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# QUANTUM LOGIC AND INDEFINITE METRIC SPACES

A. Z. Jadczyk

Institute of Theoretical Physics, University of Wrocław, Wrocław

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In the framework of general quantum-logic approach a notion of an object is introduced as a primary to that of a state. An attempt is made to generalize the notion of state. It is shown that with such a generalization, there is no troubles with "negative probabilities" in indefinite-metric spaces. Instead of this, there are new physical effects possible.

## 1. Introduction

The failure of all attempts to give a coherent explanation of the most fundamental facts of elementary particle physics brings some physicists to examine again and again the most profound basis of the quantum theory. Persistently new works appear on this subject. From time to time someone finds the new condition more "physical" or only different from the hitherto existing ones, leading to the necessity of the standard Hilbert space or $C^*$-algebra approach. It would seem that the justification of the conventional formalism is unquestionable to such an extent, that there is nothing more to do but to search for non-trivial models satisfying all the accepted axioms. Unfortunately, the nature seems to be mischievous rather than allied in all these attempts. On the other hand, none of the so far proposed non-standard formalisms has been widely accepted and not because of its non-standardness but because of the lack of proposals of a concrete and working mathematical apparatus. It is, however, well known that such a generalization of the standard formalism exists and — what is more essential — it works. We have in mind indefinite metric here. It is commonly accepted that indefinite metric offers new possibilities, removes a number of difficulties and gives concrete results (see e.g. [5]). But also, it causes the serious troubles as far as the interpretation is concerned (negative "probabilities", "non-physical" states, etc). It is for these interpretative troubles, presumably, that indefinite metric has not been taken into account in all attempts of axiomatic formulation of the quantum theory [1] (the only exception known to the author is a separate section in Wightman and Gårding [10]).

The purpose of this paper is to put forward a formalism, which embodies indefinite metric as a particular case. The generalization presented here is far from being perfect and

[1] There is a non-scientific reason besides, perhaps. It is that indefinite metric is not "crazy" enough.

complete. It is rather an outline of some ideas, which require further improvements, further investigations and a severe criticism, certainly. We put also some problems which, in our opinion, might be worthwhile to solve.

In this introducton we would like to call the reader's attention to some subtleties of the language we use. The reason why our language differs from a customary one is of philosophical nature and we wish to take the opportunity to make some general comments.

Since the appearance of the Special Theory of Relativity, physicist's minds are exposed to the influence of a philosophy called sometimes *operationism*. This attitude is still fashionable and there would be nothing wrong with that, were it for a weak form of operationism. In its restrained form operationism advises us to avoid concepts which do not correspond to any element of reality. It instructs us to remove notions we can do without. This form is nothing more than a common sense philosophy, of course. However, in its utmost form —and it is just this form which, in our opinion, causes more harm than profit—operationism claims that the physical entity *is*, in fact, the set of operations by which it is measured and also, that every theory should contain nothing but observable entities. Undoubtedly, operational concepts are the only solid foundation from which we can begin. Undoubtedly, operationism is an extremaly valuable aspect of analysis. But it can hardly be made a practicable method of science. The task of the physicists is to guess, on the basis of data they possess, what is nature really like. It is to guess what concepts and what entities are the most fundamental ones, no matter how far away from experience they are and no matter how difficult their measurements might be. It is unquestionable that a good theory should explain the known experimental data and foresee new ones. But that is the only point we have the right to demand. Our duty is to guess what is the structure of the reality, irrespectively whether and what we just measure, irrespectively whether and what we just observe.

Physics is a two-sided problem. One side is a structure of the reality, a theory. The second one is the connection of concepts and entities of the theory with experience (i.e. with a "classical", well-estabilished theory). These two sides of the problem must not be torn apart. But these are two sides and not just a one. It is the confusion of these both sides which is the main source of subjectivism in quantum theory.

Keeping these reasons in mind, we introduce from the very beginning a purely objective terminology and so, when considering quantum logic, we intentionally avoid such terms as "question" or "proposition". We prefer the term "property". But a property is always a property of something. So, we introduce the concept of an "object". It replaces the notion of the pure state in our framework. We introduce the concept of a "face" of an object, a "face" of the logic. It replaces the notion of a maximal set of commuting observables. We avoid the term "observable" and "observation" as long as possible. As a matter of fact, we should have excluded the word "logic" as well. We save it for two reasons. The first reason is a historical one. The second one, which is of the major importance, is that the term "logic" has lost by now its subjective shades and becomes a synonym of "an inner structure" (of a computer, for instance). We hope that this somewhat unusual language will not prevent the reader from assimilating the essence of this paper.



## 2. Quantum logic. Axiomatic structure

Let $S$ be a *physical system*. It is a primitive, undefined concept. It may be thought of as an abstracted part of the physical world, if such a sentence explains anything. The second undefined notion is that of "an object of $S$" (it should be, however, remarked that a precise definition of an object will be given in Definition 6 below)[2]. Let $L(S)=L$ be the set of all *properties* which different *objects of S may possess*. We write $a \in O$ if the object $O$ posseses the property $a$. In the set $L \times L$ of all ordered pairs of members of $L$ we distinguish a set $J$ by the following proposition:

(P) $(a, b) \in J$ *iff for each object* $O$, *if* $a \in O$ *then* $b \in O$.

