} \to T^*\boldsymbol{E}\,.$$

The coordinate expression of the momentum $\mathcal{P}$ is given by

$$\check{\mathcal{P}} = (G^0_{ij}\, x^j_0 + A_i)\, \check{d}^i\,, \qquad \mathcal{P} = -(G_{ij}\, x^i_0\, x^j_0 + A_i\, x^i_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_i)\, d^i\,.$$

Then, we can prove that each Poincaré–Cartan form Θ can be written as [63]

$$\Theta = \mathcal{L} + \mathcal{P} : J\boldsymbol{E} \to T^*\boldsymbol{E}\,, \tag{1.71}$$

where $\mathcal{L} \equiv д \lrcorner \Theta$ and $\mathcal{P} \equiv V_{\boldsymbol{E}}\mathcal{L}$.

Next we describe the above objects in terms of observers.

So, let us consider an observer o, the closed 2–form $\Phi[o] \equiv 2\, o^*\Omega : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$ and its local potential $A[o] : \boldsymbol{E} \to T^*\boldsymbol{E}$ (defined up to a closed form of $\boldsymbol{E}$) (see section 1.7).

We define the *Hamiltonian* associated with a Poincaré–Cartan form Θ and the observer o to be the local map

$$\mathcal{H}[o] := -o \lrcorner \Theta : J\boldsymbol{E} \to \mathbb{T}^* \otimes I\!R\,; \tag{1.72}$$

its coordinate expression in adapted coordinates is

$$\mathcal{H}[o] = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, u^0\,.$$

We can express the Hamiltonian as

$$\mathcal{H}[o] = \mathcal{K}[o] - o \lrcorner A[o]\,. \tag{1.73}$$

We observe that, by pullback with respect to $t : \boldsymbol{E} \to \boldsymbol{T}$, a Hamiltonian $\mathcal{H}[o]$ can be also regarded as a fibred morphism

$$\mathcal{H}[o] : J\boldsymbol{E} \to T^*\boldsymbol{E}\,;$$

accordingly, we shall write

$$\mathcal{H}[o] \equiv \mathcal{H}_0\, u^0 \simeq \mathcal{H}_0\, d^0\,.$$

Moreover, we define the *observed momentum* associated with the Lagrangian $\mathcal{L}$ and the observer o to be the fibred morphism

$$\mathcal{P}[o] := \nu[o]^* \circ \check{\mathcal{P}} : J\boldsymbol{E} \to T^*\boldsymbol{E}\,, \tag{1.74}$$

whose coordinate expression in adapted coordinates is

$$\mathcal{P}[o] = (G_{ij} x_0^j + A_i)\, d^i\,.$$

Hence, we obtain the following expressions (see also (1.14))

$$\begin{aligned} \check{\mathcal{P}} &= \check{\mathcal{Q}}[o] + \check{A}[o]\,, \\ \mathcal{L} &= \mathcal{K}[o] + \text{д} \lrcorner A[o]\,, \\ \mathcal{L} &= -\mathcal{H}[o] + \text{д} \lrcorner \mathcal{P}[o]\,, \\ \Theta &= -\mathcal{H}[o] + \mathcal{P}[o]\,. \end{aligned}$$

Hence, we have the coordinate expressions

$$\Theta = -\mathcal{H}_0\, d^0 + \mathcal{P}_i\, d^i\,, \tag{1.75}$$

$$\mathcal{L}_0 = (-\mathcal{H}_0 + \mathcal{P}_i\, x_0^i)\, d^0\,, \tag{1.76}$$

where $\mathcal{H}_0$ and $\mathcal{P}_i$ are the components of the Hamiltonian $\mathcal{H}[o]$ and of the observed momentum $\mathcal{P}[o]$ associated with the observer o attached to the chosen chart.

We stress that our definitions of a Poincaré–Cartan form, a Lagrangian and a momentum (defined locally up to a gauge) do not involve observers. On the other hand, each Poincaré–Cartan form, each Lagrangian and each momentum can be expressed as a combination of two terms related to an observer.

We recall that all above formulas incorporate the Planck constant because $\hbar$ has been used in the definition of the normalised metric G and electromagnetic field F, hence in the definition of Ω.

2 Quantum mechanics

Next, we formulate a covariant scheme for the quantum mechanics of a scalar particle with given mass m and charge q in the framework of classical curved spacetime $\boldsymbol{E}$ with absolute time, equipped with a given vertical metric and gravitational and electromagnetic fields. As for the classical background we shall mainly refer to the total cosymplectic phase 2–form, which encodes fully the classical spacetime structure.

2.1 Quantum structure

We start by introducing the quantum framework constituted by the quantum bundle equipped with a quantum connection. These are our only assumptions for the quantum structure.

Assumption Q.1. We assume the *quantum bundle* to be a one–dimensional complex bundle over spacetime

$$\pi : \boldsymbol{Q} \to \boldsymbol{E} \tag{2.1}$$

equipped with the Hermitian product with values in the bundle of space–like 3–forms $\Lambda^3 V^* \boldsymbol{E}$

$$h : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{C} \otimes \Lambda^3 V^* \boldsymbol{E} \,.\, \square \tag{2.2}$$

A local section

$$b \in \mathcal{S}(\mathbb{L}^{3/2} \otimes \boldsymbol{Q}) \,,$$

such that

$$h(b,\, b) = \eta \,,$$

is a base said to be *normal*. We denote the complex dual base by

$$z \in \mathcal{F}(\boldsymbol{Q}, \mathbb{L}^{*3/2} \otimes \mathbb{C}) \,.$$

Henceforth, we shall refer to a normal base b and a related fibred chart (x^λ, z). The induced local base of $\mathcal{T}(\boldsymbol{Q})$ will be denoted (by abuse of notation) by $(\partial_\lambda, \partial z)$.

We obtain the associated real base (b_1, b_2) and dual real base w^1, w^2, where

$$b_1 \,\equiv\, b \,, \qquad b_2 \,\equiv\, \mathfrak{i}\, b \,, \qquad b_1, b_2 \in \mathcal{S}(\mathbb{L}^{3/2} \otimes \boldsymbol{Q}) \,,$$
$$z = w^1 + \mathfrak{i}\, w^2 \,, \qquad w^1, w^2 \in \mathcal{F}(\boldsymbol{Q}, \mathbb{L}^{*3/2} \otimes I\!R) \,.$$

If $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ is a section, then we write

$$\Psi = \psi\, b \,, \qquad \psi \,\equiv\, z \circ \Psi \in \mathcal{F}(\boldsymbol{E}, \mathbb{L}^{*3/2} \otimes \mathbb{C}) \,.$$

Hence, the coordinate expression of h is

$$h(\Psi,\, \Psi') = \bar{\psi}\, \psi' \eta\,, \qquad \forall \Psi, \Psi' \in \mathcal{S}(\boldsymbol{Q})\,.$$

We obtain the real linear fibred isomorphism over $\boldsymbol{E}$

$$h^\sharp : \boldsymbol{Q}^* \to \mathbb{L}^3 \otimes \boldsymbol{Q}\,,$$

induced by the real component of h, with coordinate expression

$$h^\sharp = b_1 \otimes w^1 + b_2 \otimes w^2\,.$$

The Liouville vector field

$$\boldsymbol{1} : \boldsymbol{Q} \to V\boldsymbol{Q} \,\simeq\, \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} : q \mapsto (q,\, q)$$

will be identified with

$$\boldsymbol{1} = \mathrm{id}_{\boldsymbol{Q}} : \boldsymbol{E} \to \boldsymbol{Q}^* \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}\,;$$

thus, we shall write

$$\boldsymbol{1} = z\,\partial z \,\simeq\, z\, b\,.$$

A connection on the bundle $\boldsymbol{Q} \to \boldsymbol{E}$ can be regarded as a section

$$\chi : \boldsymbol{Q} \to T^*\boldsymbol{E} \underset{\boldsymbol{Q}}{\otimes} T\boldsymbol{Q}\,,$$

which projects on $1_{\boldsymbol{E}} \in T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} T\boldsymbol{E}$. Thus a connection χ can be also regarded as a section

$$\chi : \boldsymbol{Q} \to J_1\boldsymbol{Q}\,,$$

where $J_1\boldsymbol{Q} \to \boldsymbol{Q}$ is the first jet bundle of sections of $\boldsymbol{Q} \to \boldsymbol{E}$.

Hence, the coordinate expression of a connection χ is of the type

$$\chi = d^\lambda \otimes (\partial_\lambda + \chi_\lambda\, \partial z)\,, \qquad \chi_\lambda \in \mathcal{F}(\boldsymbol{Q}, \mathbb{C})\,.$$

The curvature of a connection χ is a $\boldsymbol{Q}$–valued 2–form [33]

$$R[\chi] : \boldsymbol{Q} \to \Lambda^2 T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}\,,$$

whose coordinate expression is

$$R[\chi] = (\partial_\lambda \chi_\mu + \chi_\lambda\, \frac{\partial \chi_\mu}{\partial z})\, d^\lambda \wedge d^\mu \otimes b\,.$$

A connection χ is said to be Hermitian if

$$\nabla[K']\big(h(\Psi,\,\Psi')\big) = h(\nabla[\chi]\Psi,\,\Psi') + h(\Psi,\,\nabla[\chi]\Psi')\,, \quad \forall \Psi, \Psi' \in \mathcal{S}(\boldsymbol{Q})\,. \tag{2.3}$$

The coordinate expression of a Hermitian connection is of the type

$$\chi = d^\lambda \otimes (\partial_\lambda + i\,\chi_\lambda\, z\,\partial z)\,, \qquad \chi_\lambda \in \mathcal{F}(\boldsymbol{E}, I\!R)\,.$$

Next, in view of the quantum connection, we briefly recall the notion of system of connections.

We consider the pullback bundle of the quantum bundle $\boldsymbol{Q} \to \boldsymbol{E}$ with respect to $J\boldsymbol{E} \to \boldsymbol{E}$

$$\pi^\uparrow : \boldsymbol{Q}^\uparrow := J\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J\boldsymbol{E}\,. \tag{2.4}$$

The previous definitions and results concerning connections of the bundle $\boldsymbol{Q} \to \boldsymbol{E}$ can be immediately extended to connections of the bundle $\boldsymbol{Q}^\uparrow \to J\boldsymbol{E}$.

