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Completely mixing quantum open systems and quantum fractals

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Abstract

Departing from classical concepts of ergodic theory, formulated in terms of probability densities, measures describing the mixing behavior and the loss of information in quantum open systems are proposed. As application we discuss the chaotic outcomes of continuous measurement processes in the EEQT framework. Simultaneous measurement of four noncommuting spin components is shown to lead to a chaotic jumps on the quantum spin sphere and to generate specific fractal images of a nonlinear iterated function system. © 2001 Published by Elsevier Science B.V.

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1. Introduction

In the past two decades the study of chaotic dynamical systems has attracted attention of many physicists. Roughly speaking, a system is said to be chaotic if orbits of the motion are in some sense irregularly distributed. One of the most useful measures of this irregularity is the Kolmogorov–Sinai entropy. Another important quantity relevant to control the chaotic behavior is the Lyapunov exponents, which measure the exponential instability of almost all orbits with respect to the change of initial conditions. It turns this instability leads to the loss of memory of initial conditions, decay of correlations and approach to statistical equilibrium.

It is believed that also quantum systems have different qualitative properties depending on whether the corresponding classical systems are integrable or chaotic [1]. For example, they reveal significant differences in the character of their wave functions or the distribution of their energy levels [2]. Also the numerical analysis of some finite quantum systems shows quantitative differences in regular and chaotic regime [3]. It is obvious that on the quantum level, the distinction between chaotic and quasiperiodic behavior must be blurred, nevertheless some relative measure of the degree of chaos should exist. There have been many attempts to find fingerprints of chaotic behavior in quantum dynamical systems. The most natural one is based on the correspondence principle, which states that a quantum system is chaotic if its classical limit is chaotic. However, it can only indicate some features of chaotic behavior but cannot serve as a precise definition. As was pointed out by van Kampen [4] such notions like ergodicity or mixing

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need a limit $t \to \infty$ while the correspondence principle refers to $\hbar \to 0$, and these two limits do not commute. Moreover, an example of a chaotic quantum phenomenon, which has no counterpart in the classical limit was given [5]. One of the solutions of this obstacle is based on the idea of taking the number of degrees of freedom to infinity [6]. For example, in [7] it was shown that a class of quantum dynamics of harmonic crystals becomes ergodic and mixing in the thermodynamic limit. Moreover, by taking $\hbar \to 0$, classical properties of ergodicity and mixing are recovered. Similar results, but for the ideal gas quantized according to the Maxwell–Boltzmann statistics, are presented in [8].

Another way to recognize chaotic behavior in quantum systems is to investigate the concept of entropy. Kosloff and Rice [9] introduced a generalization of Kolmogorov–Sinai entropy for the quantum case, which allows to compare the behavior of a given system when described alternatively by classical and quantum mechanics. An even simpler idea was proposed by Thiele and Stone [10]. They suggested that the von Neumann entropy of the time-averaged density matrix (minus the entropy connected with the preparation of the initial state) can measure quantum chaos. The entropy point of view was taken also in a more recent paper by Słomczyński and Życzkowski [11]. They considered a pair consisting of a quantum system and a measuring apparatus and proposed a new definition of entropy to measure chaotic behavior of such a coupled system. According to them “the approach linking chaos with the unpredictability of the measurement outcomes is the right one in the quantum case”.

On the other hand, papers concerning a possible generalization of the notion of Lyapunov exponents have appeared. Because there is no quantum analog of the classical trajectory, the starting point has to be different. Perron–Frobenius operators acting on the space of densities provide a natural frame for the construction of the quantum counterpart of classical characteristic exponents. Such a construction has been carried out, for example, in [12,13].

In the present paper we will not discuss this interesting subject in general. Leaving aside the problem of existence and definition of an intrinsic quantum chaos, we adopt the point of view of [11] and restrict ourselves to the case of a quantum system interacting with a measuring device, or, more broadly, to quantum open systems. Because, as was shown by Graham [14,15], in a class of such systems the sensitive dependence of some expectation values on initial conditions remains, and the limits $t \to \infty$, $\hbar \to 0$ may commute for a dissipative dynamics, so the notion of chaotic behavior seems to make sense there. Moreover, in the presence of dissipation, coherence effects degrade and give way to an incoherent dynamics closer to the classical behavior. One of the attempts in this direction was the investigation of properties of the quantum survival probability function in open systems [16]. For the following master equation:

$$\dot{\rho} = -\mathrm{i}[H, \rho] + \gamma([H\rho, H] + [H, \rho H]),$$

the different behavior in the regular and chaotic cases of the quantum survival probability function averaged over initial conditions and Hamiltonian ensembles was demonstrated. For a general discussion of quantum chaos with dissipation see [17]. A more general approach investigating hypercyclicity and chaos in the context of strongly continuous semigroups of bounded linear operators in Banach spaces was proposed in [18]. However, the definition of a chaotic semigroup given there cannot be applied to contractive semigroups such as Perron–Frobenius semigroups acting on the space of densities. In this case, the idea of exactness of the system proved to be fruitful in the description of chaotic Markov semigroups associated with some differential equations [19].

In this paper we generalize the notions of completely mixing and exact systems to the quantum level and propose a quantity, the quantum characteristic exponent λ_q, which, in the classical case, corresponds to the highest order Lyapunov exponent measuring the speed of convergence of an exact classical system to statistical equilibrium. The property $\lambda_q > 0$ selects a subclass of completely mixing systems which we call exponentially mixing. However, contrary to the classical case, exponentially mixing quantum open systems may not imply chaotic behavior. The relation of λ_q to that one proposed by Majewski and Kuna in [13] is also discussed. Finally, these concepts are illustrated by the examples of quantum measurements based on event enhanced quantum theory (EEQT).

2. Classical systems

Classical mechanics deals with trajectories of dynamical systems, whereas the time evolution in quantum mechanics is formulated (in Schrödinger picture) in terms of density matrices. They correspond to integrable positive functions in the commutative case. Therefore, to be closer to the framework of quantum mechanics, we formulate some classical concepts of ergodic theory in terms of densities.