If $(a, b) \in J$, we write $a \leqslant b$ and say that $a$ *implies* $b$. We assume that the relation "$\leqslant$" is reflexive, antisymmetric and transitive, i.e. "$\leqslant$" is a partial order in $L$. We also assume that there is the least element $o$ and the greatest element 1 in L.

(A1) $\langle L; \leqslant; o, 1\rangle$ *is a partially ordered set (poset) with* $o \leqslant a \leqslant 1$ *for all* $a \in L$.

For an arbitrary pair $a, b \in L$ we write $a \vee b$ and $a \wedge b$ for a least upper bound and a greatest lower bound, respectively, provided they exist. We do not assume that $L$ is a lattice, i.e. that $a \vee b$ and $a \wedge b$ exist for all $a, b \in L$. What more, it seems that the non-existence of meets and joints is essential for our approach[3].

The implication is followed by the notion of orthocomplementation. It is postulated that to each $a \in L$ there corresponds the *orthocomplement* $a^{\perp} \in L$ satisfying:

(A2) (i) $a^{\perp\perp}=a$,

(ii) *if* $a \leqslant b$, *then* $b^{\perp} \leqslant a^{\perp}$,

(iii) $a \wedge a^{\perp}=o, \; a \vee a^{\perp}=1$.

Thus $\langle L; \leqslant; o, 1; ^{\perp}\rangle$ is an orthocomplemented poset. Disjointedness is defined in the usual way: $a$ and $b$ are *disjoint* (or *orthogonal*) if and only if $a \leqslant b^{\perp}$. We write then $a \perp b$. The next assumption is

(A3) *L is an orthoposet, i.e.*

(i) *if* $a_1, \ldots, a_n$ *is any finite sequence of mutually disjoint elements of* $L$, *then* $\bigvee_{i=1}^{n} a_i$ *exists,*

(ii) *if* $a \leqslant b$ *then* $b=a \vee (b \wedge a^{\perp})$.

DEFINITION 1. We say that two elements of $L$, $a$ and $b$, are *compatible*[4] (in symbols $a \leftrightarrow b$), if there are three mutually orthogonal elements $a_0$, $b_0$ and $c$ in $L$ such that

$$a=a_0 \dotplus c, \qquad b=b_0 \dotplus c.$$

[2] It is worthwhile to draw the reader's attention to the fact that we have in mind concrete objects here. Thus, the electron with momentum $p_1$ is a different object from the electron with momentum $p_2$. Analogously, we allow the particle at an instant $t_1$ to be a different object from the particle at $t_2$.

[3] Compare Section 5.

[4] It has been repeatedly emphasized that the existence of non-compatible properties is a specific feature of quantum theory. It should be stressed that the real importance is not in the mere existence of incompatible pairs in $L$, but in the fact that *each separate* object possesses incompatible properties.

Notice that the compatibility relation is symmetric and reflexive and also, by (A3), that $a\leftrightarrow b$ is equivalent to $a\leftrightarrow b^{\perp}$ and implies $a\wedge b=c$ and $a\vee b=a_0+b_0+c$ (we write "+" for the least upper bound of orthogonal elements). Our last axiom excludes the most pathological examples and makes it possible to develop a calculus of observables (see [3] and references cited there).

(A4) *If $a, b, c$ are mutually compatible, then $a\leftrightarrow(b\vee c)$.*

Now, each family of mutually compatible elements of $L$ is contained in a Boolean subalgebra of $L$ ([3], Theorem 2.1).

DEFINITION. 2. A *face* in $L$ is any maximal Boolean subalgebra of $L$. If $C$ is any set of mutually compatible elements of $L$ and $C$ is contained in some face $F$, then $F$ is said to be a *face of* $C$. We write $\mathscr{L}(C)$ for the set of all faces of $C$ and $L(C)$ for the set of all members of $L$ which are in some $G\in\mathscr{L}(C)$. Eventually, we denote by $\mathscr{L}$ the set of all faces in $L$ [5].

Clearly we have

$$L(C)=\bigcup\{F:\ F\in\mathscr{L}(C)\},$$
$$L=\bigcup\{L(a):\ a\in L\}=\bigcup\{F:\ F\in\mathscr{L}\},$$
$$\mathscr{L}=\bigcup\{\mathscr{L}(a):\ a\in L\}.$$

We are now able to define a physical quantity [6] as a homomorphism of the Boolean algebra of all Borel sets of the real line into some face in $L$. We will not touch these problems here. The rest of this section is devoted to the precise definition of an object.