2.1. Remark. A *system of connections* of the bundle $\boldsymbol{Q} \to \boldsymbol{E}$ parametrised by $J\boldsymbol{E}$ is defined to be a fibred morphism over $\boldsymbol{E}$ (see [33])

$$\xi : J\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J\boldsymbol{Q}\,.$$

Hence, ξ associates with each observer o a connection

$$\xi \circ o^\uparrow : \boldsymbol{Q} \to J\boldsymbol{Q}\,,$$

where $o^\uparrow : \boldsymbol{Q} \to J\boldsymbol{E} \times_{\boldsymbol{E}} \boldsymbol{Q}$ is the pullback of o.

There is a distinguished inclusion

$$\iota : J\boldsymbol{E} \underset{\boldsymbol{E}}{\times} J\boldsymbol{Q} \to J(\boldsymbol{Q}^\uparrow)\,,$$

where $J(\boldsymbol{Q}^\uparrow)$ is the first jet bundle of sections of the bundle $J\boldsymbol{Q}^\uparrow \to J\boldsymbol{E}$.

Hence, we obtain the section, i.e. a connection of the bundle $J(\boldsymbol{Q}^\uparrow) \to J\boldsymbol{E}$,

$$\text{ч} := \iota \circ \xi^\uparrow : \boldsymbol{Q}^\uparrow \to J(\boldsymbol{Q}^\uparrow)\,,$$

where $\xi^\uparrow : J\boldsymbol{E} \times_{\boldsymbol{E}} \boldsymbol{Q} \to J\boldsymbol{E} \times_{\boldsymbol{E}} J\boldsymbol{Q}$ is the pullback of ξ.

The connection ч is said to be *universal*, because every connection of the system $\xi \circ o^\uparrow$ can be obtained from ч by pullback as

$$\xi \circ o^\uparrow = o^* \text{ч}\,.$$

The universal connection fulfills the following property: for every vertical vector field $X : \boldsymbol{E} \to VJ\boldsymbol{E}$

$$X \lrcorner\, \text{ч} = X\,.$$

Conversely, if ч is a connection on the bundle $J(\boldsymbol{Q}^{\uparrow}) \to J\boldsymbol{E}$, which fulfills the above property, then there is a unique system of connections parametrised by $J\boldsymbol{E}$, whose universal connection is ч.

Also the curvature of the universal connection fulfills the universal property

$$R[o^* \text{ч}] = o^* R[\text{ч}] \,. \square$$

Now we are in the position to make our main assumption of the quantum theory.

Assumption Q.2. We assume a connection

$$\text{ч} : \boldsymbol{Q}^{\uparrow} \to T^* J\boldsymbol{E} \underset{J\boldsymbol{E}}{\otimes} T\boldsymbol{Q}^{\uparrow} \tag{2.5}$$

called *quantum connection*, such that

(1) ч is Hermitian,

(2) ч is universal,

(3) the curvature of ч is given by

$$R[\text{ч}] = \mathfrak{i}\, \Omega \otimes \mathbf{1} : \boldsymbol{Q}^{\uparrow} \to \Lambda^2 T^* J\boldsymbol{E} \underset{J\boldsymbol{E}}{\otimes} \boldsymbol{Q}^{\uparrow} \,. \square \tag{2.6}$$

The closure of the phase 2–form Ω turns out here to be a necessary integrability condition because of the Bianchi identity of $R[\text{ч}]$.

2.2. Theorem. *Given a local base b, a quantum connection* ч *turns out to be locally of the type*

$$\text{ч} = \text{ч}^{\parallel} + \mathfrak{i}\, \Theta \otimes \mathbf{1} \,, \tag{2.7}$$

where $\text{ч}^{\parallel}$ *is the flat connection associated with b and Θ is a classical Poincaré–Cartan form (see section 1.9).*

Moreover, given a local base b and an observer o, a quantum connection ч *turns out to be locally of the type*

$$\text{ч} = \text{ч}^{\parallel} + \mathfrak{i}\, (-\mathcal{H}[o] + \mathcal{P}[o]) \otimes \mathbf{1} \tag{2.8}$$

$$= \text{ч}^{\parallel} + \mathfrak{i}\, (-\mathcal{K}[o] + \mathcal{Q}[o] + A[o]) \otimes \mathbf{1} \,, \tag{2.9}$$

where $\text{ч}^{\parallel}$ *is the flat connection associated with b and $\mathcal{H}[o], \mathcal{P}[o]$ are the Hamiltonian and the observed momentum associated with a classical Poincaré–Cartan form and the observer o, and $\mathcal{K}[o], \mathcal{Q}[o]$ are the kinetic energy and momentum associated with o and $A[o]$ is a potential of $\Phi[o]$.*

In other words, the coordinate expression of a quantum connection ч *turns out to be locally of the type*

$$\begin{aligned} ч &= d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathfrak{i}\, ч_\lambda\, d^\lambda \otimes (z\,\partial z) \\ &= d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathfrak{i}\, \Big(-(\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_i)\, d^i \Big) \otimes (z\,\partial z) \\ &= d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathfrak{i}\, (-\mathcal{H}_0\, d^0 + \mathcal{P}_i\, d^i) \otimes (z\,\partial z)\,, \end{aligned}$$

where $A[o]$ *is a local potential and* $\mathcal{H}[o], \mathcal{P}[o]$ *are the corresponding Hamiltonian and observed momentum associated with the observer* o *attached to the chosen chart.* □

We stress that classically Poincaré–Cartan form, the potential, the Hamiltonian and the momentum are locally determined up to a gauge (see 1.9), but here ч determines this local gauge.

Given a quantum connection ч, let us analyse the transition maps for the local potential $A[o]$ occurring in the above formula, with respect to a change of the base b and of the observer o.

2.3. Proposition. If b, b' are local bases and $b' = \exp(i\theta)$, with $\theta \in \mathcal{F}(\boldsymbol{E})$, and o is an observer, then we obtain (with reference to formula (2.9))

$$A'[o] = A[o] - d\theta\,. \;\square \tag{2.10}$$

2.4. Proposition. If b is a local base, o, o' are observers, with $o' = o + V$, with $V : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$, then we obtain (with reference to (2.9))

$$A[o'] = A[o] - \tfrac{1}{2}\, G(V, V) + \nu[o]^* \lrcorner\, G^\flat(V)\,. \tag{2.11}$$

In other words, if (x^λ) and (x'^λ) are spacetime charts adapted to o and o', respectively, and we set

$$A[o'] = A'_\lambda d'^\lambda\,, \qquad A[o] = A_\lambda d^\lambda\,, \qquad V = V^i u^0 \otimes \partial_i\,,$$

then we obtain

$$A'_0 = A_0 + (-\tfrac{1}{2}\, V_i + A_i)\, V^i\,, \qquad A'_i = \frac{\partial x^j}{\partial x'^i}\,(A_j + V_j)\,. \;\square$$

A pair $(Q, ч)$ is said to be a *quantum structure*.

In [63] a topological necessary and sufficient condition for the existence of a quantum structure has been found.

2.5. Theorem. *The following conditions are equivalent:*

(1) there exists a quantum structure $(\boldsymbol{Q}, ч)$*;*

(2) the closed form Ω determines a cohomology class in the subgroup

$$[\Omega] \in \iota(H^2(\boldsymbol{E}, \mathbb{Z})) \subset H^2(\boldsymbol{E}, I\!R)\,,$$

where $\iota : (H^2(\boldsymbol{E}, \mathbb{Z})) \to H^2(\boldsymbol{E}, I\!R)$ is the canonical group morphism. □

The two simple assumptions of a quantum bundle over spacetime and of a universal quantum connection enable us to avoid the intricate problems related to polarisations, which are typical in geometric quantisation.

The quantum structure is the source of all further developments, including the quantum Lagrangian, the Schrödinger equation, the quantum momentum, the probability current, the quantum operators and the quantum Hilbert bundle. Now, the condition on the quantum connection to be linked with the cosymplectic form Ω has forced us to start with the enlarged quantum bundle $\boldsymbol{Q}^{\uparrow}$, but we expect that most physically significant objects live on $\boldsymbol{Q}$; actually, a projection method will yield these objects on $\boldsymbol{Q}$.

2.2 Covariant differential

Now, we analyse the covariant differential and related operators acting on quantum sections.

In order to perform the covariant differential of a quantum section $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$, we need to take its pullback

$$\Psi^{\uparrow} : J\boldsymbol{E} \to \boldsymbol{Q}^{\uparrow} : j_e \mapsto \Big(j_e,\ \Psi(e)\Big)\,.$$

On the other hand, the covariant differential turns out to be valued just in $T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} \boldsymbol{Q}$ because of the universality of ч.

Thus, for each $\Psi \in \mathcal{S}(\boldsymbol{Q})$, we obtain the *covariant differential*

$$\nabla\Psi \;\equiv\; \nabla[\text{ч}]\Psi : J\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}\,, \tag{2.12}$$

with coordinate expression

$$\nabla\Psi = \nabla_\lambda\psi\, d^\lambda \otimes b\,,$$

where

$$\nabla_\lambda\psi\, d^\lambda \;\equiv\; (\partial_\lambda\psi - \mathfrak{i}\ \text{ч}_\lambda\, \psi)\, d^\lambda = (\partial_0\psi + \mathfrak{i}\ \mathcal{H}_0\, \psi)\, d^0 + (\partial_j\psi - \mathfrak{i}\ \mathcal{P}_j\, \psi)\, d^j\,.$$

Moreover, we define the *time-like* and *space-like differentials* of a $\Psi \in \mathcal{S}(\boldsymbol{Q})$ to be the maps

$$\bar\nabla\Psi := \text{д} \lrcorner \nabla\Psi : J\boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q}\,, \qquad \overset{\vee}{\nabla}\Psi : J\boldsymbol{E} \to V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q} \tag{2.13}$$

with coordinate expressions

$$\bar{\nabla}\Psi = (\partial_0\psi + \dot{x}_0^j\,\partial_j\psi - \mathfrak{i}\;\mathcal{L}_0\,\psi)\,d^0 \otimes b\,, \qquad \overset{\vee}{\nabla}\Psi = (\partial_j\psi - \mathfrak{i}\;\mathcal{P}_j\,\psi)\,\check{d}^j \otimes b\,.$$

Furthermore, we define the *quantum Laplacian* of a $\Psi \in \mathcal{S}(\boldsymbol{Q})$ to be the section

$$\Delta\Psi \;\equiv\; \Delta[\text{ч}]\Psi \;\equiv\; \langle \bar{G},\; \nabla[K,\text{ч}]\nabla[\text{ч}]\Psi\rangle : J\boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q}\,, \tag{2.14}$$

with coordinate expression

$$\Delta\Psi = G^{hk}\,\Big((\partial_h - \mathfrak{i}\;\text{ч}_h)(\partial_k - \mathfrak{i}\;\text{ч}_k) + K_h{}^l{}_k\,(\partial_l - \mathfrak{i}\;\text{ч}_l)\Big)\,\psi\,u^0 \otimes b\,.$$

Eventually, let us consider an observer o.

