On Berry's phase*

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Abstract

We discuss geometro-algebraic aspects of the Berry's phase phenomenon. In particular we show how to induce parallel transport along states via Kaluza-Klein mechanism in infinite dimensions.

1 Classical and Quantum Worlds

In recent times, an increasing amount of work is being devoted to developing a new mathematical framework of noncommutative geometry, also called quantum geometry. As a result, we have:

- quantum groups (Connes, Drinfeld, Ocneanu, Woronowicz, ...)
- quantum nets (Finkelstein,...)
- quantum logic (Birkhoff, Jauch, Piron, von Neumann, ...)
- quantum probability (Accardi, Davies, ...)
- quantum information (Ingarden, Urbanik, ...)
- quantum computers (Deutsch, ...)
- ...

So, it becomes natural to ask this question: is *everything* going to be *quantum* ? A monist would probably answer this question "yes, that is our lot, the 'true' description is the quantum description ...". Being neither a monist nor a dualist, I

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would like to reiterate the well known and so often repeated position taken by Niels Bohr: that there is a cut between the classical and the quantum; a necessary cut; where classical is, roughly, everything that can be expressed in terms of an ordinary language; which can be communicated.
Thus we have the following

Dualistic Picture

Quantum	Classical
Matter	Language (= Knowledge)

Today, this is nothing but a picture. What we need is a theory, a theory that will make the picture mathematically precise and quantitatively predictive. The dualism we have in mind is similar to the one we know from General Relativity, namely the dualism between Matter and Geometry. There is no better way to express its idea than as is done by John Archibald Wheeler, who puts the story into these words:

- Geometry tells Matter how to Move while Matter tells Geometry how to Curve.

An analogous deep and simple statement concerning the picture above is still lacking[1]. What we need, in particular, is an adequate mathematical theory of the substance of (objective) knowledge - then a statement such as "the wave function describes state of knowledge" would have a precise meaning, devoid of any subjectivity. Although it is not directly related to our subject, perhaps it is, worth our while to note that the picture above necessarily implies that we should deal, as strongly stressed by Prigogine [2], with open systems.[2]

2 Classical Geometry of the Quantum Space

In quantum theory one associates operators with yes-no questions and with observables. However the question "what is the state of the quantum system" is NOT a quantum question. It pertains rather to the right hand side of the picture above, to the classical domain. Of course, one could think of some "quantum meta-system", where the question of the state of a "system" would be represented by an operator (or 'super-operator') - but then we could ask of the state of the meta-system, meta-meta system, etc., ad infinitum. Until a self-consistent theory that closes this

[1]Cf. however the idea of interaction between World I and World III by Popper and Eccles [1]
[2]I am indebted to Rudolph Haag for an illuminating discussion on this subject

infinite chain[3] is established, we have, in the quantum framework, a distinguished classical object: the State Space of the quantum system. Much of the properties[4] of the system are determined by geometry of this space. Here we would like to discuss only one topic related to this geometry: parallel transport along states. It is convenient, for doing this, to use the algebraic language.

2.1 Observables and States

Observables and, more generally, operations are described, in this language, by the elements of a von Neumann (i.e, weakly closed unital) operator algebra $\mathcal{A}$ (cf. e.g. [6]). States of the system are described by normal positive forms on $\mathcal{A}$. [5] We wish to restrict our attention to faithful states, i.e. such states ω that $\omega(A^\star A) = 0$ implies $A = 0$. The reason for this is that, anyhow, such states are dense in the space of all states. In particular pure states can be approximated by faithful states, although a faithful state will (except in trivial cases) never be pure. [6] Let $\mathcal{S}$ denote the state space (if necessary one can normalize the elements of $\mathcal{S}$ by demanding $\omega(I) = 1$, where I is the unit element of $\mathcal{A}$). The most characteristic concept of a quantum theory is that of a superposition of states. But in order to make superpositions we need state-vectors rather than states. Now, state-vectors are vectors in a representation space of the algebra $\mathcal{A}$. And even in the particular case where all the faithful representations of $\mathcal{A}$ are equivalent, it does not mean that there is a unique way of making a superposition of a state-vector from one representation with a state-vector from another one. One needs a method of *transporting* state vectors from one representation to another one. A possible framework for discussing such questions is provided by the Gelfand-Naimark-Segal construction, where one constructs a fibration of Hilbert spaces over the state space $\mathcal{S}$.