DEFINITION 3. A *preobject in* $L$ is any (non-empty) family $G$ of elements of $L$ with the following properties

(i) if $a, b\in G$, then $a\leftrightarrow b$;

(ii) if $a, b\in G$, then $a\wedge b\in G$;

(iii) $o\notin G$.

We denote by $\mathscr{P}$ the set of all preobjects in $L$.

The set-inclusion "$\subset$" is now a partial order in $\mathscr{P}$.

PROPOSITION 1. *Let $G\in\mathscr{P}$. Then the following three assertions are pairwise equivalent:*

(i) *$H\in\mathscr{P}$ and $G\subset H$ implies $G=H$ (i.e. $G$ is maximal),*

(ii) *if $b\leftrightarrow G$ then either $b$ or $b^{\perp}$ is in $G$,*

(iii) *if $b\leftrightarrow G$ and $b\wedge a_1\wedge\ldots\wedge a_n\neq o$ for any finite sequence $a_1,\ldots,a_n$ of elements of $G$, then $b\in G$.*

*Proof*: Suppose that (i) holds and let $b\leftrightarrow G$. Let $F\in\mathscr{L}(G)$ be such that $b\in F$. Clearly $b^{\perp}\in F$. Define

$$G_b=\{c\in F:\ c\vee b\geqslant a \text{ for some } a\in G\},$$

If $c\in G$, then $c\in F$ and $c\vee a\geqslant c$, thus $c\in G_b$ and so $G\leqslant G_b$. We show that $G_b$ is a preobject. Let $c_1, c_2\in G_b$. Since $G_b\subset F$, it follows that $c_1\leftrightarrow c_2$. Also $c_i\vee b\geqslant a_i$ for some $a_i\in G$, $i=1,2$. Thus $(c_1\wedge c_2)\vee b\geqslant(a_1\wedge a_2)\in G$ and therefore $(c_1\wedge c_2)\in G_b$. Analogously for the set $G_{b\perp}$ obtained from $G_b$ by the exchange of $b$ with $b^{\perp}$ in the definition. We claim that $o\notin G_b\cap G_{b\perp}$. Indeed, $o$ being in $G_b$ and in $G_{b\perp}$ we have $b\geqslant a_1$ and $b^{\perp}\geqslant a_2$ for some $a_1$, $a_2\in G$, hence $o\geqslant a_1\wedge a_2$ contrary to the definition of preobject. We may thus assume that $o$ is not in $G_b$. Then $G_b$ is a preobject, and by (i), $G_b=G$. However, by the definition, $b^{\perp}\in G_b$. Thus (ii) holds. Assume (ii) and let $b$ satisfy the assumption of (iii). Then $b$ or $b^{\perp}$ is in $G$. However, were $b^{\perp}$ in $G$, then $b\wedge b^{\perp}\neq o$ what is an absurd. Hence $b\in G$. To prove (iii)$\Rightarrow$(i), let $G_1$ be a preobject which contains $G$. If $b\in G_1$, then, by the definition, $b$ satisfies the assumption of (iii) and therefore $b\in G$. We conclude that $G=G_1$, i.e. $G$ is a maximal preobject.

DEFINITION 4. A preobject which satisfies one of the equivalent conditions of Proposition 1 is called a *filter*. $\mathscr{F}$ is the set of all filters in $L$.

COROLLARY 1. *Let $F_0$ be a filter in $L$. There is a unique Boolean algebra $F\subset L$ such that $F_0\subset F$. $F$ is a face of $F_0$. Conversely, for each face $F$ in $L$ one can find a filter $F_0$ such that $F\in\mathscr{L}(F_0)$.*

*Proof*: Let $F$ be any Boolean subalgebra of $L$ which contains $F_0$. Assume $a\leftrightarrow F$. Then $a\leftrightarrow F_0$ and so, by Proposition 1, $a$ or $a^{\perp}$ is in $F_0$ and thus in $F$. Now, since $F$ is a Boolean algebra, it follows that both $a$ and $a^{\perp}$ are in $F$. Hence $F$ is maximal. Suppose that $F_1$ and $F_2$ are two faces of $F_0$. Then $F=F_1\cap F_2$ is a Boolean algebra which also contains $F_0$ and, by the above, it is a face of $F_0$ too. Now, by the maximality of $F_i$ $(i=1,2)$ it follows that $F_1=F=F_2$. This proves the first part of the statement. To prove the second one, let $F\in\mathscr{L}$, and $a$ be any element of $F$. The set $\{a\}\subset F$ is a preobject provided $a\neq o$. Let $F_0$ be a maximal preobject with $\{a\}\subset F_0\subset F$. We show that $F_0$ is maximal in $L$. Let $F_1$ be any filter with $F_0\subset F_1$ and let $G$ be the face $F_1$. Then $F\subset G$ and thus $F=G$. Therefore $F_1\subset F$ and so $F_0=F_1$. The proof is complete.