According to the discussion on systems of connections (see 2.1) we obtain the *observed quantum connection*

$$o^*\text{ч} : \boldsymbol{Q} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}\,, \tag{2.15}$$

with coordinate expression in adapted coordinates

$$o^*\text{ч} = d^\lambda \otimes \partial_\lambda + \mathfrak{i}\,A_\lambda\,z\,d^\lambda \otimes \partial z\,,$$

where $A_\lambda d^\lambda = A[o]$ is the potential determined locally by ч.

Accordingly, we obtain the *observed covariant differential* of a $\Psi \in \mathcal{S}(\boldsymbol{Q})$

$$(\nabla[\text{ч}]\Psi) \circ o = \nabla[o^*\text{ч}]\Psi : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}\,, \tag{2.16}$$

with coordinate expression in adapted coordinates

$$\nabla[o^*\text{ч}]\Psi \;\equiv\; \nabla[o^*\text{ч}]_\lambda\psi\,d^\lambda \otimes b\,,$$

where

$$\nabla[o^*\text{ч}]_\lambda\psi \;\equiv\; (\partial_\lambda\psi - \mathfrak{i}\;A_\lambda\,\psi)\,.$$

Analogously, we obtain the *observed Laplacian* of a $\Psi \in \mathcal{S}(\boldsymbol{Q})$

$$\Delta[o^*\text{ч}]\Psi \;\equiv\; \langle \bar{G},\; \nabla[K,o^*\text{ч}]\nabla[o^*\text{ч}]\Psi\rangle : \boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q}\,, \tag{2.17}$$

with coordinate expression in adapted coordinates

$$\Delta[o^*\text{ч}]\Psi = G^{hk}\,\Big((\partial_h - \mathfrak{i}\;A_h)(\partial_k - \mathfrak{i}\;A_k) + K_h{}^l{}_k\,(\partial_l - \mathfrak{i}\;A_l)\Big)\psi\,u^0 \otimes b\,.$$

2.3 Quantum dynamics

Now, we look for a distinguished Lagrangian on the quantum bundle as source of the quantum dynamics. Actually, we find such a Lagrangian and derive from it the quantum momentum, the generalised Schrödinger equation and the conserved probability current. We are also able to express all above objects by a direct geometrical way beyond the Lagrangian formalism.

For each $\Psi \in \mathcal{S}(\boldsymbol{Q})$, we have the two distinguished maps

$$\tfrac{1}{2}\, dt \wedge \Big(h(\Psi, \mathfrak{i}\, \bar{\nabla}\Psi) + h(\mathfrak{i}\, \bar{\nabla}\Psi, \Psi) \Big) : J\boldsymbol{E} \to \Lambda^4 T^* \boldsymbol{E}\,,$$
$$\tfrac{1}{2}\, dt \wedge \Big((\bar{G} \otimes h)(\overset{\vee}{\nabla}\Psi, \overset{\vee}{\nabla}\Psi) \Big) : J\boldsymbol{E} \to \Lambda^4 T^* \boldsymbol{E}\,,$$

which live on $J\boldsymbol{E}$. But we are looking for a map living on $\boldsymbol{E}$. The following theorem yields such a distinguished map.

2.6. Theorem. *There is a unique linear combination (up to a multiplicative factor) of the above maps which projects on $\boldsymbol{E}$, namely the form*

$$\text{(2.18)}\quad \mathsf{L}[\Psi] = \tfrac{1}{2}\, dt \wedge \Big((h(\Psi, i\bar{\nabla}\Psi) + h(\mathfrak{i}\, \bar{\nabla}\Psi, \Psi) - (\bar{G} \otimes h)(\overset{\vee}{\nabla}\Psi, \overset{\vee}{\nabla}\Psi) \Big) : \boldsymbol{E} \to \Lambda^4 T^* \boldsymbol{E}\,,$$

with coordinate expression

$$\mathsf{L}[\Psi] = \tfrac{1}{2} \Big(i\,(\bar{\psi}\, \partial_0 \psi - \psi\, \partial_0 \bar{\psi}) - G^{hk}\, \partial_h \bar{\psi}\, \partial_k \psi$$
$$+\, \mathfrak{i}\; G^{hk} A_h\, (\psi\, \partial_k \bar{\psi} - \bar{\psi}\, \partial_k \psi) + \bar{\psi}\, \psi\, (2\, A_0 - G^{rs}\, A_r A_s) \Big)\, \upsilon^0\,,$$

where we have set

$$\upsilon^0 \;\equiv\; \upsilon(u^0) = \sqrt{|g|} d^0 \wedge d^1 \wedge d^2 \wedge d^3\,. \square$$

In [35] all natural quantum Lagrangians have been classified by methods of gauge-natural bundles and natural operators, [41, 43]. It has been proved that all natural quantum Lagrangians are multiples of the canonical volume form υ, where multiplicative factors are invariant functions. A base of these invariant functions is constituted by three functions and the unique non trivial function in the base is just given by the above formula.

Assumption Q.3. We assume the 4–form L as the *quantum Lagrangian* for the quantum dynamics. □

2.7. Proposition. The quantum Lagrangian yields, according to the standard procedure, the *quantum momentum*, which, for each $\Psi \in \mathcal{S}(\boldsymbol{Q})$, is the map

$$\mathsf{P}[\Psi] := \mathfrak{i}\, h^\sharp \circ (V_{\boldsymbol{Q}} \mathsf{L})[\Psi] : \boldsymbol{E} \to \mathbb{L}^3 \otimes (T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \Lambda^4 T^* \boldsymbol{E})\,,$$

with coordinate expression

$$\mathsf{P}[\Psi] = \Big(\psi\,\partial_0 - \mathfrak{i}\; G^{hk}(\partial_h\psi - \mathfrak{i}\; A_h\,\psi)\,\partial_k\Big)\, b\otimes \upsilon^0\,.\,\square$$

Moreover, by considering the linear fibred isomorphism over $\boldsymbol{E}$

$$T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \Lambda^4 T^*\boldsymbol{E} \,\simeq\, \Lambda^3 T^*\boldsymbol{E}\,,$$

we can regard, for each $\Psi \in \mathcal{S}(\boldsymbol{Q})$, the quantum momentum as a section

$$\mathsf{P}[\Psi] : \boldsymbol{E} \to \mathbb{L}^3 \otimes (\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \Lambda^3 T^*\boldsymbol{E})\,. \tag{2.19}$$

For each $\Psi \in \mathcal{S}(\boldsymbol{Q})$, we have the two distinguished maps (by applying the above isomorphism)

$$\text{д}\otimes\Psi : J\boldsymbol{E} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q})\,,$$
$$G^\sharp(\overset{\vee}{\nabla}\Psi) : J\boldsymbol{E} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q})\,,$$

which live on $J\boldsymbol{E}$. But we are looking for a map living on $\boldsymbol{E}$. The following theorem yields such a distinguished map.

2.8. Proposition. There is a unique linear combination (up to a multiplicative factor) of the above maps which projects on $\boldsymbol{E}$, namely the section

$$\mathsf{P}[\Psi] = \Big(\text{д}\otimes\Psi - \mathfrak{i}\; G^\sharp(\overset{\vee}{\nabla}\Psi)\Big)\otimes\upsilon : \boldsymbol{E} \to \mathbb{L}^3 \otimes (\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \Lambda^3 T^*\boldsymbol{E})\,.\,\square \tag{2.20}$$

2.9. Proposition. The quantum Lagrangian yields, according to the standard procedure, the *quantum Euler-Lagrange operator*, which, for each $\Psi \in \mathcal{S}(\boldsymbol{Q})$, is the map

$$h^\sharp(\mathsf{E}[\Psi]) : \boldsymbol{E} \to \mathbb{L}^3 \otimes (\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \Lambda^4 T^*\boldsymbol{E})\,, \tag{2.21}$$

with coordinate expression

$$h^\sharp(\mathsf{E}[\Psi]) = 2\Big(\mathfrak{i}\;(\partial_0 - \mathfrak{i}\,A_0 + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}})\psi + \tfrac{1}{2}\,G^{hk}\,(\partial_h - \mathfrak{i}\,A_h)\,(\partial_k - \mathfrak{i}\,A_k)\,\psi$$
$$+ \tfrac{1}{2}\,\frac{\partial_h(G_0^{hk}\sqrt{|g|})}{\sqrt{|g|}}\,(\partial_k - \mathfrak{i}\,A_k)\,\psi\Big) b\otimes\upsilon^0\,.\,\square$$

In order to obtain a more concise expression of the above formula, let us make a general observation.

2.10. Lemma. Let $\boldsymbol{F} \to \boldsymbol{B}$ be a vector bundle, $s : \boldsymbol{B} \to \boldsymbol{F}$ a section and $Y : \boldsymbol{F} \to T\boldsymbol{F}$ a vector field, which is projectable on the base space and linear over its projection. Then, s can be naturally regarded as a vertical vector field on $\boldsymbol{F}$; hence the Lie bracket between Y and this vertical vector field yield a vertical vector field, which can be naturally regarded as a section

$$Y.s : \boldsymbol{B} \to \boldsymbol{F} \,.$$

If (x^λ, y^i) is a linear fibred chart of $\boldsymbol{F}$ and (b_i) the associated local base, then we obtain the coordinate expressions

$$Y = Y^\lambda \, \partial_\lambda + Y^i_j \, y^j \partial_i \,, \qquad Y.s = (Y^\lambda \, \partial_\lambda s^i - Y^i_j \, s^j) \, b_i \,, \qquad Y^\lambda, Y^i_j \in \mathcal{F}(\boldsymbol{B}) \,. \square$$

Now, let us consider an observer o.