2.2 GNS-construction.

Given a state $\omega \in \mathcal{S}$ one constructs[7] , out of $\mathcal{A}$ and ω, a triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, where $\mathcal{H}_\omega$ is a Hilbert space, π_ω is a *-representation of $\mathcal{A}$ in $\mathcal{H}_\omega$, and Ω_ω is a vector $\mathcal{H}_\omega$, cyclic

[3]Some ideas in this direction can be found in Wheeler [3], see also Rössler [4]

[4]According to Mielnik [5], perhaps even *all* properties of the quantum system

[5]Let us recall that a complex linear form $\phi : \mathcal{A} \to \mathbb{C}$ is positive if $\phi(A^\star A) \geq 0, \forall A \in \mathcal{A}$, and ϕ is normal if $A_i \geq 0, A \uparrow A$ implies $\phi(A_i) \uparrow \phi(A)$

[6]Notice that our negligence of pure states is in accord with the remark above about open systems.

[7]Cf. [6], especially Appendix A, for the relevant concepts

with respect to $\pi_\omega(\mathcal{A})$ [8], and such that for all $A \in \mathcal{A}$ we have $\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega)$.

2.3 The fibration $\mathcal{P}$

In this way one gets a fibration of Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$ over $\mathcal{S}$ with a distinguished cross-section $\omega \mapsto \Omega_\omega$. There is a particular parallel transport in this fibration related to Berry's phase. To discuss this question it is convenient to introduce an associated principal fibration according to the idea of Uhlmann [7, 8].

For each ω in $\mathcal{S}$ let P_ω denote the set of all vectors ψ in $\mathcal{H}_\omega$ which represent ω i.e. such that

$$\omega(A) = (\psi, \pi_\omega(A)\psi) \tag{1}$$

for all $A \in \mathcal{A}$. Let $\mathcal{P} = \cup\{P_\omega : \omega \in \mathcal{S}\}$, then , by invoking the Tomita-Takesaki theory [9], $p : \mathcal{P} \to \mathcal{S}$ can be made into a principal fibration with base $\mathcal{S}$ and structure group $\mathcal{U}(\mathcal{A})$ - the group of unitary elements of the algebra $\mathcal{A}$. This fibration has, by the construction, a global cross-section $\omega \mapsto \Omega_\omega$, therefore one could think of it as of a trivial fibration: $\mathcal{P} \equiv \mathcal{S} \times \mathcal{U}(\mathcal{A})$. However, and this is still to be understood better, there are other, apparently more relevant, local trivializations (and thus "smooth structures") of $\mathcal{P}$, which lead naturally to a non-trivial geometry.[10]

3 Example: The case of $\mathcal{A} = \mathcal{B}(\mathcal{H})$

Let $\mathcal{H}$ be a fixed (complex, separable) Hilbert space, and let $\mathcal{A}$ be the von Neumann algebra of all bounded, operators on $\mathcal{H}$. By the Gleason theorem, states on $\mathcal{A}$ are represented by density matrices i.e. positive trace-class operators ρ on $\mathcal{H}$. The GNS construction can be, in this particular case, realized as follows:

Let $\mathcal{HS}$ denote the space of all Hilbert-Schmidt operators on $\mathcal{H}$; then $\mathcal{HS}$ itself is a Hilbert space when endowed with the scalar product: $(A, B) = \mathrm{Tr}(A^\star B)$. If $\rho \in \mathcal{S}$ is a state (i.e. a positive trace class operator on $\mathcal{H}$ or, in short, a density matrix), then $\rho^{\frac{1}{2}}$ – the positive square root of the positive operator ρ – is of Hilbert-Schmidt class. Thus we can put, for each ρ in $\mathcal{S}$

$$\mathcal{H}_\rho = \mathcal{HS}, \tag{2}$$

[8] i.e. $\pi_\omega(\mathcal{A})\Omega_\omega$ is dense in $\mathcal{H}_\omega$

[9] For a brief exposition of the Tomita's theory see e.g. [9]

[10] These problems will be discussed in Ref. [10]

$$\Omega_\rho = \rho^{\frac{1}{2}}, \tag{3}$$

and

$$\pi_\rho(A)B = AB \tag{4}$$

for all $A \in \mathcal{A}(\equiv \mathcal{B}(\mathcal{H})), \mathcal{B} \in \mathcal{H}_\rho(\equiv \mathcal{HS})$.

Then automatically

$$\rho(A) = \mathrm{Tr}(\rho A) = \mathrm{Tr}(\rho^{\frac{1}{2}} A \rho^{\frac{1}{2}}) = (\Omega_\rho, \pi_\rho(A)\Omega_\rho) \tag{5}$$

i.e: the vector $\Omega_\rho = \rho^{\frac{1}{2}}$ is the GNS representative of the state ρ. The total space of the fibration $\mathcal{P}$ consists now of all invertible (with an unbounded inverse, in general) Hilbert-Schmidt operators. The fiber projection $p : \mathcal{P} \to \mathcal{S}$ is given by

$$\mathcal{HS} \ni \mathcal{W} \mapsto \mathcal{W}\mathcal{W}^\star \in \mathcal{S} \tag{6}$$