We start with recalling an intimate connection between the behavior of trajectories of a dynamical system and the evolution of its densities. For a general discussion of this point see [19]. We consider mainly the discrete time case. The reformulation of appropriate formulas to the case of continuous time systems is straightforward. Suppose $(X, \mathcal{A}, \mu)$ is a normalized measure space ($\mu(X) = 1$) and $S : X \to X$ a measure preserving transformation of X, that is:

- $\forall A \in \mathcal{A} \quad S^{-1}(A) \in \mathcal{A}$,
- $\forall A \in \mathcal{A} \quad \mu(S^{-1}(A)) = \mu(A)$.

Having defined S we choose an initial point $x_0 \in X$ and observe its trajectory $(x_0, S(x_0), S^2(x_0), \dots)$. The chaotic behavior of the system can be recognized by a high sensitivity of the trajectory with respect to a slight change of the initial state. It means that if we start with a set of initial conditions located in a small region then, after a large number of iterations of the transformation S, the points fill completely the whole space X. More precisely, for $S^n(A) \in \mathcal{A}$

$$\lim_{n\to\infty} \mu(S^n(A)) = 1 \quad \forall A \in \mathcal{A}, \quad \mu(A) > 0. \tag{1}$$

A transformation satisfying (1) is called *exact*. Notice, that invertible transformations cannot be exact. A typical example of such a behavior is the r-adic transformation $S : [0, 1) \to [0, 1)$ given by $S(x) = rx \pmod 1$, $r = 2, 3, \dots$.

The behavior of trajectories can be described in terms of densities, i.e. nonnegative, integrable functions on X with integral being equal to one. The space of densities will be denoted by D:

$$D = \{f \in L^1(X, \mathcal{A}, \mu) : f \geq 0 \text{ a.e.}, \quad \|f\|_1 = 1\}.$$

It is well known that densities and trajectories are related in the following sense.

Theorem 2.1 ([19]). *S is exact if and only if for all $f \in D$*

$$\lim_{n\to\infty} \|P^n f - \mathbf{1}\|_1 = 0,$$

where P is the Perron–Frobenius operator corresponding to the transformation S, $P : L^1(X, \mathcal{A}, \mu) \to L^1(X, \mathcal{A}, \mu)$, given by

$$\int_A (Pf)(x)\, \mathrm{d}\mu(x) = \int_{S^{-1}(A)} f(x)\, \mathrm{d}\mu(x) \quad \forall A \in \mathcal{A}.$$

For example, in the case of r-adic transformation and for the density $f(x) = 2x$, we have

$$P^n f(x) = \frac{1}{r^n} \sum_{i=0}^{r^n-1} 2\left(\frac{x+i}{r^n}\right) = \frac{2x}{r^n} + \frac{r^n - 1}{r^n},$$

and so $\|P^n f - \mathbf{1}\|_1 = 1/2r^n$.

The concept of entropy gives a useful condition for a system to be exact. Let

$$H(f) = \int_X \eta(f(x))\, \mathrm{d}\mu(x), \tag{2}$$

where $\eta(x) = -x \log x$ for $x > 0$ and $\eta(0) = 0$. Then $\lim_{n\to\infty} H(P^n f) = 0$ for all bounded $f \in D$ implies that system S is exact. Observe that 0 is the maximum value of the entropy H. If the measure μ is finite but not normalized, then the above theorem should be modified as follows. If

$$\lim_{n\to\infty} H(P^n f) = H_{\max} = H\left(\frac{1}{\mu(X)} \cdot \mathbf{1}\right) \tag{3}$$

then S is exact. If $\mu(X) = \infty$, then there is no constant density and we replace the condition (3) by the following one:

$$\lim_{n\to\infty} H(P^n f | P^n g) = 0 \quad \text{for} \quad f, g \in D. \tag{4}$$

Here $H(f|g)$ is the relative entropy defined by

$$H(f|g) = \int_X f(x) \log \frac{f(x)}{g(x)} \, \mathrm{d}\mu(x) \quad \text{if } \operatorname{supp} f \subset \operatorname{supp} g.$$

It has the following properties. $H(f|g) \ge 0$ and $H(Pf|Pg) \le H(f|g)$. Observe that, if there is a constant invariant density, then (4) implies (3). In fact, putting $g = (1/\mu(X)) \cdot \mathbf{1}$ we get

$$H(P^n f | P^n g) = \left[-H(P^n f) + H\left(\frac{1}{\mu(X)} \cdot \mathbf{1}\right)\right] \to 0.$$

We take conditions (3) or (4) as qualitative indicators of the chaotic behavior of dynamical systems. In general, condition (4) gives less than (3) (see [20]):

$$\lim_{n\to\infty} H(P^n f | P^n g) = 0 \Rightarrow \lim_{n\to\infty} \|P^n f - P^n g\|_1 = 0.$$

Such a system, in which $\|P^n f - P^n g\|_1 \to 0$, $n \to \infty$, for any two densities, is called *completely mixing*. It is clear that every exact system is completely mixing. The converse is true if there is an invariant density for the Perron–Frobenius operator P. However, the heat equation generates a semigroup which is completely mixing but has no invariant density what shows that the converse does not hold in general.

To describe quantitatively the chaotic behavior one can use the Lyapunov characteristic exponents. The Lyapunov exponents of a given trajectory characterize the mean exponential rate of divergence of nearby trajectories. In m-dimensional phase space there are m (possibly nondistinct) characteristic exponents

$$\sigma_1(x_0) \ge \sigma_2(x_0) \ge \cdots \ge \sigma_m(x_0).$$

The largest one is defined by

$$\sigma_1(x_0) = \lim_{n\to\infty} \limsup_{d(x_1,x_0)\to 0} \frac{1}{n} \log \frac{d(S^n x_1, S^n x_0)}{d(x_1, x_0)},$$

where $d(x, y)$ is a metric on the phase space X. For example, in the r-adic case $\sigma(x_0) = \log r$ and does not depend on the initial point x_0.