Then, we obtain the vector field

$$X[o] := o \lrcorner (o^* \text{ч}) : \boldsymbol{Q} \to \mathbb{T}^* \otimes T\boldsymbol{Q} \,,$$

with coordinate expression in adapted coordinates

$$X[o] = u^0 \otimes (\partial_0 + \mathfrak{i} \; A_0 \, z \, \partial z) \,.$$

Hence, for each $\Psi \in \mathcal{S}(\boldsymbol{Q})$, we obtain the Lie derivative

$$L[o, \text{ч}]\Psi := L(X[o])(\Psi \sqrt{\upsilon}) \, / \, \sqrt{\upsilon} : \boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q} \,,$$

with coordinate expression

$$L[o, \text{ч}]\Psi = (\partial_0 - \mathfrak{i} \, A_0 + \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}) \, \psi \, u^0 \otimes b \,.$$

2.11. Proposition. Given an observer o, the Euler-Lagrange operator is expressed, for each $\mathcal{S}(\boldsymbol{Q})$, by

$$h^\sharp(\mathsf{E}[\Psi]) = 2 \Big(\mathfrak{i} \; L[o, \text{ч}]\Psi + \tfrac{1}{2} \, \Delta[o^* \text{ч}]\Psi \Big) \otimes \upsilon \,. \square \tag{2.22}$$

For each $\Psi \in \mathcal{S}(\boldsymbol{Q})$, we have the two distinguished maps

$$\bar{\nabla}\Psi \otimes \upsilon : J\boldsymbol{E} \to \mathbb{L}^3 \otimes (\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \Lambda^4 T^* \boldsymbol{E}) \,,$$
$$d[\text{ч}]\mathsf{P}[\Psi] : J\boldsymbol{E} \to \mathbb{L}^3 \otimes (\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \Lambda^4 T^* \boldsymbol{E}) \,,$$

which live on $J\boldsymbol{E}$. But we are looking for a map living on $\boldsymbol{E}$. The following theorem yields such a distinguished map.

2.12. Proposition. There is a unique linear combination (up to a multiplicative factor) of the above maps, which projects on $\boldsymbol{E}$, namely the section

$$h^\sharp(\mathsf{E}[\Psi]) = \bar\nabla\Psi \otimes \upsilon + d\mathsf{P}[\Psi] : \boldsymbol{E} \to \mathbb{L}^3 \otimes (\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \Lambda^4 T^*\boldsymbol{E}) \,. \square \tag{2.23}$$

Thus, the quantum dynamical equation in the unknown $\Psi \in \mathcal{S}(\boldsymbol{Q})$ is assumed to be the *generalised Schrödinger equation*

$$h^\sharp(\mathsf{E}[\Psi]) = 0 \,.$$

2.13. Proposition. By considering the invariance of the quantum Lagrangian under the action of the group $U(1)$, the Nöther theorem yields the conserved *probability current*, which, for each $\Psi \in \mathcal{S}(\boldsymbol{Q})$ solution of the Schrödinger equation, is the closed 3–form

$$\mathsf{j}[\Psi] = \tfrac{1}{2}\Big(h(\Psi\,,\mathsf{P}[\Psi]) - h(\mathsf{P}[\Psi]\,,\Psi)\Big)/, \eta : \boldsymbol{E} \to \Lambda^3 T^*\boldsymbol{E}\,, \tag{2.24}$$

with coordinate expression

$$\mathsf{j}[\Psi] = \bar\psi\psi\,\upsilon^0_0 - G^{hk}\Big(\mathfrak{i}\,\tfrac{1}{2}\,(\bar\psi\,\partial_k\psi - \partial_k\bar\psi\,\psi) + A_k\,\bar\psi\psi\Big)\,\upsilon^0_h\,,$$

where $\upsilon^0_\alpha \equiv i_{\partial_\alpha}\upsilon^0$. $\square$

2.4 Quantisable functions

In view of quantum operators, we present a Lie algebra of functions which depends only on the classical spacetime structure. The bracket of this algebra is not the Poisson bracket. Moreover, these functions are characterised by the fact that their Hamiltonian lift is projectable over spacetime; so we get a tangent lift for these functions.

Let us consider a time scale $\tau \in \mathbb{T} \otimes I\!R$ and the vector subbundles over $J\boldsymbol{E}$

$$T_\tau J\boldsymbol{E} \subset TJ\boldsymbol{E} \qquad \text{and} \qquad T^*_\gamma J\boldsymbol{E} \subset T^*J\boldsymbol{E}\,,$$

which project on τ and vanish on γ, respectively.

The cosymplectic form Ω yields a linear fibred isomorphism over $J\boldsymbol{E}$

$$\Omega^\flat{}_\tau : T_\tau J\boldsymbol{E} \to T^*_\gamma J\boldsymbol{E}\,,$$

whose inverse will be denoted by $\Omega^\sharp{}_\tau$.

Now, for any function $f \in \mathcal{F}(J\boldsymbol{E})$, we obtain the 1–form

$$d_\gamma f := df - \gamma \lrcorner df : J\boldsymbol{E} \to T^*_\gamma J\boldsymbol{E}\,, \tag{2.25}$$

and, given a time scale τ, the vector field

$$H_\tau[f] \equiv \Omega^\sharp{}_\tau(d_\gamma f) : J\boldsymbol{E} \to T_\tau J\boldsymbol{E}\,, \tag{2.26}$$

with coordinate expression

$$H_\tau[f] = \tau^0 \, (\partial_0 + x^h_0 \, \partial_h + \gamma_0{}^h_0 \, \partial^0_h) + \\ + G^{hk} \Big(-\partial^0_k f \, \partial_h + \Big(\partial_k f + (\Gamma_k{}^l_0 - G_{kr} G^{ls} \, \Gamma_s{}^r_0) \, \partial^0_l f \Big) \, \partial^0_h \Big) \,,$$

which is said to be the τ–*Hamiltonian lift* of f.

The sheaf $\mathcal{F}(J\boldsymbol{E})$ turns out to be a sheaf of $I\!R$–Lie algebras with respect to the generalised Poisson bracket

$$\{f_1, f_2\} := i(H_\tau[f_2]) \, i(H_\tau[f_1]) \, \Omega \,. \tag{2.27}$$

The Hamiltonian lifts are not yet sufficient for our purposes (in view of next developments) because they live on $J\boldsymbol{E}$, while we are looking for vector fields on $\boldsymbol{E}$. Moreover, the construction of these vector fields depends on an arbitrary time scale, while we would like to get a natural construction. Both problems can be solved by means of a projectability method, according to the following result [31, 33].

2.14. Theorem. *Let* $\tau \in \mathbb{T} \otimes I\!R$ *and* $f \in \mathcal{F}(J\boldsymbol{E})$*. Then, the vector field* $H_\tau[f]$ *is projectable on* $\boldsymbol{E}$ *if and only if* f *is, with respect to the fibres of* $J\boldsymbol{E} \to \boldsymbol{E}$*, a polynomial of degree 2, whose second derivative is of the type*

$$D^2 f = f'' \otimes G : \boldsymbol{E} \to \mathbb{T}^2 \otimes (V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E}) \,, \qquad \text{with} \quad f'' \in \mathbb{T} \otimes I\!R \,,$$

and

$$\tau = f'' \,. \square$$

2.15. Definition. Functions of the above kind are said to be *quantisable*. □

2.16. Remark. A function $f \in \mathcal{F}(J\boldsymbol{E})$ is quantisable if and only if its coordinate expression is of the type

$$f = \tfrac{1}{2} \, f^0 \, G_{hk} \, x^h_0 x^k_0 + f_j \, x^j_0 + f_0 \,, \qquad f_j, f_0 \in \mathcal{F}(\boldsymbol{E}) \,, \quad f^0 \in I\!R \,. \tag{2.28}$$

For a quantisable function f as above we obtain

$$f'' = f^0 \, u_0 \,.$$

Let us consider an observer o. A function $f \in \mathcal{F}(J\boldsymbol{E})$ is quantisable if and only if it is of the type

$$f = f'' \, \mathcal{K}[o] + f' \circ \nabla[o] + f_o \,,$$

where $f'' \in \mathbb{T} \otimes I\!R$, $f' \in \mathcal{F}(\mathbb{T} \otimes V^*\boldsymbol{E})$ is a function linear with respect to the fibres and $f_o \in \mathcal{F}(\boldsymbol{E})$. For a quantisable function f as above we obtain

$$f_o = f \circ o. \square$$

We stress that in the coordinate expression of a quantisable function f there is no relation at all between the coefficients f^0 and f_0.

The sheaf of quantisable functions is denoted by

$$\mathcal{Q}(J\boldsymbol{E}) \subset \mathcal{F}(J\boldsymbol{E}) .$$

2.17. Theorem. *The sheaf $\mathcal{Q}(J\boldsymbol{E})$ turns out to be a sheaf of $I\!R$–Lie algebras with respect to the bracket*

$$[f_1, f_2] := \{f_1, f_2\} + \gamma(f_1{}'').f_2 - \gamma(f_2{}'').f_1 \,, \tag{2.29}$$

with coordinate expression

$$\begin{aligned}
[f_1, f_2]^0 &= 0 \\
[f_1, f_2]_i &= G_{ij}\,(f_1{}^0\,\partial_0 f_2{}^j - f_2{}^0\,\partial_0 f_1{}^j - f_1{}^h\,\partial_h f_2{}^j + f_2{}^h\,\partial_h f_1{}^j) \\
[f_1, f_2]_0 &= f_1{}^0\,\partial_0 f_{2\,0} - f_2{}^0\,\partial_0 f_{1\,0} - (f_1{}^h\,\partial_h f_{2\,0} - f_2{}^h\,\partial_h f_{1\,0}) \\
&\quad - (f_1{}^0\, f_2{}^j - f_2{}^0\, f_1{}^j)\,\Phi_{0j} + f_1{}^i\, f_2{}^j\,\Phi_{ij} \,,
\end{aligned}$$

where

$$f_1{}^i \;\equiv\; G^{ij}\, f_{1\,j} \,, \qquad f_2{}^i \;\equiv\; G^{ij}\, f_{2\,j} \,.\square$$

2.18. Remark. Let $\mathcal{Q}_a(J\boldsymbol{E}) \subset \mathcal{Q}(J\boldsymbol{E})$ be the subsheaf of affine quantisable functions (i.e. quantisable functions f with $f'' = 0$). We can easily see that $\mathcal{Q}_a(J\boldsymbol{E})$ is closed with respect to the Lie bracket. $\square$

Thus, for any quantisable function $f \in \mathcal{Q}(J\boldsymbol{E})$, the vector field

$$X^{\uparrow}[f] := H_{f''}[f]$$

projects on a vector field, which will be denoted by

$$X[f] : \boldsymbol{E} \to T\boldsymbol{E} \,,$$

and said to be the *tangent lift* of f.