The unitary group $\mathcal{U}(\mathcal{A}) \equiv \mathcal{U}(\mathcal{H})$ acts on $\mathcal{P}$ via the right action: $W \mapsto WU$, so that

$$p(WU) = WU(WU)^\star = WUU^\star W^\star = WW^\star = p(W). \tag{7}$$

The space $\mathcal{P}$ inherits from the Hilbert space $\mathcal{HS}$, in which it is embedded, a Riemannian metric. Given a curve $\gamma(t)$ through $W = \gamma(0)$, a tangent vector to γ at W is given by an operator $X = (d\gamma(t)/dt)|_{t=0}$. The scalar product of two tangent vectors X, Y at W is then defined by

$$(X, Y)_W = \mathrm{Re}\mathrm{Tr}(X^\star Y) = \frac{1}{2}\mathrm{Tr}(X^\star Y + Y^\star X). \tag{8}$$

This scalar product makes $\mathcal{P}$ into a Riemannian space with $\mathcal{U}(\mathcal{H})$ -invariant Riemannian metric. Therefore, as in Kaluza-Klein theories, we get automatically a connection, i.e, a parallel transport[11], in $\mathcal{P}$. The horizontal subspaces are defined as orthogonal complements to the vertical ones, where vertical means 'tangent to a fiber'. It is instructive to look into an explicit formula which characterizes the horizontal vectors, A vertical tangent vector at W is of the form $(d\gamma(t)/dt)_{t=0}$, where

$$\gamma(t) = W\exp(tY),\ Y = -Y^\star \tag{9}$$

[11] Or, in other words, $\mathcal{U}(\mathcal{H})$ gauge field

Thus X is horizontal at W if and only if for all $Y = -Y^\star$

$$0 = 2(WY, X)_W = \mathrm{Tr}((WY)^\star X + X^\star WY) \; = \mathrm{Tr}(X^\star WY - YW^\star X) \qquad (10)$$
$$= \mathrm{Tr}((X^\star W - W^\star X)Y)$$

which implies

$$X^\star W - W^\star X = 0. \qquad (11)$$

This condition is automatically satisfied if X is of the form $X = HW, H = H^\star$. The latter condition being not only sufficient but, in fact, also a necessary for Eq. (11) to hold, we get the following, rather simple, characterization:

$$\begin{array}{lccl} X \text{ vertical at } & W & \longleftrightarrow & X = WA, A = -A^\star \quad \text{(anti-Hermitean)} \\ X \text{ horizontal at } & W & \longleftrightarrow & X = WH, H = H^\star \quad \text{(Hermitean)} \end{array}$$

It is this characterization that generalizes naturally from the particular case of $\mathcal{A} = \mathcal{B}(\mathcal{H})$ to a more general case of an arbitrary $\mathcal{A}$.[12] The following argument, due to Uhlmann, indicates the relation of the parallel transport defined by Eq.(11) to the Grassmann connection responsible for the Berry's phase phenomenon.

4 Berry's Phase.

Consider a path $t \mapsto E(t)$, where $E(t)$ are orthogonal projections on $\mathcal{H}$. Although this path is not, strictly speaking, in $\mathcal{P}$ (projections are not invertible, even if they are of trace class, i.e, finite dimensional), nevertheless our path can be, with an arbitrary accuracy, approximated by a path in $\mathcal{P}$. For each t, let[13] $|i, t>$, $i = l, 2, \ldots$ be an orthonormal basis in $E(t)\mathcal{H}$, i.e, we have

$$E(t) = \sum_i |i, t >< i, t| \qquad (12)$$

Suppose we want to choose $t \mapsto |i, t>$ in such a way that the path

$$W(t) = \sum_i |i, t >< t, 0| \qquad (13)$$

[12]See the previous footnote

[13]In the following we use Dirac's *bra* and *ket* notation

is horizontal. Notice that

$$p(W(t)) = W(t)W(t)^\star = \sum_{i,j} |i,t >< i,0|j,0 >< j,t| = E(t). \tag{14}$$

The horizontality condition (11) gives then

$$0 = (dW)^\star W - W^\star dW = \sum |i,0 >< i,t| \overleftarrow{d} |j,t > -|i,0 >< i,t| \overrightarrow{d} |j,t >< j,0|, \tag{15}$$

which, by taking into account $< i,t|j,t >= \delta_{ij}$, what implies

$$< i,t| \overleftarrow{d} |j,t > \equiv - < i,t| \overrightarrow{d} |j,t > \tag{16}$$

gives

$$< i,t| \overrightarrow{d} |j,t >= 0, \tag{17}$$

or, equivalently,

$$E(t) \overrightarrow{d} |j,t > \equiv 0. \tag{18}$$

The last equation is the determining equation for the Grassmann connection. Its relation to the adiabatic transport is discussed, in Ref. [11, 12, 13, 14].

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