Higher order exponents can be also defined [21]. The mean exponential growth of the phase space volume given by

$$\sigma^{(m)}(x_0) = \lim_{n\to\infty} \frac{1}{n} \log \frac{|S^n V(x_0)|}{|V(x_0)|} = \sum_{i=1}^{m} \sigma_i(x_0), \tag{5}$$

where $V(x_0)$ is an m-dimensional parallelepiped with one vertex placed in x_0, is of particular interest. In view of Theorem 2.1, the growth of the phase space volume corresponds to decrease of any density to the constant density. Therefore, for completely mixing systems we define another quantity

$$\lambda(f_0) = \liminf_{\|f-f_0\|_1\to 0} \lim_{n\to\infty} \left[-\frac{1}{n} \log \frac{\|P^n f - P^n f_0\|_1}{\|f - f_0\|_1} \right] \tag{6}$$

for $f_0, f \in D_0$, D_0 being some appropriately chosen dense subspace in D. Observe that for measure preserving flows both $\sigma^m(x_0)$ and $\lambda(f_0)$ are equal to zero. We now discuss the elements of the above formula more precisely. The minus sign reflects the property that for fixed f and f_0 $\|P^n f - P^n f_0\|_1 \to 0$ when $n \to \infty$. We have put the minus sign because our objective here is to replace characteristic exponents, which are defined in terms of trajectories, by a quantity expressed in terms of densities in such a way that they would measure the same property of the given dynamics, and hence coincide for simple one-dimensional systems. As will be shown in Proposition 2.2, in the r-adic case the Lyapunov characteristic exponent $\sigma(x_0) = \log r$ for any $x_0 \in (0, 1)$, indeed equals to our quantity $\lambda(\mathbf{1})$. Therefore, although this minus sign may seem to be artificial, it is necessary in order to describe the same feature of the dynamics. Moreover, we use the lim inf expression because for some densities, even very regular ones, the action of the Perron–Frobenius operator may produce a constant density in a finite number of iterations, what would result in an infinite limit. The restriction of D to some subspace is a crucial point. It is known from the analysis of mixing systems that the decay of correlations depends on the functions chosen. As was shown in [22], in the case of a chaotic area preserving map on the torus, there are examples of functions for which the correlations decay faster than exponentially or only algebraically. Hence the rate of decay is sensitive to the choice of functions, and so the restriction to a subset of “nice” functions is unavoidable. Finally, observe that formula (6) is equivalent to

$$\lambda(f_0) = \inf_{f\in D_0, f\neq f_0} \lim_{n\to\infty} \left[-\frac{1}{n} \log \frac{\|P^n f - P^n f_0\|_1}{\|f - f_0\|_1} \right].$$

In fact, if f_k is an arbitrary sequence from D_0, then the sequence $g_k = f_0 + (f_k - f_0)/k$ satisfies $\|g_k - f_0\|_1 \to 0$ and

$$\frac{\|P^n g_k - P^n f_0\|_1}{\|g_k - f_0\|_1} = \frac{\|P^n f_k - P^n f_0\|_1}{\|f_k - f_0\|_1}.$$

Moreover, because we calculate $n \to \infty$ limit first, the normalization can be dropped, and so

$$\lambda(f_0) = \inf_{f\in D_0, f\neq f_0} \lim_{n\to\infty} \left[-\frac{1}{n} \log \|P^n f - P^n f_0\|_1 \right]. \tag{7}$$

To show that formula (6) is indeed related with the Lyapunov exponents we calculate $\lambda(\mathbf{1})$ in the r-adic case. At first we rescaled the interval $[0, 1)$ into $[0, 2\pi)$, replace $\mathrm{d}x$ by $\mathrm{d}x/2\pi$ and define $S(x) = rx (\mathrm{mod}\, 2\pi)$. Hence X can be seen as the circle S^1.

Proposition 2.2. *Let* $D_0 = D \cap C^1(S^1)$. *Then* $\lambda(\mathbf{1}) = \log r$.

Proof. Let $\hat{f}$ denote the Fourier transform of a function $f \in D_0$, $f \neq f_0$, i.e.

$$\hat{f} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kx} f(x)\, \mathrm{d}x, \quad k \in \mathbf{Z}.$$

Let P be the Perron–Frobenius operator corresponding to S. Because

$$Pf(x) = \frac{1}{r} \sum_{j=0}^{r-1} f\left(\frac{x}{r} + \frac{2\pi j}{r} \right),$$

so

$$(Pf)^\wedge(k) = \frac{1}{2\pi r}\sum_{j=0}^{r-1}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kx} f\left(\frac{x}{r}+\frac{2\pi j}{r}\right)\,\mathrm{d}x = \frac{1}{2\pi}\sum_{j=0}^{r-1}\int_{2\pi j/r}^{2\pi(j+1)/r} \mathrm{e}^{-\mathrm{i}k(ry-2\pi j)} f(y)\,\mathrm{d}y = \hat{f}(kr),$$

and $(P^n f)^\wedge(k) = \hat{f}(kr^n)$. Hence

$$\|P^n f - \mathbf{1}\|_{L^1} \le \|P^n f - \mathbf{1}\|_{L^2} = \|(P^n f)^\wedge - \hat{\mathbf{1}}\|_{l^2} = \left[2\sum_{k=1}^{\infty}|\hat{f}(kr^n)|^2\right]^{1/2}. \tag{8}$$

On the other hand

$$\|f - \mathbf{1}\|_{L^1} \ge \|\hat{f} - \hat{\mathbf{1}}\|_{l^\infty} = \sup_{k\ge 1}|\hat{f}(k)|. \tag{9}$$

Because $\lim_{k\to\infty}|\hat{f}(k)| = 0$, so $\|\hat{f} - \hat{\mathbf{1}}\|_{l^\infty} = |\hat{f}(m)|$ for some natural m. Since f' is continuous, hence integrable, so $\lim_{k\to\infty} k|\hat{f}(k)| = 0$. It implies that there exists k_0 such that for any $k \ge k_0$ we have $k|\hat{f}(k)| \le |\hat{f}(m)|$. Let us choose n such that $r^n \ge k_0$. Then

$$\left[2\sum_{k=1}^{\infty}|\hat{f}(kr^n)|^2\right]^{1/2} \le \frac{A|\hat{f}(m)|}{r^n}, \quad \text{where } A = \left[2\sum_{k=1}^{\infty}\frac{1}{k^2}\right]^{1/2}. \tag{10}$$

Combining (8)–(10) we arrive at

$$\frac{\|P^n f - \mathbf{1}\|_{L^1}}{\|f - \mathbf{1}\|_{L^1}} \le \frac{A}{r^n}.$$

Hence $\lambda(\mathbf{1}) \ge \log r$. To finish the proof one only has to find a sequence of densities f_k such that $\lim_{k\to\infty}\|f_k - \mathbf{1}\|_1 = 0$ and

$$\lim_{k\to\infty}\lim_{n\to\infty}\left[-\frac{1}{n}\log\frac{\|P^n f_k - \mathbf{1}\|_1}{\|f_k - \mathbf{1}\|_1}\right] = \log r.$$

It may be easily checked that the sequence $f_k(x) = 1 + (1/k\pi)(x-\pi)$, $k \in \mathbf{N}$, fulfills the required conditions. □

Completely mixing systems in which $\lambda(f) > 0$ for some $f \in D$ will be called *exponentially mixing*. As the example of r-adic transformation shows they are closely related to chaotic systems. Having discussed, the signatures of the chaotic behavior in terms of probability densities for classical systems we now consider the quantum case.