2.19. Remark. Let us consider a quantisable function f.

Then, its tangent lift is expressed, in coordinates, by

$$X[f] = f^0\,\partial_0 - f^i\,\partial_i \,. \tag{2.30}$$

and, with respect to an observer o, by

$$X[f] = o(f'') - G^{\sharp}(f' \circ \nabla[o]) \,.\square$$

2.20. Remark. Let us consider a quantisable function f.

Then, the time component of $X[f]$ is just f''.

Moreover, if the time scale $f'' \equiv \pm u_0 \in \mathbb{T} \otimes I\!R$ is non vanishing, then we obtain the map

$$\pm u^0 \otimes X[f] : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$$

which is projectable on $1 \in \mathbb{T}^* \otimes \mathbb{T}$, hence can be regarded as an observer. □

2.21. Theorem. *The map*

$$\mathcal{Q}(J\boldsymbol{E}) \to \mathcal{T}(\boldsymbol{E}) : f \mapsto X[f]$$

turns out to be an epimorphism of sheaves of $I\!R$*–Lie algebras; hence, we have*

$$X\Big[[f_1,\, f_2]\Big] = \Big[X[f_1],\, X[f_2]\Big]\,. \square \tag{2.31}$$

2.22. Example. Let us consider an observer o and a time scale u_0 and let us refer to a chart adapted to the observer and to the time scale. Then, the functions x^0, x^i, x^i_0, $\mathcal{P}_i$, $\mathcal{K}_0$, $\mathcal{H}_0$, $\mathcal{L}_0$ turn out to be quantisable. Moreover, we obtain the following coordinate expressions

$$X[x^\alpha] = 0\,, \qquad X[x^i_0] = -\,G^{ij}\,\partial_j\,, \qquad X[\mathcal{P}_i] = -\partial_i\,, \tag{2.32}$$

$$X[\mathcal{K}_0] = X[\mathcal{H}_0] = \partial_0\,, \qquad X[\mathcal{L}_0] = \partial_0 - A^i\,\partial_i\,. \tag{2.33}$$

We notice that the observer associated with $\mathcal{K}_0$ turns out to be o itself. On the other hand, the observer associated with $\mathcal{L}_0$ moves with respect to o with velocity

$$-G^\sharp(\check{A}) : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}\,.$$

Moreover, for each $f \in \mathcal{Q}(J\boldsymbol{E})$, we obtain

$$[x^0, f] = f^0\,, \qquad [x^i, f] = f^i$$

$$[x^i, \mathcal{P}_j] = \delta^i_j\,, \qquad [\mathcal{P}_i, \mathcal{P}_j] = 0\,, \qquad [\mathcal{H}_0, \mathcal{P}_j] = \Phi_{0j}\,.$$

We observe that all constructions of this section could be extended by considering variable time scales of the type $\tau : J\boldsymbol{E} \to \mathbb{T} \otimes I\!R$ (see [33]).

2.5 Quantum vector fields

Next, we classify the vector fields on the pullback quantum bundle, which are compatible with the quantum structure, and show that their projections on the quantum bundle constitute a Lie algebra of vectors fields, which is isomorphic to the Lie algebra of quantisable functions. These projected vector fields will yield pre–quantum operators later.

2.23. Lemma. Let us consider a vector field $Y : \boldsymbol{E} \to T\boldsymbol{E}$ whose time component is constant and a vertical form $\alpha : \boldsymbol{E} \to \Lambda^r V^* \boldsymbol{E}$. Then, we can easily see that the vertical restriction of the Lie derivative $L[Y]\tilde{\alpha}$, where $\tilde{\alpha} : \boldsymbol{E} \to \Lambda^r T^* \boldsymbol{E}$ is an extension of α, does not depend on the choice of the extension $\tilde{\alpha}$. Hence, the above procedure well defines a vertical form

$$L[Y]\alpha : \boldsymbol{E} \to \Lambda^r V^* \boldsymbol{E} \,.\,\square$$

Let $Y : \boldsymbol{Q} \to T\boldsymbol{Q}$ be a vector field which projects on vector fields $X : \boldsymbol{E} \to T\boldsymbol{E}$, $\underline{X} : \boldsymbol{T} \to T\boldsymbol{T}$ and which is (real) linear over X. Then, we say that Y is Hermitian if

$$L[Y]\big(h(\Psi, \Psi')\big) = h(L[Y]\Psi,\, \Psi') + h(\Psi,\, L[Y]\Psi')\,, \qquad \forall \Psi, \Psi' \in \mathcal{S}(\boldsymbol{Q})\,.$$

A vector field Y is Hermitian if and only if its coordinate expression is of the type

$$Y = X^\lambda \, \partial_\lambda + (\mathfrak{i}\, f - \tfrac{1}{2}\operatorname{div} X)\, z\, \partial z\,, \qquad X^0 \in \mathcal{F}(\boldsymbol{T})\,,\ f, X^i \in \mathcal{F}(\boldsymbol{E})\,. \tag{2.34}$$

Let $Y^\uparrow : \boldsymbol{Q}^\uparrow \to T\boldsymbol{Q}^\uparrow$ be a vector field which projects on vector fields $X^\uparrow : J\boldsymbol{E} \to TJ\boldsymbol{E}$, $\underline{X}^\uparrow : \boldsymbol{T} \to T\boldsymbol{T}$ and which is (real) linear over $X^\uparrow$. Then, we say that $Y^\uparrow$ is Hermitian if

$$L[Y^\uparrow]\big(h(\Psi, \Psi')\big) = h(L[Y^\uparrow]\Psi,\, \Psi') + h(\Psi,\, L[Y^\uparrow]\Psi')\,, \qquad \forall \Psi, \Psi' \in \mathcal{S}(\boldsymbol{Q})\,.$$

A vector field $Y^\uparrow$ is Hermitian if and only if it is of the type

$$Y^\uparrow = \text{ч}(X^\uparrow) + (\mathfrak{i}\, f - \tfrac{1}{2}\operatorname{div} X^\uparrow)\,\mathbf{1}\,, \tag{2.35}$$

where $f \in \mathcal{F}(J\boldsymbol{E})$ and $X^\uparrow \in \mathcal{T}(J\boldsymbol{E})$ is projectable on $\underline{X} \in \mathcal{T}(T)$.

The Hermitian vector fields of $\boldsymbol{Q}$ and of $\boldsymbol{Q}^\uparrow$ constitute Lie algebras.

Now, let us consider the sheaf $\mathcal{Q}(\boldsymbol{Q}^\uparrow)$ of Hermitian vector fields

$$Y^\uparrow : \boldsymbol{Q}^\uparrow \to T\boldsymbol{Q}^\uparrow\,,$$

which project on vector fields $Y \in \mathcal{T}(\boldsymbol{Q})$ and whose time component is constant.

Then, modifying some results of [31, 33], we obtain the following theorem.

2.24. Theorem. *We have the natural $I\!R$–linear sheaf isomorphism*

$$\mathcal{Q}(J\boldsymbol{E}) \to \mathcal{Q}(\boldsymbol{Q}^\uparrow) : f \mapsto Y^\uparrow[f]\,,$$

given by

$$Y^\uparrow[f] = \text{ч}(X^\uparrow[f]) + (\mathfrak{i}\, f - \tfrac{1}{2}\operatorname{div} X[f])\,\mathbf{1}\,.\,\square \tag{2.36}$$

We have been forced to consider vector fields on the pullback quantum bundle $\boldsymbol{Q}^\uparrow$ because we wanted to relate them to the quantum connection ч and this lives on the bundle $\boldsymbol{Q}^\uparrow$. However, the above vector fields are not yet good candidates as operators on quantum sections because they would transform sections of $\boldsymbol{Q}$ into sections of $\boldsymbol{Q}^{\uparrow}$, but their projections on $\boldsymbol{Q}$ are suitable for our purpose.

Thus, for each quantisable function $f \in \mathcal{Q}(J\boldsymbol{E})$, the vector field $Y^{\uparrow}[f]$ projects on a (local) vector field, which will be denoted by

$$Y[f] : \boldsymbol{Q} \to T\boldsymbol{Q} ,$$

said to be the *quantum lift* of f and called a *quantum vector field*. We denote the sheaf of quantum vector fields by

$$\mathcal{Q}(\boldsymbol{Q}) \subset \mathcal{T}(\boldsymbol{Q}) .$$

2.25. Remark. If $f \in \mathcal{Q}(J\boldsymbol{E})$, then $Y[f]$ turns out to be projectable on $X[f]$. □

2.26. Proposition. Let us consider a quantisable function f, then its quantum lift is expressed, in coordinates, by

$$Y[f] = f^0 \, \partial_0 - f^j \, \partial_j + \Big(\mathfrak{i} \; (f^0 \, A_0 - f^h \, A_h + f_0) - \tfrac{1}{2} \operatorname{div} X[f] \Big) \, z \, \partial z \, , \tag{2.37}$$

where

$$\operatorname{div} X[f] = \frac{\partial_0 (f^0 \sqrt{|g|})}{\sqrt{|g|}} - \frac{\partial_j (f^j \sqrt{|g|})}{\sqrt{|g|}} \, ,$$

and, with respect to any observer o, by

$$Y[f] = (o^* \text{ч})(X[f]) + \Big(\mathfrak{i} \; (f \circ o) - \tfrac{1}{2} \operatorname{div} X[f] \Big) \, \boldsymbol{1} \, .$$

2.27. Theorem. *The sheaf $\mathcal{Q}(\boldsymbol{Q})$ is closed with respect to the Lie bracket. Moreover, the map*

$$\mathcal{Q}(J\boldsymbol{E}) \to \mathcal{Q}(\boldsymbol{Q}) : f \mapsto Y[f]$$

turns out to be an isomorphism of sheaves of $\mathbb{R}$–Lie algebras; hence, we have

$$Y\Big[[f_1, \, f_2]\Big] = \Big[Y[f_1], \, Y[f_2]\Big] \, . \; \square \tag{2.38}$$

2.6 Pre–quantum operators

Next, we let the above vector fields associated with quantisable functions act on the quantum sections as Lie derivatives.