DEFINITION 5. Two filters $F_1$ and $F_2$ are said to be *equivalent* ($F_1\sim F_2$ in symbols) if and only if the following condition is satisfied: *for each $a_1\in F_1$ there exists $a_2\in F_2$ with $a_2\leqslant a_1$* and conversely, *to each $a_2\in F_2$ there exists $a_1\in F_1$ with $a_1\leqslant a_2$.*

It is clear that "$\sim$" is an equivalence relation in $\mathscr{F}$.

DEFINITION 6. Any equivalence class in $\mathscr{F}$ is called an *object* [7].

A justification of the last definition is given in the following statement:

PROPOSITION 2. *Let $\mathbf{F}$ be an object. Then the following two statements are equivalent:*

(i) *there exists $o\neq a\in L$ and $F\in\mathbf{F}$ such that $a\leqslant b$ for all $b\in F$,*

(ii) *there exists $o\neq a\in L$ such that for each $F\in\mathbf{F}$ and all $b\in F$, $a\leqslant b$ holds.*

[5] As far as the relation of the above notions to experience and classical theory is concerned, to each element of $\mathscr{L}$ there should correspond some type of the apparatus-system interaction. To different faces of the same property $a$ there should correspond different experimental arrangements. According to this point of view it is insufficient to say "device which measures $a$". One has also to point out what face of $a$ is to be examined.

[6] It is *convenient* to use the term "observable" for such a quantity.

[7] Our definition of the object is based upon Jauch's and Piron's definition of the state [4].

*If one of these conditions is satisfied, then a is a unique element of L satisfying* (i) *or* (ii), *a is an atom of L and* $a = \bigwedge \{ \bigwedge \{b : b \in F\} : F \in \boldsymbol{F}\}$. *Conversely, to every atom a of L there corresponds a unique object* $\boldsymbol{F}$ *such that* $a = \bigwedge \{ \bigwedge b : b \in F\} : F \in \boldsymbol{F}\}$.

*Proof*: (i)⇒(ii). Assume that $a$ satisfies (i) for a filter $F \in \boldsymbol{F}$, and let $c \in E$ for some other filter $E \in \boldsymbol{F}$. Then, one can find $b$ in $F$ with $b \leqslant c$ and so $a \leqslant b \leqslant c$. The implication (ii)⇒(i) is obvious. To prove the second part of the Proposition we assume that (ii) holds. Then $a$ or $a^{\perp}$ is in $F$ for each $F \in \boldsymbol{F}$. Suppose $a^{\perp} \in F$ for some $F$ in $\boldsymbol{F}$. Then, by (ii), $a \leqslant a^{\perp}$, i.e. $a = o$, contrary to the hypothesis. Thus $a \in F$ for each $F \in \boldsymbol{F}$ and therefore $a \in \bigwedge \{b : b \in F\}$ for each $F$ in $\boldsymbol{F}$. It thus follows, that $a$ is a unique element of $L$ satisfying (i) or (ii). Now, $c \leqslant a$ implies that $c$ also satisfies (ii) provided $c \neq o$. Hence $a$ is an atom of $L$. We prove the last statement. Let $a$ be any atom of $L$. Then, the one-point set $\{a\}$ is a preobject and is contained in some filter $F$ in $L$. Let $\boldsymbol{F}$ be the equivalence class of $F$. Since $a$ is an atom, it easily follows that $a \leqslant b$ for all $b \in F$, and so, by the above, $a = \bigwedge \{ \bigwedge \{b : b \in G\} : G \in \boldsymbol{F}\}$. Let $\boldsymbol{F}_1$ be another object with this property. Put $b_{F_1} = \bigwedge \{b : b \in F_1\}$, $F_1 \in \boldsymbol{F}_1$. Then $b_{F_1} \leqslant b$ for all $b \in F_1$ and thus $b_{F_1} \in F_1$ or $b_{F_1} = o$. Now, since $a \neq o$, we have $b_{F_1} \neq o$ and therefore $b_{F_1} \in F_1$. Let $G, H \in \boldsymbol{F}_1$. Then, there exists $b \in H$ such that $b \leqslant b_G$. Hence, $b_H \leqslant b_G$ and similarly $b_G \leqslant b_H$. Thus $b_G = b_H$. We conclude that $a = \bigwedge \{b : b \in F_1\}$ for all $F_1 \in \boldsymbol{F}_1$. But then $F$ is equivalent to $F_1$ for each pair $F \in \boldsymbol{F}$, $F_1 \in \boldsymbol{F}_1$. Thus $\boldsymbol{F} = \boldsymbol{F}_1$. The proof is complete.

DEFINITION 7. An object satisfying one of the equivalent conditions of Proposition 2 is said to be *proper*. Otherwise it is said to be *improper*.

*Remark*. Equipped in a natural orthogonality relation the set of all objects is an orthogonality space and thus defines a quasilogic (see [2]). It would be interesting to known what connection is there between the quasilogic defined like that and the logic $L$ itself. We only indicate the existence of such a problem here.