3. Quantum open systems

As we mentioned in Section 1 we discuss only dissipative quantum systems. Dissipation in quantum theory appears by the coupling of a system to a reservoir. The tracing over classical variables leads to the reduced dynamics. Using certain approximation technics [23] it can be shown that the reduced dynamics is given by a Markov semigroup acting on the set of reduced density matrices. The Hamiltonian evolution equation is replaced by the master equation. Such an evolution does not conserve energy and maps pure states into mixed states. For technical reason we confine the discussion to N-level quantum systems. Hence a dynamical semigroup $T_t : M_{N\times N}(\mathbf{C}) \to M_{N\times N}(\mathbf{C})$ maps positive definite matrices into positive definite matrices and preserves the trace tr. As the composition of a Hamiltonian evolution and a conditional expectation T_t is also completely positive.

As the first indicator of the mixing property we consider the relative von Neumann entropy $H(\rho|\sigma)$ defined by

$$H(\rho|\sigma) = \mathrm{tr}(\rho\log\rho - \rho\log\sigma)$$

for any density matrices such that $\mathrm{supp}\,\rho \subset \mathrm{supp}\,\sigma$. It is well known that H is nonincreasing with respect to T_t, i.e. $H(T_t\rho|T_t\sigma) \le H(\rho|\sigma)$. Following Eq. (4) we define a system to be completely mixing if

$$\lim_{t\to\infty} H(T_t\rho|T_t\sigma) = 0 \tag{11}$$

holds. If a totally mixed state $(1/N)\mathbf{1}$ is T_t-invariant, then (11) implies that $H(T_t\rho) \to H_{\max}$. Since in the classical case it leads to the exactness of a system we take this property as the definition of exactness in the quantum case as well. Let us notice that not every dissipative system is completely mixing. For example, for the following master equation:

$$\dot{\rho} = \sigma_1\rho\sigma_1 - \rho,$$

where ρ is a 2×2 density matrix and σ_1 denotes the first Pauli matrix, the $t\to\infty$ limit of the relative entropy of two neighboring one-dimensional projectors

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix},$$

$\phi \in (0, \pi/4)$, is given by $\lim_{t\to\infty} H(T_t e_1|T_t e_2) = -\log\cos 2\phi \ne 0$. Let us also point out that Hamiltonian dynamics cannot be completely mixing since in this case the relative entropy is constant in time.

Because $H(\rho|\sigma) \ge \frac{1}{2}\|\rho-\sigma\|_1^2$, where $\|A\|_1 = \mathrm{tr}|A|$, so

$$\lim_{t\to\infty} H(T_t\rho|T_t\sigma) = 0 \Rightarrow \lim_{t\to\infty} \|T_t\rho - T_t\sigma\|_1 = 0. \tag{12}$$

Therefore, for a completely mixing system one can ask the question how fast the limit $\|T_t\rho - T_t\sigma\|_1$, $t\to\infty$, tends to zero. Guided by the classical experience (compare formula (7)) we propose the following quantity to measure the exponential rate of convergence:

$$\lambda_{\mathrm{q}}(\rho) = \inf_{\sigma\ne\rho} \lim_{t\to\infty} \left[-\frac{1}{t}\log\|T_t\rho - T_t\sigma\|_1\right], \tag{13}$$

and call it the quantum characteristic exponent. It is similar to that one proposed by Majewski and Kuna [13] (see also [24]). Indeed, $\lambda_{\mathrm{q}}(\rho) = \inf_{\sigma\ne\rho}\lambda_{\mathrm{q}}(\rho;\sigma)$ and $\lambda_{\mathrm{q}}(\rho;\sigma)$ coincides (up to the minus sign) with their formula if we notice that T_t is linear and calculate the corresponding derivative in a tangent direction to the space of all density matrices, i.e. for δx such that $\mathrm{tr}\,\delta x = 0$. As in the classical case completely mixing systems with $\lambda_{\mathrm{q}}(\rho) > 0$, for some ρ, will be called exponentially mixing. Finally, let us point out that if a completely mixing system has a stationary density matrix ρ_0, that is $T_t(\rho_0) = \rho_0$ for all $t \ge 0$, then ρ_0 is unique and λ_{q} does not depend on the choice of an initial statistical state ρ. In other words $\lambda_{\mathrm{q}}(\rho) = \lambda_{\mathrm{q}}(\rho_0)$, where

$$\lambda_{\mathrm{q}}(\rho_0) = \inf_{\sigma\ne\rho_0} \lim_{t\to\infty} \left[-\frac{1}{t}\log\|T_t\sigma - \rho_0\|_1\right].$$

It is also worth noting that the entropy production rate for an evolving reduced density matrix of an open quantum system proved to be a fruitful indicator of the character of dynamical behavior. As was shown in [25], the classical unpredictability corresponds to the rapid entropy production on the Lyapunov time-scale in quantum analogs of classical systems which exhibit chaotic behavior. By contrast, dynamics of analogs of integrable systems leads to a much slower evolution towards a dynamical equilibrium of the system. Therefore, the rate of increase of entropy or, more generally, of decrease of relative entropy, can indeed distinguish between chaotic and regular quantum evolutions, in a similar way as in the classical case.