Let us recall Lemma 2.10 and state the following additional result.

2.28. Lemma. Let $\boldsymbol{F} \to \boldsymbol{B}$ be a vector bundle equipped with a linear connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes_{\boldsymbol{B}} T\boldsymbol{F}$, a vector field $X : \boldsymbol{B} \to T\boldsymbol{B}$ and its horizontal prolongation

$Y \equiv X \lrcorner c : \boldsymbol{F} \to T\boldsymbol{F}$. Then, for each section $s : \boldsymbol{B} \to \boldsymbol{F}$, we obtain

$$Y.s = \nabla[c]_X s \,. \square$$

2.29. Definition. Let f be a quantisable function. Then, we define the *pre–quantum operator* associated with f to be the sheaf morphism

$$Z[f] : \mathcal{S}(\boldsymbol{Q}) \to \mathcal{S}(\boldsymbol{Q})$$

given by

$$Z[f](\Psi) := i\, Y[f].\Psi \,. \square \tag{2.39}$$

Here, the imaginary coefficient i has been inserted just to obtain symmetric quantum operators, later.

2.30. Proposition. Let f be a quantisable function. Then, the pre–quantum operator is expressed, in coordinates, by

$$\begin{aligned} Z[f](\Psi) = i\Big(f^0\, \nabla[o^*ч]_0 \psi - f^j\, \nabla[o^*ч]_j \psi - i\, f_0\, \psi \\ + \tfrac{1}{2}\big(\frac{\partial_0(f^0\sqrt{|g|})}{\sqrt{|g|}} - \frac{\partial_j(f^j\sqrt{|g|})}{\sqrt{|g|}}\big)\,\psi\Big) \otimes b \,. \end{aligned} \tag{2.40}$$

and, for each observer o, by

$$Z[f](\Psi) = \mathfrak{i}\,\Big(X[f] \lrcorner \nabla[o^*ч]\Psi\Big) + \Big(f \circ o + \mathfrak{i}\,\tfrac{1}{2}\ \operatorname{div} X[f]\Big)\Psi \,. \square$$

We denote the sheaf of pre–quantum operators by

$$\mathcal{O}(\boldsymbol{Q}) := \{Z[f] \mid f \in \mathcal{Q}(J\boldsymbol{E})\} \,.$$

The sheaf $\mathcal{O}(\boldsymbol{Q})$ becomes a sheaf of $I\!R$–Lie algebras with respect to the bracket

$$[Z[f_1],\, Z[f_2]] := -i\,(Z[f_1] \circ Z[f_2] - Z[f_2] \circ Z[f_1]) \,.$$

2.31. Theorem. *The map*

$$\mathcal{Q}(J\boldsymbol{E}) \to \mathcal{O}(\boldsymbol{Q}) : f \mapsto Z[f]$$

is an isomorphism of sheaves of $I\!R$–Lie algebras; hence, we have

$$Z\Big[[f_1,\, f_2]\Big] = \Big[Z[f_1],\, Z[f_2]\Big] \,. \square \tag{2.41}$$

2.32. Example. Let us consider an observer o and a time scale u_0 and let us refer to a chart adapted to the observer and to the time scale. Then, the pre–quantum operators associated with the quantisable functions $x^\alpha, x^i_0, \mathcal{P}_i, \mathcal{H}_0$ are given, for each $\Psi \in \mathcal{S}(\boldsymbol{Q})$, by

$$Z[x^\alpha](\Psi) = x^\alpha \, \Psi \,, \tag{2.42}$$

$$Z[x^j_0](\Psi) = -\, i \, (\partial_j - \mathfrak{i} \; A_j) \psi \, b \,, \qquad Z[\mathcal{P}_j](\Psi) = -\, \mathfrak{i} \, \Big(\partial_j + \tfrac{1}{2} \, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}} \Big) \psi \, b \,, \tag{2.43}$$

$$Z[\mathcal{H}_0](\Psi) = \mathfrak{i} \, \Big(\partial_0 + \tfrac{1}{2} \, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} \Big) \psi \, b \,. \tag{2.44}$$

2.7 F–smooth systems of sections

In view of the quantum Hilbert bundle and the quantum operators, we introduce the concept of system of sections of a double fibred manifold by using the non standard concept of F–smoothness due to Frölicher. This setting enables us to make several geometrical constructions on an infinite dimensional bundle whose elements are smooth sections of a finite dimensional bundle, skipping hard topological methods. In particular, we analyse the connections on this infinite dimensional bundle.

First of all, we recall the notion of *smooth space* originally introduced by A. Frölicher [22] and then studied by other authors along lines which are useful for our purposes ([9, 33]).

An *F–smooth space* is defined to be a set $\boldsymbol{S}$, along with a set $C \equiv \{c : \boldsymbol{I}_c \to \boldsymbol{S}\}$ of curves (where $\boldsymbol{I}_c \subset I\!\!R$ is an open interval which depends on c), which will be called *F–smooth*, and which fulfill the following conditions:

- each constant curve $c : \boldsymbol{I}_c \to \boldsymbol{S}$ belongs to C;

- if $c : \boldsymbol{I}_c \to \boldsymbol{S}$ is F–smooth and $\gamma : \boldsymbol{I}_\gamma \to \boldsymbol{I}_c$ is a smooth map, then $c \circ \gamma : \boldsymbol{I}_\gamma \to \boldsymbol{S}$ is F–smooth.

If $\boldsymbol{S}$ and $\boldsymbol{S}'$ are F–smooth spaces, then a map $f : \boldsymbol{S} \to \boldsymbol{S}'$ is said to be *F–smooth* if, for each F–smooth curve $c : \boldsymbol{I}_c \to \boldsymbol{S}$, the curve $c' \equiv f \circ c : \boldsymbol{I}_c \to \boldsymbol{S}'$ is F–smooth.

In particular, each smooth manifold $\boldsymbol{M}$ turns out to be F–smooth by assuming as F-smooth curves just the smooth curves. Moreover, a map between smooth manifolds is smooth if and only if it is F–smooth.

If $\boldsymbol{S}$ and $\boldsymbol{S}'$ are F-smooth spaces, then $\boldsymbol{S} \times \boldsymbol{n} S'$ turns out to be F–smooth in a natural way.

Now, let us consider a double smooth fibred manifold

$$\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B} \,.$$

A typical fibred chart of the double fibred manifold will be denoted by (x^λ, y^i, z^a).

2.33. Definition. We define a *system* associated with $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$ to be a pair

$$(\sigma, \, \epsilon) \equiv \Big(\sigma : \boldsymbol{S} \to \boldsymbol{B}, \; \epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G} \Big) \,,$$

where $\boldsymbol{S}$ is an F-smooth set, σ is a surjective map, ϵ is a fibred morphism over $\boldsymbol{F}$, which is injective with respect to $\boldsymbol{S}$, and the F-smooth structure of $\boldsymbol{S}$ is given by the F-smooth curves $\hat{c} : \boldsymbol{I}_{\hat{c}} \to \boldsymbol{S}$, such that the induced maps

$$c \equiv \sigma \circ \hat{c} : \boldsymbol{I}_{\hat{c}} \to \boldsymbol{B} : \lambda \mapsto \sigma(\hat{c}(\lambda))$$
$$c^*(\epsilon) : c^*(\boldsymbol{F}) \to \boldsymbol{G} : f_\lambda \mapsto \epsilon\Big(\hat{c}(\lambda),\, f_\lambda\Big)$$

are smooth. □

So, σ and ϵ turn out to be F–smooth.

2.34. Remark. Let (σ, ϵ) be a system and $\hat{\mathcal{S}}$ the sheaf of local F–smooth sections $\hat{s} : \boldsymbol{B} \to \boldsymbol{S}$.

Then, we obtain a sheaf isomorphism

$$\tilde{\epsilon} : \hat{\mathcal{S}} \to \mathcal{S} : \hat{s} \mapsto s \equiv \epsilon \circ \hat{s}^{\uparrow} ,$$

where $\hat{s}^{\uparrow} : \boldsymbol{F} \to \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F}$ is the pullback of $\hat{s}$, onto a sheaf $\mathcal{S}$ of local smooth sections $s : \boldsymbol{F} \to \boldsymbol{G}$, which are defined on tube–like open subsets of $\boldsymbol{F}$ (i.e. , on preimages of open subsets of $\boldsymbol{B}$). □

We stress that, in the above remark and from now on, the notion of sheaf of local smooth sections $s : \boldsymbol{F} \to \boldsymbol{G}$, which are defined on tube–like open subsets of $\boldsymbol{F}$ is referred to the tube–like topology of $\boldsymbol{F}$.

Conversely, let us consider a subsheaf

$$\mathcal{S} \subset \{s : \boldsymbol{F} \to \boldsymbol{G}\}$$

of the sheaf of local smooth sections $s : \boldsymbol{F} \to \boldsymbol{G}$, which are defined on tube–like open subsets of $\boldsymbol{F}$.

2.35. Proposition. The sheaf $\mathcal{S}$ defines a system as follows:

i) $\boldsymbol{S}$ is the set

$$\boldsymbol{S} \equiv \bigcup_{b \in \boldsymbol{B}} \boldsymbol{S}_b ,$$

where, for each $b \in \boldsymbol{B}$,

$$\boldsymbol{S}_b \equiv \{s_b : \boldsymbol{F}_b \to \boldsymbol{G}_b \mid s \in \mathcal{S}\}$$

is the set consisting of the restrictions to $\boldsymbol{F}_b$ of the sections belonging to $\mathcal{S}$;

ii) σ is the surjective map

$$\sigma : \boldsymbol{S} \to \boldsymbol{B} : s_b \mapsto b ;$$

iii) ϵ is the evaluation fibred morphism over $\boldsymbol{F}$

$$\epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G} : (s_b, f) \mapsto s_b(f) . \square$$

2.36. Remark. The correspondence

$$(\sigma,\, \epsilon) \mapsto \mathcal{S}$$

between systems and sheaves is bijective.

Moreover, we have the mutually inverse sheaf isomorphisms

$$\tilde{\epsilon} : \hat{\mathcal{S}} \to \mathcal{S} : \hat{s} \mapsto s \,\equiv\, \epsilon \circ s^{\uparrow} \,, \qquad \tilde{\epsilon}^{-1} : \mathcal{S} \to \hat{\mathcal{S}} : s \mapsto \hat{s} \,,$$

where $s^{\uparrow} : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G}$ is the pullback of s and $\hat{s} : b \mapsto s_b \,\equiv\, s|\boldsymbol{F}_b$. $\square$

So, now we consider a system $(\sigma,\, \epsilon)$ and make some geometrical constructions related to it.