## 3. Objects and states

We now pass on to a relationship between the object and state. It is fundamental for the quantum theory. Consider an object $\boldsymbol{F}$. If $F$ is any filter of $\boldsymbol{F}$, denote by $F''$ a unique face of $F$. Denote by $\mathscr{L}(\boldsymbol{F})$ the set of all such faces and by $\boldsymbol{F}''$ the union of such ones. Now, for any $a \in \boldsymbol{F}''$ we have: either the object $\boldsymbol{F}$ possesses the property $a$ or *non-a* is a property of $\boldsymbol{F}$. The behaviour of elements of $\boldsymbol{F}''$ with respect to the object $\boldsymbol{F}$ is thus, to a certain degree, like a classical one. But then, what can be said about properties which are not in $\boldsymbol{F}''$? In the standard quantum theory, it is assumed, that the object $\boldsymbol{F}$ determines a probability $p_{\boldsymbol{F}}(a)$ of the fact that $\boldsymbol{F}$ possesses a property $a$ and that for each $a$ [8]. It is tacitly assumed that the above probability depends on an object $\boldsymbol{F}$ and property $a$ only. Or, in more practical terms, that each apparatus selecting objects with a property $a$ shall give the same fraction, irrespectively of the principle of its action. This unspoken postulate is a direct consequence of the usual Hilbert-space formalism on the one hand and is justifiable, if one assumes that the quantum object is nothing more but a statistical mixture of classical ones, on the other. The last point of view consequently leads to a hidden-variables theory. Such an approach is comprehensible and may account for an inertia of the human mind (still) accustomed to classical pictures. A question of an open-minded man would rather sound as follows: why each property $a$ is to be characterized by a definite probability? And even if there are some reasons for that, then why such a probability is to be experimentally verifiable [9]? And even if it is so, then why the probability does not depend on details of the experiment [10]?

[8] We recall here that during the period of forty years, since a consistent interpretation of quantum theory has been given, up to the recent time, the question has been raised again and again: why only probabilities? And the answer, if given at all, was that it is because something is unknown. Because one does not know the value of some parameters or, because one does not know the details of interaction between object and apparatus, between object and vacuum and so on. The inference was either constructive—one does not know but should try to know—or agnostic—one does not know and never will know.



It is plain that the last possibility being allowed, it should be in agreement with all the known experimental facts in the first place and, in the second, the possible dependence of averages on the measuring apparatus should be describable by the theory and sufficiently regular. In the standard theory a state is a probabilistic measure on $L$. It should be, however, observed that it is hardly possible to compare frequencies of non-compatible properties. With this in mind, it is to be expected that a state is to be rather a family of measures then one measure on the whole $L$.

DEFINITION 1. Let $m$ be a function defined on an orthomodular orthoposet with values in $[o, \infty]$. If for each finite sequence of mutually orthogonal elements $a_1, \ldots, a_n$, $m(\sum a_i) = \sum m(a_i)$, then $m$ is said to be a *finitely additive measure* (*f.a.-measure*). A f.a.-measure is finite, if $m(a)$ is finite for each $(a)$, and is probabilistic if $m(1) = 1$.

PREDEFINITION 2. A *state* on $L$ is a mapping $m$: $F \to m_F$, where $F$ runs over the set of all faces in $L$ and $m_F$ is a f.a.-measure on $F$ for each $F \in \mathscr{L}$.

CONJECTURE. *Let* $\boldsymbol{F}_0$ *be an object. Then there is a* (*unique?*) *state* $m$ *on* $L$ *such that* $m_F(a) = m_F(1)$ *for each* $a \in F \cap (\bigcup \{G : G \in \boldsymbol{F}_0\})$ *and* $m_F(a) = o$, *if a is orthogonal to* $\boldsymbol{F}_0$.

To convert the predefinition above into a definition one has to answer the question how do different measures $m_F$ intertwine. In the simplest case all $m_F$ agree on intersections of different faces and define a unique, probabilistic measure on $L$. If there is sufficiently many such states, then one deals with a standard case. If it is not so, then two possibilities may happen:

(a) *there exists exactly one probabilistic measure on L*,

(b) *L does not admit probabilistic measures at all* (see [2], p. 24).

[9] It is not quite clear what is the experimental meaning of the statement: given (unstable) particle at time $t$, the probability that it does not exist at $t_0 < t$ is $p$.

[10] Everywhere in this work the word "probability" is to be understood in the sense of the propensity interpretation due to K. R. Popper (see [6]).

In case (a) we expect that a unique probabilistic measure on $L$ is a trace-like measure and thus $L$ is finite-dimentional. The most interesting case is case (b). Let us observe, in the first place, that even if there are no positive measures on $L$, then there may be sufficiently many signed measures (see sec. 4). In such a case a signed measure may play the role of intertwinning function for a state. We feel that a solution of intertwinning problem, given below, is far from being a definite one. In any case it would be desirable to have a more impressive formulation.

DEFINITION 3. A function $m$ defined on an orthomodular orthoposet is said to be a *(finitely additive) signed measure*, if $m(\sum a_i)=\sum m(a_i)$ for any finite sequence of mutually orthogonal elements $a_i$. (We assume that $m(a)$ is finite for each $a$.)