4. Continuous quantum measurements

EEQT is a minimal extension of quantum theory that accounts for events [26,27]. It postulates that a quantum measurement process is a particular coupling between a quantum and classical system. The time evolution of such a hybrid system is determined by a completely positive semigroup of linear operators on the space of density matrices of the total system. Moreover, it provides the interpretation of the continuous evolution of ensembles in terms of a piecewise deterministic process with values in the pure state space of the total system. The process, after averaging, reproduces the dynamical equation for statistical states. Such a process turned out to be unique [28], what allows for deducing the algorithm generating sample histories of an individual quantum system [29]. The sensitive dependence of sample paths of the associated process on the initial conditions together with fractal structure of the limit set of a sample path indicate chaotic properties of the corresponding dynamical semigroup. In this way they constitute the same counterpart for the evolution of density matrices as trajectories do for density functions in the classical case of $L^1(X)$ spaces.

Let us now briefly describe the framework of EEQT. Suppose that possible states of the measuring apparatus C form a discrete set labeled by $\alpha = 1, 2, \ldots, m$. The algebra of observables of C is the algebra $\mathcal{A}_\mathrm{c}$ of complex finite sequences $f_\alpha, \alpha = 1, \ldots, m$. For technical reason we use Hilbert space language even for the description of the classical system. Thus we introduce an m-dimensional Hilbert space $\mathcal{H}_\mathrm{c}$ with a fixed basis, and we realize $\mathcal{A}_\mathrm{c}$ as the algebra of diagonal matrices $F = \mathrm{diag}(f_1, \ldots, f_m)$. Statistical states of C are then diagonal density matrices $\mathrm{diag}(p_1, \ldots, p_m)$, and pure states of C are vectors of the fixed basis of $\mathcal{H}_\mathrm{c}$. Suppose, further, Q is the quantum system whose bounded observables form the algebra $\mathcal{A}_\mathrm{q}$ of bounded operators on a Hilbert space $\mathcal{H}_\mathrm{q}$. We assume that $\mathcal{H}_\mathrm{q}$ is finite-dimensional. Pure states of Q form a complex projective space $\mathbf{C}P(\mathcal{H}_\mathrm{q})$ over $\mathcal{H}_\mathrm{q}$. Statistical states of Q are given by nonnegative density matrices $\hat{\rho}$, with $\mathrm{tr}(\hat{\rho}) = 1$. The algebra $\mathcal{A}_\mathrm{T}$ of observables of the total system $T = Q \times C$ is given by the tensor product of algebras of observables of Q and C: $\mathcal{A}_\mathrm{T} = \mathcal{A}_\mathrm{q} \otimes \mathcal{A}_\mathrm{c}$. It acts on the tensor product $\mathcal{H}_\mathrm{q} \otimes \mathcal{H}_\mathrm{c} = \oplus_{\alpha=1}^{m} \mathcal{H}_\alpha$, where $\mathcal{H}_\alpha \approx \mathcal{H}_\mathrm{q}$. Thus $\mathcal{A}_\mathrm{T}$ can be thought of as algebra of diagonal $m \times m$ matrices $A = (a_{\alpha\beta})$, whose entries are quantum operators: $a_{\alpha\alpha} \in \mathcal{A}_\mathrm{q}, a_{\alpha\beta} = 0$ for $\alpha \neq \beta$. Statistical states of $Q \times C$ are given by $m \times m$ diagonal matrices $\rho = \mathrm{diag}(\rho_1, \ldots, \rho_m)$ whose entries are positive operators on $\mathcal{H}_\mathrm{q}$, with the normalization $\mathrm{tr}(\rho) = \sum_\alpha \mathrm{tr}(\rho_\alpha) = 1$. Duality between observables and states is provided by the expectation value $\langle A \rangle_\rho = \sum_\alpha \mathrm{tr}(A_\alpha \rho_\alpha)$. The coupling of Q to C is specified by a matrix $V = (g_{\alpha\beta})$, where $g_{\alpha\beta}$ are linear operators: $g_{\alpha\beta} : \mathcal{H}_\beta \to \mathcal{H}_\alpha$. We assume $g_{\alpha\alpha} = 0$. This condition expresses the simple fact: there is no dissipation without receiving information. The evolution equation for states is given by the Lindblad form

$$\dot{\rho}_\alpha = -\mathrm{i}[H, \rho_\alpha] + \sum_\beta g_{\alpha\beta} \rho_\beta g_{\alpha\beta}^\star - \frac{1}{2}\{\Lambda_\alpha, \rho_\alpha\}, \tag{14}$$

where $\Lambda_\alpha = \sum_\beta g_{\beta\alpha}^\star g_{\beta\alpha}$ and $\{\cdot, \cdot\}$ stands for the anticommutator. We apply now the above scheme to three concrete measurement processes.

4.1. Measurement of noncommuting observables

In this example we model a simultaneous measurement of several noncommuting observables, like different spin projections [30]. In such a case we calculate the quantum exponent λ_q for the reduced dynamics given by tracing out classical parameters, and show that it leads to a chaotic and fractal structure on the space of pure states of the quantum system. An interesting investigation of quantum chaos in open dynamical systems, in which absorption leads to the appearance of a fractal set in the underlying classical phase space was presented in [31]. In the quantum case fractal structure appears in the Husimi functions of eigenstates of a nonunitary evolution operator. However, there is no apparent relation between the approach taken in Ref. [31] and our discussion of quantum fractal sets (on the set of pure states) arising from a continuous simultaneous observation of several noncommuting observables.

The measuring apparatus consists of four yes–no polarizers corresponding to spin directions $\vec{n}_i$, $i = 1, \dots, 4$, arranged at the vortices of a regular tetrahedron

$$\vec{n}_1 = (1, 0, 0), \qquad \vec{n}_2 = \left(-\frac{1}{3}, 0, \frac{2\sqrt{2}}{3}\right), \qquad \vec{n}_3 = \left(-\frac{1}{3}, \sqrt{\frac{2}{3}}, -\frac{\sqrt{2}}{3}\right), \qquad \vec{n}_4 = \left(-\frac{1}{3}, -\sqrt{\frac{2}{3}}, -\frac{\sqrt{2}}{3}\right).$$