Let us make a preliminarly remark. If $b \in \boldsymbol{B}$, $u \in T_b\boldsymbol{B}$, $f \in \boldsymbol{F}_b$, $s \in \mathcal{S}$, then $(T_f\boldsymbol{F})_u$ and $(T_{s(f)}\boldsymbol{G})_u$ are affine spaces associated with the vector spaces $V_f\boldsymbol{F}$ and $V_{s(f)}\boldsymbol{G}$, respectively.

2.37. Remark. We define the *tangent space* of the F–smooth set $\boldsymbol{S}$ as the set [33]

$$T\boldsymbol{S} := \bigcup_{\hat{s}_b \in \boldsymbol{S}} T_{\hat{s}_b}\boldsymbol{S} \,,$$

where, for each $\hat{s}_b \in \boldsymbol{S}_b \subset \boldsymbol{S}$,

$$T_{\hat{s}_b}\boldsymbol{S} \,\equiv\, \{\zeta_u\}$$

is the set consisting of smooth sections

$$\zeta_u : (T\boldsymbol{F})_u \to (T\boldsymbol{G})_u \,, \qquad \text{with} \quad u \in T_b\boldsymbol{B} \,,\, b \,\equiv\, \sigma(\hat{s}_b) \in \boldsymbol{B} \,,$$

such that, for each $f \in \boldsymbol{F}_b$, the restriction of ζ_u to $(T_f\boldsymbol{F})_u$ is an affine map

$$\zeta_{(u,f)} : (T_f\boldsymbol{F})_u \to (T_{s(f)}\boldsymbol{G})_u \,,$$

whose derivative is

$$D(\zeta_{(u,f)}) = V_f s : V_f\boldsymbol{F} \to V_{s(f)}\boldsymbol{G} \,.$$

In other words, ζ_u is a map which fulfills the equality

$$\zeta_{(u,f)}(v + w) \mapsto \zeta_{(u,f)}(v) + V_f s(w) \,, \qquad \forall v \in (T_f\boldsymbol{F})_u \,, \forall w \in (V_f\boldsymbol{F})_b \,, \forall f \in \boldsymbol{F}_b \,;$$

i.e. , ζ_u is a map whose coordinate expression is of the type

$$(x^\lambda, y^i, z^a; \dot{x}^\lambda, \dot{y}^i, \dot{z}^a) \circ \zeta_u = (x^\lambda, y^i, s^a; u^\lambda, \dot{y}^i, \zeta^a + \partial_i s^a\, \dot{y}^i) \,, \qquad \text{with} \qquad \zeta^a \in \mathcal{F}(\boldsymbol{F}) \,.$$

We obtain the surjective maps

$$\pi_{\boldsymbol{S}} : T\boldsymbol{S} \to \boldsymbol{S} : \zeta_u \mapsto \hat{s}_b \,, \qquad T\sigma : T\boldsymbol{S} \to T\boldsymbol{B} : \zeta_u \mapsto u \,,$$

and the evaluation fibred morphism over $T\boldsymbol{F}$

$$T\epsilon : T\boldsymbol{S} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F} \to T\boldsymbol{G} : (\zeta_u, v) \mapsto \zeta_u(v)\,,$$

which is injective with respect to $T\boldsymbol{S}$.

The fibres of $\pi_{\boldsymbol{S}} : T\boldsymbol{S} \to \boldsymbol{S}$ are naturally equipped with a vector structure; moreover, $T\sigma$ turns out to be a linear fibred morphism over σ.

The space $T\boldsymbol{S}$ has a natural F–smooth structure, which makes the maps $T\sigma, T\epsilon$, $\pi_{\boldsymbol{S}}$ and the vector structure of the fibres F–smooth.

Thus, $(T\sigma : T\boldsymbol{S} \to T\boldsymbol{B}$, $T\epsilon)$ is a system of the double fibred manifold $T\boldsymbol{G} \to T\boldsymbol{F} \to T\boldsymbol{B}$. Hence, we can apply to this system results analogous to those obtained for the original system. We denote by $T\hat{\mathcal{S}}$ and $T\mathcal{S}$ the sheaves associated with $T\boldsymbol{S}$ and $T\boldsymbol{F}$, respectively. □

2.38. Remark. Let $\hat{s} \in \hat{\mathcal{S}}$ be an F–smooth section. Then we can easily see that $Ts \equiv T(\tilde{\epsilon}(s)) \in T\mathcal{S}$. Then, we define the *tangent prolongation* of $\hat{s}$ to be the F–smooth section

$$\widehat{Ts} \equiv (\widetilde{T\epsilon})^{-1}(Ts) : T\boldsymbol{B} \to T\boldsymbol{S}\,. \square$$

An F-smooth *connection* on the fibred set $\boldsymbol{S} \to \boldsymbol{B}$ is defined to be an F–smooth section

$$\chi : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T\boldsymbol{S}\,,$$

which is linear over $\boldsymbol{S}$ and projectable over $\mathrm{id}_{T\boldsymbol{B}} : T\boldsymbol{B} \to T\boldsymbol{B}$.

A connection can be regarded as an operator on sections of the double fibred manifold in the following way.

2.39. Remark. Let χ be a connection.

Then, χ yields a first sheaf morphism

$$\hat{\chi}. : \hat{\mathcal{S}} \to T\hat{\mathcal{S}} : \hat{s} \mapsto \hat{\chi}.\hat{s}\,,$$

where the section $\hat{\chi}.\hat{s} : T\boldsymbol{F} \to T\boldsymbol{G}$ is given by the composition

$$T\boldsymbol{B} \xrightarrow{\hat{s}^{\uparrow}} \boldsymbol{S} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \xrightarrow{\chi} T\boldsymbol{S}\,,$$

where $\hat{s}^{\uparrow}$ is the pullback of $\hat{s}$ given by

$$\hat{s}^{\uparrow} : T\boldsymbol{B} \to \boldsymbol{S} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} : u_b \mapsto \Big(s(b),\, u_b\Big)\,.$$

Moreover, χ yields a second sheaf morphism

$$\chi. : \mathcal{S} \to T\mathcal{S} : s \mapsto \chi.s\,,$$

given by the composition

$$T\boldsymbol{F} \xrightarrow{\hat{s}^{\uparrow}} \boldsymbol{S} \underset{\boldsymbol{B}}{\times} T\boldsymbol{F} \xrightarrow{\chi^{\uparrow}} T\boldsymbol{S} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F} \xrightarrow{T\tilde{\epsilon}} T\boldsymbol{G},$$

where $\hat{s}^{\uparrow}$ and $\chi^{\uparrow}$ are the pullbacks of $\hat{s}$ and χ given by

$$\hat{s}^{\uparrow} : T\boldsymbol{F} \to \boldsymbol{S} \underset{\boldsymbol{B}}{\times} T\boldsymbol{F} : f_b \mapsto \big(s(b),\, f_b\big),$$
$$\chi^{\uparrow} : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} T\boldsymbol{F} \to T\boldsymbol{S} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F} : (a_b, v_u) \mapsto \big(\chi(a_b, u),\, v_u\big).$$

Actually, for each $s \in \mathcal{S}$, we have

$$\widehat{\chi.s} = \hat{\chi}.\hat{s}.$$

The coordinate expression of $\chi.s$ is of the type

$$(x^{\lambda}, y^i, z^a; \dot{x}^{\lambda}, \dot{y}^i, \dot{z}^a) \circ (\chi.s) = (x^{\lambda}, y^i, s^a; \dot{x}^{\lambda}, \dot{y}^i,\, \chi^a_{\mu}(s)\, \dot{x}^{\mu} + \partial_j s^a\, \dot{y}^j),$$

where $\chi^a_{\mu}(s) \in \mathcal{F}(\boldsymbol{F})$. □

A connection induces the covariant differential in the standard way.

2.40. Proposition. Let χ be a connection and $\hat{s} \in \hat{\mathcal{S}}$.

Then, the map

$$\hat{\nabla}[\chi]\hat{s} := T\hat{s} - \hat{\chi}.\hat{s} = \widehat{Ts - \chi.s} : T\boldsymbol{B} \to T\boldsymbol{S}$$

takes its values in $V\boldsymbol{S}$, is projectable over $\hat{s}$ and is a linear fibred morphism over $\hat{s}$.

Moreover, the map

$$\nabla[\chi]s := Ts - \chi.s : T\boldsymbol{F} \to T\boldsymbol{G}$$

takes its values in $V\boldsymbol{G}$ and is a smooth linear local linear morphism over s, which factorises through a smooth linear local morphism over s

$$\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to V\boldsymbol{G}.$$

We have

$$\hat{\nabla}[\chi]\hat{s} = \widehat{\nabla[\chi]s}.$$

We have the coordinate expression

$$\nabla[\chi]s = \big(\partial_{\lambda} s^a - \chi^a_{\lambda}(s)\big) d^{\lambda} \otimes (\partial_a \circ s). \; \square$$

We call $\hat{\nabla}[\chi]\hat{s}$, or, equivalently, $\nabla[\chi]s$ the *covariant differential* of s with respect to χ.

A connection χ is said to be of *order* k if, for each $s \in \mathcal{S}$, the map $\chi(s)$ depends on s through its vertical k–jet.

Next, let us consider the case when $\boldsymbol{G} \to \boldsymbol{F}$ is a vector bundle, hence $\boldsymbol{S} \to \boldsymbol{B}$ and $T\boldsymbol{S} \to T\boldsymbol{B}$ are F–smooth vector bundles.

A connection χ is said to be *linear* if it is a linear fibred morphism over $\boldsymbol{B}$, i.e. if the sheaf morphism $\chi.$ is linear.

2.41. Proposition. Let χ be a linear connection of order k.

Then, the coordinate expression of $\chi.s$ is of the type

$$\chi^a_\lambda(s) = \chi^a_{\lambda b}\, s^b + \chi^a_{\lambda b}{}^j\, \partial_j s^b + \cdots + \chi^a_{\lambda b}{}^{j_1 \dots j_r}\, \partial_{j_1 \dots j_r} s^b\,,$$

where $\chi^a_{\lambda b}\,, \chi^a_{\lambda b}{}^j\,, \dots\,, \chi^a_{\lambda b}{}^{j_1 \dots j_r} \in \mathcal{F}(\boldsymbol{E})$.