THEOREM 1. *Let $m$ be a signed measure on a Boolean algebra $F$. Then there exists the smallest element in the (non-empty) set of all positive measures $p$ such that $|m(a)|\leqslant p(a)$ for each $a\in F$.* (We denote it by $v(m, F)$ and call the *total variation* of $m$[11]).

The proof of this theorem is given, for instance, in [1], Chapter III, sec. 1.

Let $m$ be a signed measure on $L$. Then $m$ restricted to any face $F$ of $L$ is a signed measure on $F$. Let $m_F=v(m, F)$. The mapping $F\to m_F$ is now a state on $L$ in the sense of the pre-definition of a state. We say in this case that $m$ intertwines different measures $m_F$ or that $m$ is an intertwinning function for the state $F\to m_F$.

DEFINITION 2. A mapping $m$: $F\to m_F$ is said to be a *state* if (not: only if!) there is an intertwinning function for $m$. A convex combination of states is a state.

*Remark.* This, somewhat awkward, definition of a state should be certainly replaced by a more convenient one. It would be desirable to know whether a state in the sense of Definition 2 satisfies the following property: $m_F$ and $m_G$ are equivalent on $F\cap G$ for every pair $F, G\in \mathscr{L}$.

## 4. Logics of *-algebras

In this section we recall some facts concerning a theory of *-algebras. It is useful to compile them together and to discuss them from a point of view we are interested in.

Let $A$ be a complex algebra with unit. We say that $A$ is a **-algebra* provided $A$ is equipped with an operation of involution $a\to a^*$ which satisfies:

(i) $(a^*)^*=a$,

(ii) $(\alpha a+\beta b)^*=\bar{\alpha}a^*+\bar{\beta}b^*$ *for any complex* $\alpha, \beta$,

(iii) $(ab)^*=b^*a^*$,

We say that $a\in A$ is *hermitian* if $a=a^*$ and $a$ is *positive* if $a=b^*b$ for some $b\in A$. Any positive element of $A$ is hermitian and each member of $A$ is a linear combination of some hermitian elements. A linear form $f$ on $A$ is said to be *real* if $f(a)$ is a real number for each hermitian $a\in A$. If $f(a)\geqslant 0$ for each positive $a\in A$, then $f$ is positive. Every linear form on $A$

[11] Notice that if $F$ is a $\sigma$-algebra and $m$ is $\sigma$-additive, then $v$ is bounded and thus, can be normalized.



is a linear combination of some real forms. It is not, in general, true that each real form is a difference of two positive forms. There are *-algebras admitting no positive forms.

Let $A$ be a *-algebra and $L$ the set of all selfadjoint idempotents of $A$, i.e. $a\in L$ if and only if $a^2=a^*=a\in A$. Observe that $o$ and 1 are in $L$. We now introduce an implication in $L$: $a\leqslant b$ iff $ab=a$. If is easy to check that "$\leqslant$" is a partial order in $L$, $o$ and 1 are the least and the greatest elements in $L$, respectively. For each $a\in L$, let $a^\perp=1-a$. The operation $a\to a^\perp$ is an orthocomplementation on $L$. Moreover, $L$ is an orthomodular orthoposet. It is easy to see that $a\to b$ if and only if $ab=ba$. Property (A4) of Section 2 is clearly satisfied. Taking into account the fact that *-operation seems to have some physical meaning (particle-antiparticle symmetry) one may expect that logics of *-algebras are the most general logics of quantum systems. It is a remarkable fact that every *-algebra can be realized as an algebra of operators on some linear space.

DEFINITION 1. Let $X$ be a complex linear space and let $\langle x, y\rangle$ be a non-degenerate bilinear hermitian form on $X$, i.e.

(i) $\langle x, \alpha y+\beta z\rangle=\alpha\langle x, y\rangle+\beta\langle x, z\rangle$,

(ii) $\langle x, y\rangle=\overline{\langle y, x\rangle}$,

(iii) $\langle x, y\rangle=0$ for all $x\in X$ implies $y=0$.

Then $X$ is called a *self-dual space*. The form $\langle x, y\rangle$ is called a *metric on $X$*. If there is at least one pair of vectors $x, y\in X$ with $\langle x, x\rangle\langle y, y\rangle<0$, then $X$ is said to be an *indefinite metric space.*

Given a self-dual space $X$, the metric $\langle x, y\rangle$ determines on $X$ a weak topology $T^w$ given by a family of seminorms

$$p_N(x)=\sup\{|\langle x, y\rangle| : y\in N\},$$

where $N$ is any finite sequence of vectors of $X$. In this topology, each linear form $f_x$: $y\to\langle x, y\rangle$ is continuous and, in fact, $T^w$ is the weakest topology in $X$ with such a property. The convergence in $T^w$ is simply given by: $x_n\to x$ if and only if $\langle x_n-x, y\rangle\to 0$ for all $y\in X$.