Because the quantum system is a two-state system, so the quantum algebra is given by 2×2 complex matrices. We assume it evolves according to Hamiltonian $H = \frac{1}{2}\omega\sigma_3$, $\omega \geq 0$, where $\sigma_1, \sigma_2, \sigma_3$ denote Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The coupling is specified by choosing four operators a_i which correspond to four vectors $\vec{n}_i$

$$a_i = \tfrac{1}{2}(I + \alpha\vec{n}_i \cdot \vec{\sigma}), \tag{15}$$

where $\alpha \in [0, 1]$. Notice, that for $\alpha = 1$, a_i are projection operators. The evolution equation is given by

$$\dot{\rho}_i = -\mathrm{i}[H, \rho_i] + \kappa \sum_j a_i \rho_j a_i - \kappa a^2 \rho_i, \tag{16}$$

what implies the following master equation for the reduced density matrix $\hat{\rho} = \sum_i \rho_i$:

$$\dot{\hat{\rho}} = -\mathrm{i}[H, \hat{\rho}] + \kappa \sum_i a_i \hat{\rho} a_i - \kappa a^2 \hat{\rho}, \tag{17}$$

where $\kappa \geq 0$ is the coupling constant and $a^2 = \sum_i a_i^2$. Eq. (17) implies that a projection valued measure corresponding to a sharp measurement has been replaced by a positive operator valued measure. Clearly, the totally mixed state $\frac{1}{2}I$ is stationary. At first we show that the dynamics given by Eq. (17) leads to an exponentially mixing system.

To solve (17) we assume the initial state is a pure one, i.e. $\hat{\rho}(0) = \frac{1}{2}(I + \vec{m}_0 \cdot \vec{\sigma})$, $\vec{m}_0 = (c_1, c_2, c_3)$ with $c_1^2 + c_2^2 + c_3^2 = 1$. Because the dynamical semigroup preserves positivity and trace, so a general solution is of the form $\hat{\rho}(t) = \frac{1}{2}(I + \vec{m}(t) \cdot \vec{\sigma})$ with $\|\vec{m}(t)\| \leq 1$. By direct calculations we obtain the following system of differential equations:

$$\dot{m}_1 = -\omega m_2 - \tfrac{4}{3}\kappa\alpha^2 m_1, \qquad \dot{m}_2 = \omega m_1 - \tfrac{4}{3}\kappa\alpha^2 m_2, \qquad \dot{m}_3 = -\tfrac{4}{3}\kappa\alpha^2 m_3,$$

and so

$$m_1(t) = (-c_2 \sin\omega t + c_1 \cos\omega t)\,\mathrm{e}^{-(4/3)\kappa\alpha^2 t},$$
$$m_2(t) = (c_1 \sin\omega t + c_2 \cos\omega t)\,\mathrm{e}^{-(4/3)\kappa\alpha^2 t},$$
$$m_3(t) = c_3\,\mathrm{e}^{-(4/3)\kappa\alpha^2 t}$$

is the solution with the initial condition $\vec{m}(0) = \vec{m}_0$. Because

$$\|\hat{\rho}(t) - \tfrac{1}{2}I\|_1 = \tfrac{1}{2}\|\vec{m}(t) \cdot \vec{\sigma}\|_1 = \|\vec{m}(t)\| = \mathrm{e}^{-(4/3)\kappa\alpha^2 t},$$

and the convergence above does not depend on the choice of $\vec{m}_0$, so

$$\lambda_q\left(\frac{1}{2}I\right) = \inf_{\hat{\rho}(0)\neq(1/2)I} \lim_{t\to\infty}\left[-\frac{1}{t}\log\|\hat{\rho}(t)-\frac{1}{2}I\|_1\right] = \frac{4}{3}\kappa\alpha^2. \tag{18}$$

We describe now a sample path of the process associated with the dynamical semigroup determined by Eq. (17). Assume that at time $t = 0$ the quantum system is in the state $\vec{r}(0) \in S^2$ (we identify here the space of pure states of the quantum system with a two-dimensional sphere S^2 with radius 1). Under the time evolution it evolves to the state $\vec{r}(t)$ which is given by the rotation of $\vec{r}(0)$ with respect to z-axis. Then, at time t_1 a jump occurs. The time rate of jumps is governed by a homogeneous Poisson process with rate κ. When jumping $\vec{r}(t)$ moves to

$$\vec{r}_i = \frac{(1-\alpha^2)\vec{r}(t) + 2\alpha(1+\alpha\vec{r}(t)\cdot\vec{n}_i)\vec{n}_i}{1+\alpha^2+2\alpha\vec{r}(t)\cdot\vec{n}_i}$$

with probability

$$p_i(\vec{r}(t)) = \frac{1+\alpha^2+2\alpha\vec{r}(t)\cdot\vec{n}_i}{4(1+\alpha^2)}.$$

And the process starts again. The iterations lead to a self-similar structure with sensitive dependence on the initial state. Figs. 1–3 depict sample paths of the quantum particle for different values of α in the case when $H = 0$ and $\kappa = 1$. Fig. 4a–c presents $\alpha = 0.7$ case together with its zooms 2× and 4×.

Numerical simulations performed for $H = 0$ and $\kappa = 1$ [32] show that when α increases from 0.75 to 0.95, then the Hausdorff dimension of the limit set decreases from 1.44 to 0.49. It is worth noting that at the same time the quantum characteristic exponent λ_q increases as α^2 (see formula (18)). This suggests an intimate relation between λ_q and the Hausdorff dimension of the limit set. In the classical case such a relation was established in [33,34].

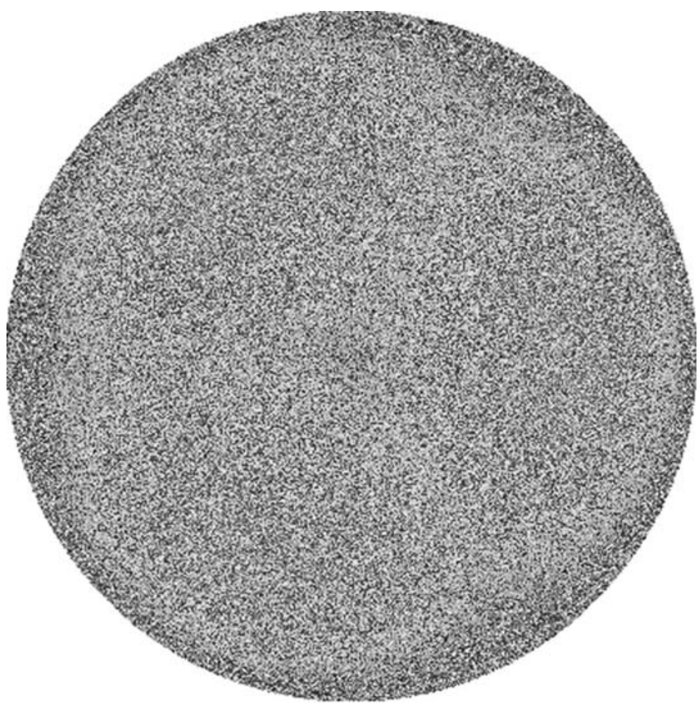

Fig. 1. Quantum state trajectory for $\alpha = 0.2$, 10 000 000 points.