Hence, the coordinate expression of $\nabla[\chi]s$ is

$$\nabla[\chi]s = (\partial_\lambda s^a - \chi^a_{\lambda b}\, s^b - \chi^a_{\lambda b}{}^j\, \partial_j s^b - \cdots - \chi^a_{\lambda b}{}^{j_1 \dots j_r}\, \partial_{j_1 \dots j_r} s^b) d^\lambda \partial_a\,.$$

Thus, the operator $\nabla[\chi]$ turns out to be a linear differential operator of order 1 in the base derivatives and of order k in the fibre derivatives of of s. □

2.8 Quantum Hilbert bundle

Eventually, we apply the geometrical constructions of the above sections to the quantum bundle and obtain an F–smooth infinite dimensional bundle of time, equipped with a pre–Hilbert structure. Moreover, we interpret the Schrödinger operator as a connection on this infinite dimensional bundle. Furthermore, we show how the pre–quantum operators and the above connection yield symmetric quantum operators.

The true Hilbert bundle could be obtained by a standard completion procedure and the above quantum operators would turn out to be self–adjoint if the concrete spacetime and the potential involved in the quantum connection are sufficiently good from a topological viewpoint.

So, we consider the double fibred manifold

$$\boldsymbol{Q} \to \boldsymbol{E} \to \boldsymbol{T}$$

and the sheaf $\mathcal{S}$ of sections $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$, which are defined on tube-like open subsets of $\boldsymbol{E}$ and have compact support on the fibres of $\boldsymbol{E} \to \boldsymbol{T}$.

Then, according to the previous section (see definition 2.33 and proposition 2.35), we obtain the *quantum system*

$$\Big(\sigma : \boldsymbol{S} \to \boldsymbol{T},\ \epsilon : \boldsymbol{S} \underset{\boldsymbol{T}}{\times} \boldsymbol{E} \to \boldsymbol{Q}\Big)\,.$$

The F–smooth complex vector bundle $\sigma : \boldsymbol{S} \to \boldsymbol{T}$ inherits the Hermitian structure on its fibres

$$\hat{h} : \boldsymbol{S} \underset{\boldsymbol{T}}{\times} \boldsymbol{S} \to \mathbb{C} : (\Psi_\tau, \Psi'_\tau) \mapsto \langle \Psi_\tau \,|\, \Psi'_\tau \rangle \equiv \int_{\boldsymbol{E}_\tau} h(\Psi_\tau, \Psi'_\tau) , \qquad \forall \tau \in \boldsymbol{T} , \tag{2.45}$$

which makes it a pre-Hilbert bundle.

2.42. Definition. We call $\sigma : \boldsymbol{S} \to \boldsymbol{T}$ the *quantum pre–Hilbert bundle*. □

We define the *Schrödinger operator* to be the sheaf morphism

$$S : \mathcal{S}(\boldsymbol{Q}) \to \mathcal{S}(\mathbb{T}^* \otimes \boldsymbol{Q}) : \Psi \mapsto - i \, \tfrac{1}{2} \, \mathsf{E}[\Psi]/\upsilon , \tag{2.46}$$

with coordinate expression

$$S.\Psi = (\overset{o}{\nabla}_0 + \tfrac{1}{2} \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \mathfrak{i} \, \tfrac{1}{2} \, \overset{o}{\Delta}_0) \, \psi \, b \otimes u^0 \, . \, \square$$

2.43. Theorem. *There is a unique linear F–smooth connection χ on the F–smooth bundle $\boldsymbol{S} \to \boldsymbol{T}$, such that, for each $\Psi \in \mathcal{S}(\boldsymbol{Q})$,*

$$\nabla[\chi] = S \, .$$

The coordinate expression of χ is given by

$$\chi_0(\Psi) =$$

$$= \Big(- \tfrac{1}{2} \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} + \mathfrak{i} \, \tfrac{1}{2} \, G^{hk} \Big((\partial_h - \mathfrak{i} \; A_h)(\partial_k - \mathfrak{i} \, A_k) + K_h{}^l{}_k \, (\partial_l - \mathfrak{i} \; A_l) \Big) + \mathfrak{i} \, A_0 \Big) \psi \, b \, . \, \square$$

The connection χ turns out to be Hermitian with respect to $\hat{h}$.

2.44. Remark. A fibred morphism over $\boldsymbol{T}$

$$\hat{\xi} : \boldsymbol{S} \to \boldsymbol{S}$$

can be regarded as a sheaf morphism

$$\hat{\xi}. : \hat{\mathcal{S}} \to \hat{\mathcal{S}} : \hat{\Psi} \mapsto \hat{\xi}.\hat{\Psi} \equiv \hat{\xi} \circ \hat{\Psi} \, .$$

Moreover, $\hat{\xi}$ yields the sheaf morphism

$$\xi : \mathcal{S} \to \mathcal{S}$$

characterised by

$$\widehat{\xi(\Psi)} = \hat{\xi}.\hat{\Psi}\,.$$

The map

$$\hat{\xi} \mapsto \xi$$

is bijective. □

Henceforth, $\mathcal{Q}(J\boldsymbol{E})$ will denote the sheaf of quantisable functions, which are defined on tube–like open subsets of $\boldsymbol{E}$.

Now we are in the position to exhibit the quantum operator on the pre–Hilbert bundle associated with every quantisable function, as the final result of a fully geometrical an covariant procedure.

Let $f \in \mathcal{Q}(J\boldsymbol{E})$. By considering the results of section 2.6 and theorem 2.43, we obtain two distinguished sheaf morphisms

$$Z[f] : \hat{\mathcal{S}} \to \hat{\mathcal{S}}\,,$$
$$f'' \lrcorner \nabla[\hat{\chi}] : \hat{\mathcal{S}} \to \hat{\mathcal{S}}\,,$$

which do not act pointwisely on the bundle $\boldsymbol{S} \to \boldsymbol{T}$ as they involve the time derivative of sections. But we are looking for a fibred endomorphism of the bundle $\boldsymbol{S} \to \boldsymbol{T}$. The following theorem yields such a distinguished map.

2.45. Theorem. *Let $f \in \mathcal{Q}(J\boldsymbol{E})$. Then, there is a unique linear combination (up to a multiplicative factor) of the above maps which acts pointwisely on the section of $\boldsymbol{S}$, namely the sheaf morphism*

$$\hat{f} := Z[f] - \mathfrak{i}\;(f'' \lrcorner \nabla[\hat{\chi}]) : \hat{\mathcal{S}} \to \hat{\mathcal{S}} \tag{2.47}$$

which can hence be regarded as a local fibred morphism

$$\hat{f} : \boldsymbol{S} \to \boldsymbol{S}\,. \tag{2.48}$$

We have the coordinate expression

$$\begin{aligned}\hat{f}(\Psi) = \Big((f_0 - \mathfrak{i}\, f^h(\partial_h - \mathfrak{i}\, A_h) - \mathfrak{i}\, \tfrac{1}{2}\, \frac{\partial_h(f^h\sqrt{|g|})}{\sqrt{|g|}} \\ - \tfrac{1}{2}\, f^0\, G^{hk}\, \big((\partial_h - \mathfrak{i}\, A_h)(\partial_k - \mathfrak{i}\; A_k) + K_h{}^l{}_k\,(\partial_l - \mathfrak{i}\, A_l)\big)\Big)\psi\, b\,. \end{aligned} \tag{2.49}$$ □

2.46. Definition. For each quantisable function f, we say that

$$\hat{f} : \boldsymbol{S} \to \boldsymbol{S} \tag{2.50}$$

is the associated *quantum operator*. □

2.47. Theorem. *For each $f \in \mathcal{Q}(J\boldsymbol{E})$, the operator $\hat{f}$ is symmetric with respect to the Hermitian metric $\hat{h}$.* □

We denote the sheaf of quantum operators by

$$\mathcal{Q}(\boldsymbol{S}) := \{\hat{f} \quad | \quad f \in \mathcal{Q}(J\boldsymbol{E})\}$$

and the subsheaf of quantum operators associated with affine quantisable functions by

$$\mathcal{Q}_a(\boldsymbol{S}) := \{\hat{f} \quad | \quad f \in \mathcal{Q}_a(J\boldsymbol{E})\} .$$

If $f_1, f_2 \in \mathcal{Q}(J\boldsymbol{E})$, then we define the bracket of the associated quantum operators as

$$[\widehat{f_1}, \widehat{f_2}] := -\mathfrak{i}\, (\widehat{f_1} \circ \widehat{f_2} - \widehat{f_2} \circ \widehat{f_1}) .$$

2.48. Theorem. *The sheaf $\mathcal{Q}_a(\boldsymbol{S})$ turns out to be an $I\!R$–Lie algebra.*
Moreover, the map

$$\mathcal{Q}_a(J\boldsymbol{E}) \to \mathcal{Q}_a(\boldsymbol{S})$$

is an $I\!R$–Lie algebra homomorphism. □

2.49. Example. Let us consider an observer o and a time scale u_0 and let us refer to a chart adapted to the observer and to the time scale. Then, the quantum operators associated with the quantisable functions $x^\alpha, x^i_0, \mathcal{P}_i, \mathcal{H}_0$ are given, for each $\Psi \in \mathcal{S}(\boldsymbol{Q})$, by

$$\widehat{x^\alpha}(\Psi) = x^\alpha\, \Psi \,, \tag{2.51}$$

$$\widehat{x^j_0}(\Psi) = -\,\mathfrak{i}\; (\partial_j - \mathfrak{i}\; A_j)\, \psi\, b \,, \qquad \widehat{\mathcal{P}}_j(\Psi) = -\,\mathfrak{i}\; \Big(\partial_j \psi + \tfrac{1}{2}\, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}} \psi \Big)\, b \,, \tag{2.52}$$

$$\widehat{\mathcal{H}_0}(\Psi) = -\Big(\tfrac{1}{2}\, G^{hk} \Big((\partial_h - \mathfrak{i}\; A_h)(\partial_k - \mathfrak{i}\; A_k) + K_h{}^l{}_k\, (\partial_l - \mathfrak{i}\; A_l) \Big) + A_0 \Big)\, \psi\, b \,. \;\square \tag{2.53}$$

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