THEOREM 1. *Let $f$ be a continuous linear form on $X$. There exists a unique vector $x\in X$ such that $f=f_x$.*

For the proof see e.g. [7]. Let $a$ be a continuous, linear operator on $X$. By Theorem 1 there exists a unique operator $a^*$ which satisfies $\langle x, ay\rangle=\langle a^*x, y\rangle$. The set $A(X)$ of all continuous linear operators on $X$ is now a *-algebra with unit. The following theorem is a direct consequence of the closed-graph theorem (see [7]).

THEOREM 2. *Assume that two linear operators $a$, $b$ on $X$ satisfy:*

$$\langle ax, y\rangle=\langle x, by\rangle \quad \textit{for all } x, y \textit{ in } X.$$

*Then $a$, $b\in A$ and $a=b^*$.*

Let $A$ be a $a^*$-algebra and let $X$ be a self-dual space. An algebraic homomorphism $\pi$: $A\to A(X)$ is said to be a **-representation* of $A$ if and only if $\pi(a^*)=\pi(a)^*$ for all $a\in A$. If $\pi$

is one-to-one, then $A$ is said to be *realized* as a **-subalgebra* of $A(X)$. With these definitions we have the following important result:

THEOREM 3. *Any *-algebra can be realized as a *-subalgebra of some* $A(X)$.

The proof, based on the fact that there is always sufficiently many real forms on a *-algebra, is essentially given in [8]. It follows from the last theorem that one may restrict himself to the study of $A(X)$. The situation is just similar to the case of $C^*$-algebras. It is well known that every $C^*$-algebra can be realized as a *-subalgebra of $A(H)$, where $H$ is a Hilbert space. There is, however, an important difference also. All Hilbert spaces with the same dimension are isomorphic. It is not the case for self-dual spaces.

## 5. Logics of indefinite metric spaces

In this section we shall study in some detail a geometry of indefinite metric spaces. Throughout the section, $X$ is a fixed indefinite metric space, the letters $a, b, c, \ldots$ stand for linear subspaces of $X$, letters $x, y, z, \ldots$ stand for vectors. Let $L_1$ be the set of all linear subspaces of $X$. If $a \in L_1$, then $a^\perp = \{x \in X: \langle x, y\rangle = 0 \text{ for all } y \in a\}$ is called the *orthogonal complement* of $a$. Clearly, $a^\perp$ is always in $L_1$. We list below the most important properties of the operation $y \to a^\perp$. The reader is referred to [9] for the proofs.

THEOREM 1. *Let* $a, b \in L_1$. *Then*

(1) *if* $a \leqslant b$, *then* $b^\perp \leqslant a^\perp$,

(2) $a \leqslant a^{\perp\perp}$,

(3) *if* $a \leqslant b$, *then* $a^{\perp\perp} \leqslant b^{\perp\perp}$,

(4) $a^\perp = a^{\perp\perp\perp} = \ldots$,

(5) $(a+b)^\perp = a^\perp \cap b^\perp$,

(6) $(a \cap b)^\perp \supset a^\perp + b^\perp$,

(7) $a^{\perp\perp} = \bar{a}$,

(8) *a is closed if and only if* $a = a^{\perp\perp}$,

(9) *a is dense if and only if* $a^\perp = o$,

(10) $a + a^\perp$ *is dense if and only if* $a^{\perp\perp} \cap a^\perp = o$.

The set $L_1$ is now a poset with the least element $o = \{0\}$ and the greatest element $1 = X$. In fact, $L_1$ is a lattice with $a \wedge b = a \cap b$ and $a \vee b = a + b$. However, $\langle L_1, \leqslant, o, 1, {}^\perp\rangle$ is not an orthocomplemented lattice. To obtain an orthocomplemented structure we have to restrict ourselves to the subset $L_2$ of all regular (i.e. $a \wedge a^\perp = o$) and closed (i.e. $a = a^{\perp\perp}$) members of $L_1$. $L_2$ is an orthocomplemented poset by the definition. It is not a lattice if $\dim(X) \geqslant 4$.

Let $a \in L_2$. Then, by Theorem 1, $a + a^\perp$ is dense in $X$. For each vector $x \in d(a) = a + a^\perp$, there is a unique decomposition $x = x_a + x_{a^\perp}$ with $x_a \in a$ and $x_{a^\perp} \in a^\perp$. We define two linear operators on $d(a)$ by $ax = x_a$ and $a^\perp x = x_{a^\perp}$. Clearly, $a + a^\perp = 1$ and $aa^\perp = a^\perp a = o$ on $d(a)$. Both, $a$ and $a^\perp$, are selfadjoint idempotents on $d(a)$. If $X$ is finite-dimensional, then $d(a) = X$ for each $a \in L_2$, and by Theorem 2 of Section 4, $a$ is continuous, i.e. $a \in A(X)$. Things are more complicated if $X$ is not finite-dimensional. We do not know whether $L_2$ is orthomodular in that case. To make an algebraic analysis possible, we have to restrict ourselves to the subset $L$ of $L_2$ consisting of those $a \in L_2$ for which $a + a^\perp = 1$, i.e. for which a correspondent projection is defined on the whole $X$ and thus, continuous. Conversely, to each hermitian idempotent on $X$ there corresponds a unique regular closed subspace of $X$ with $a + a^\perp = X$. We identify $L$ with the logic of *-algebra $A = A(X)$. It is easy to see that for each $x \in X$ with $\langle x, x\rangle \neq 0$, the mapping $a_x$: $y \to (\langle x, y\rangle / \langle x, x\rangle) \cdot x$ is an element of $L$. In fact, $a_x$ is an atom of $L$ and $L$ is atomic.