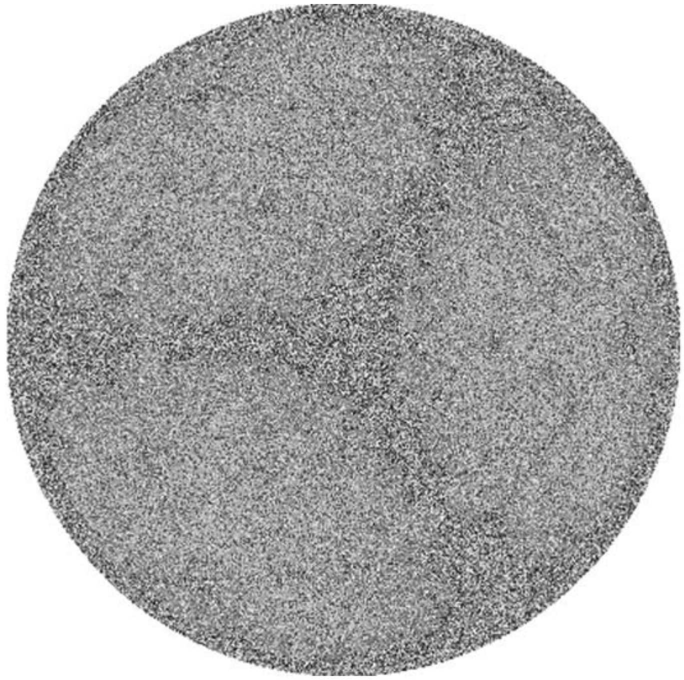

Fig. 2. Quantum state trajectory for $\alpha = 0.4$, 10 000 000 points.

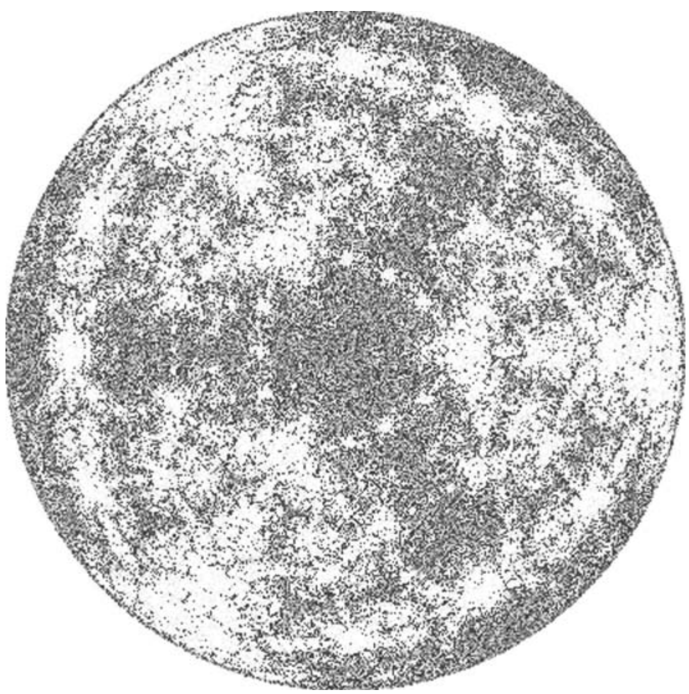

Fig. 3. Quantum state trajectory for $\alpha = 0.6$, 10 000 000 points.