CONJECTURE. *If the dimension of X is finite and sufficiently large, then there exists precisely one probabilistic measure on L:* $a \to \dim(a)$. *If X is of infinite dimension, then L admits no positive measures*[12].

Let now $x \in X$ and $\langle x, x\rangle \neq 0$. The atom $a_x$ determines a unique object $F(x)$. On the other hand, the mapping $m^x$: $a \to \langle x, ax\rangle / \langle x, x\rangle$ is a signed measure on $L$. Thus, according to Section 3, $m^x$ is an intertwining function for some state $m^x$: $F \to m^x_F$.

PROPOSITION. *If a is orthogonal to* $F(x)$, *then* $m^x_F(a) = 0$ *for every F such that* $a \in F$. *If a is a property of the object* $F(x)$, *then* $m^x_F(a) = m^x_F(1)$ *for each face F with* $a \in F$.

*Proof*: Let $F$ be any face in $L$. Then $m^x_F$ is defined by (see [8], p. 97)

$$m^x_F(a) = \sup\Big\{\sum_{i=1}^{n} |m(a_i)|\Big\},$$

where the supremum is taken over all finite sequences $a_i \in F$ of mutually orthogonal elements of $F$ with $a_i \leqslant a$. Now, if $a$ is orthogonal to $F(x)$, then $a$ is orthogonal to $a_x$ and the same holds for each $b \leqslant a$. Hence, $m^x_F(a) = 0$. If $a$ is a property of $F_x$ then $a^\perp$ is orthogonal to $F(x)$. Thus $m^x_F(a^\perp) = 0$ and therefore $m^x_F(a) = m^x_F(1)$.

In this way, to each non-isotropic vector $x \in X$ there corresponds a state satisfying Conjecture of Section 3 up to the uniqueness. We have to show that our definition of a state is in agreement with the usual one. First of all, let us recall that in usual treatments of indefinite metric one always distinguishes a "physical" subspace with positive metric. But now, if $x \in X$ and there exists a positive-metric subspace $X^+$ such that $x \in X^+$ and $a \subset X^+$, then $m^x_F(a)$ is nothing more but simply $\langle x, ax\rangle / \langle x, x\rangle$ in agreement with conventional calculations. In other words, the dependence of averages on a measuring apparatus may appear only for "observables" which connect positive and negative parts of $X$. There are also troubles with $\sigma$-additivity for such observables. These technical problems require further investigations.

[12] The author proved that there is exactly one probabilistic measure in case of $X$ being the Minkovski space. However, he was not patient enough to check the proof (very laborious indeed) with proper care. If the conjecture is right, then a proof based on methods developed in [2] should exist.

## 6. Summary and conclusions

In this paper we tried to show that the application of indefinite metric in quantum theory may prove to be an essential step forward. In order to see the true essence of indefinite-metric formalism, we used the most general language of quantum logic here. The necessary notions were introduced in Section 2 where the concept of an object has been defined. Our definition of an object differs from an analogous definition given by Jauch and Piron in [4]. We have demonstrated, however, that it agrees with the corresponding intuitive notion. In Section 3 the connection between objects and states was discussed. From our point of view, an object is a primary notion and a state—a secondary one. According to the definition of Section 3, a state is a family of intertwined positive measures on maximal Boolean subalgebras of the logic. Physically speaking, we permit a dependence of the observable's averages on a class of measuring devices. In Section 4, the axioms of Section 2 were realized on a logic of *-algebra. We also showed that each *-algebra can be realized as an algebra of operators on some self-dual space. The logic of *-algebra of all continuous linear operators acting on indefinite-metric space was considered in Section 5. We demonstrated that every proper object of this logic determines a state with reasonable properties. We thus gave the meaning of indefinite metric. "Negative probabilities" have been replaced with experimentally veritiable effects (in each concrete model). Simultaneously, we have demonstrated that our approach is in accordance with the so far used one. Some general problems (spectral decomposition, automorphisms of the logic, etc.) have not been touched at all. The results of the present paper suggest a possibility of further generalization. One may expect that the quantum logic of indefinite-metric spaces is a particular case of more general structures. We have in mind general "locally Hilbertian" logics. Our "predefinition" of a state in Section 3 seems to be sufficiently general to apply to such structures as well.

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