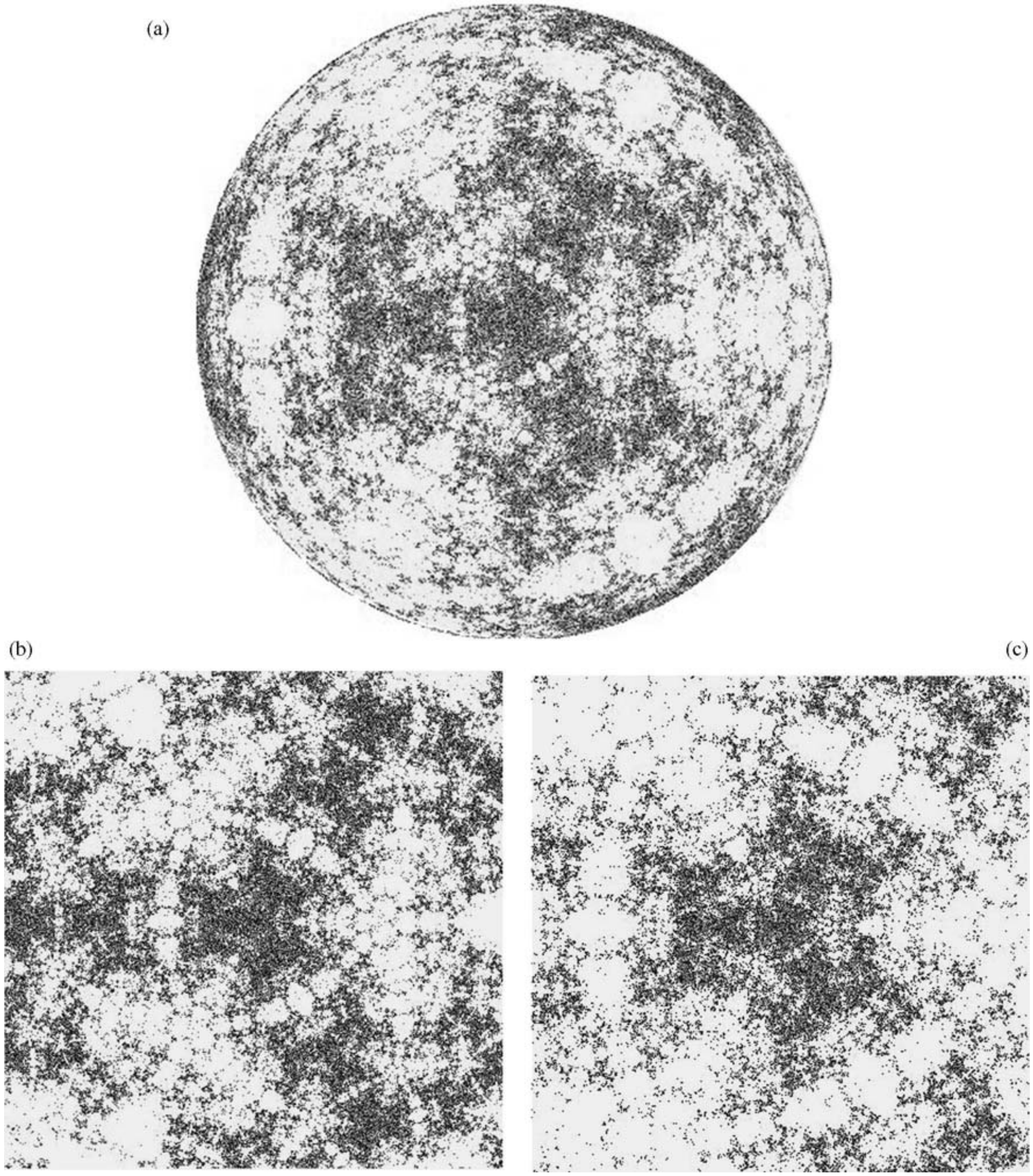

Fig. 4. (a) Quantum state trajectory for $\alpha = 0.7$, 1 000 000 000 points; (b) quantum state trajectory for $\alpha = 0.7$, zoom $= 2\times$, 1 000 000 000 points; (c) quantum state trajectory for $\alpha = 0.7$, zoom $= 4\times$, 1 000 000 000 points.

4.2. Quantum Zeno effect

A phenomenon of keeping a quantum state from evolving by performing a sequence of frequent measurements was discussed by Misra and Sudarshan [35] and named quantum Zeno effect (paradox). Roughly speaking, it means that the constant observation freeze the quantum system in its initial state. Such a situation was described, for example, in [36] for the case of the neutron placed in a static magnetic field and interacting with a device which selects one component of its spin. By this example we demonstrate a nontrivial dependence of the quantum characteristic exponent λ_q on the value of the coupling constant. Because the quantum system is a two-state system, so the quantum algebra is just the algebra of 2×2 complex matrices, while the classical system being an yes–no device consists of two distinct points. The evolution equation for the reduced density matrix $\hat{\rho} \in M_{2\times 2}$, $\hat{\rho} \geq 0$, $\mathrm{tr}\hat{\rho} = 1$, is given by

$$\dot{\hat{\rho}} = -\mathrm{i}[H, \hat{\rho}] + \kappa(e\hat{\rho}e - \tfrac{1}{2}\{e, \hat{\rho}\}), \tag{19}$$

where Hamiltonian $H = \frac{1}{2}\omega\sigma_3$ and the matrix $\mathrm{e} = \frac{1}{2}(I + \sigma_1)$ is a projector onto the first eigenvector of the Pauli matrix σ_1. The coupling constant κ measures the frequency of "checking" whether the quantum system is in the eigenstate of σ_1 or not. Again, as in the previous example, the totally mixed state $\frac{1}{2}I$ is stationary. Now Eq. (19) leads to the following system of differential equations

$$\dot{x}_1 = -\omega x_2, \qquad \dot{x}_2 = \omega x_1 - \tfrac{1}{2}\kappa x_2, \qquad \dot{x}_3 = -\tfrac{1}{2}\kappa x_3, \tag{20}$$

where $\hat{\rho}(t) = \frac{1}{2}(I + \vec{x} \cdot \vec{\sigma})$. For a given initial condition $\hat{\rho}(0)$ it is a routine exercise to solve (20), and hence to obtain the following result.

Theorem 4.1. *Let* $\kappa = 4\alpha\omega$, $\alpha \geq 0$. *Then*

$$\lambda_q\left(\frac{1}{2}I\right) = \begin{cases} \omega \cdot \alpha & \text{if} \quad \alpha \in [0, 1], \\ \omega \cdot \dfrac{1}{\alpha + \sqrt{\alpha^2 - 1}} & \text{if} \quad \alpha > 1. \end{cases}$$

It follows that the degree of mixing (measured in terms of λ_q) increases from zero to the maximum value ω when the frequency of measurements increases to the critical value 4ω. The further increase of number of observations results in the decrease of λ_q to 0 like $1/\kappa$. Hence we can conclude that the interaction with the measuring apparatus makes the evolution of the quantum system exact (and so mixing), but constant observation suppresses this property.

4.3. Two-level atom driven by a laser

Let us consider the fluorescent photons emitted by a single, two-level atom that is coherently driven by an external electromagnetic field. It is known that the quantum system evolves from the ground state in a dissipative way. When a photoelectric count is recorded by a photoelectric detector (we assume the detector efficiency to be equal to one), the atom returns to the ground state with the emission of one photon. The master equation for such a system is given by the following formula:

$$\dot{\rho} = -\mathrm{i}[H, \rho] + \gamma A\rho A^* - \tfrac{1}{2}\gamma\{A^*A, \rho\}, \tag{21}$$

where

$$H = -\frac{\Omega}{2}\sigma_1, \qquad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

Ω is Rabi frequency and $\gamma > 0$ is the relaxation rate. Because the unique stationary state ρ_0 is given by $\rho_0 = \frac{1}{2}(I + n_2\sigma_2 + n_3\sigma_3)$, where

$$n_2 = \frac{2\Omega\gamma}{4\omega^2 + \gamma^2}, \qquad n_3 = \frac{-\gamma^2}{4\Omega^2 + \gamma^2},$$

so the system is not exact. However, it is exponentially mixing. Solving Eq. (21) and using formula (13) we obtain that $\lambda_q(\rho_0) = \frac{1}{2}\gamma$. Hence, although $\lambda_q(\rho_0) > 0$ it would be difficult to consider this system chaotic. The property of being exponentially mixing is a new feature in quantum open systems.

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