COVARIANT REPRESENTATIONS OF THE CAUSAL LOGIC

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ABSTRACT. An algebra is found whose representations are in a one-to-one correspondence with covariant, current generated, representations of the causal logic.

Whereas quantum mechanics of a free Galilean extended particle is described by a covariant representation of the Galilean logic of space-time (see [1]), a relativistic object should be described by a covariant representation of the causal logic of Minkowski space [2]. Since a structure of relativistic causal logic is much less trivial than the Galilean one, it is not so easy to classify its covariant representations. However, the results of [3] indicate that a rich class (if not all of interest) of representations can be generated by conserved currents. We discuss this class below.

Let $\mathcal{H}$ be a Hilbert space of a relativistic, causal system, and let U be a unitary possibly projective representation of the Poincaré group in $\mathcal{H}$, with selfadjoint generators M_r, N_r, P_r, P_0. In particular,

$$[N_r, P_0] = iP_r, \qquad [N_r, P_s] = i\delta_{rs}P_0, \text{ etc.} \tag{1}$$

We also assume that there is given an operator-valued current $j_\mu(x)$, covariant and conserved ($\partial^\mu j_\mu = 0$). If j is to generate a representation of the causal logic, we must also assume that $j_0(0, x) = \rho(x)$ is a density of a spectral measure on R^3, so that $X^r = \int x^r \rho(x) \mathrm{d}^3 x$ are position operators at time $x^0 = 0$. From covariance of the current under translations we then get $[X^r, P_s] = i\delta^r_s$. On the other hand, covariance under Lorentz rotations gives

$$[N_r, \rho(\mathbf{0})] = ij_r(0)$$

and

$$[N_r, j_s(0)] = i\delta_{rs}\rho(\mathbf{0}).$$

Now, since

$$j(x) = \exp(-iPx)j(0)\exp(iPx),$$

and

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$$\exp(iPx)N\exp(-iPx) = N + P_0 \boldsymbol{x},$$

it follows that

$$[N_r, \rho(x)] = ij_r(x) + [P_0, \rho(x)]\, x_r, \tag{2}$$

$$[N_r, j_s(x)] = i\delta_{rs}\rho(x) + [P_0, j_s(x)]\, x_r, \tag{3}$$

and conservation of j implies

$$[P_0, \rho(x)] = [P^s, j_s(x)]\,. \tag{4}$$

To analyse these commutation relations it is convenient to use Imprimitivity Theorem, and realize $\mathcal{H}$ as $L^2(R^3, \mathcal{K}, \mathrm{d}^3 p)$ of square integrable functions of the momenta, with values in a Hilbert space $\mathcal{K}$ (of internal degrees of freedom), so that P_r are realized as multiplications by arguments, and $X_r = i\partial/\partial p_r$. One can then define spin by

$$m = M - X \times P, \tag{5}$$

so that

$$[m_r, m_s] = i\epsilon_{rst} m_t, \tag{6}$$

and m_r can be considered as acting on $\mathcal{K}$ only. It is also convenient to define n by

$$n = N - \tfrac{1}{2}(P_0 X + X P_0), \tag{7}$$

so that $[n_r, P_s] = 0$, and $n_r = n_r(p)$ are multiplications by Hermitean, $\mathcal{K}$-operator valued functions of the momenta. Substitution of $j_r(x)$ from (2) into (5) implies then that n_r are linear, and P_0 is at most quadratic in the momenta. Making now use of (3) implies that n_r are constant, and P_0 is linear. Therefore

$$P_0 = aP + a, \tag{8}$$

where a_r and a are Hermitean operators in $\mathcal{K}$. Substitution back to the commutation relations of the Poincaré group gives us the following algebra in $\mathcal{K}$ (see [4]):

$$\begin{aligned}
&[m_r, m_s] = i\epsilon_{rst} m_t, && [n_r, n_s] = -i\epsilon_{rst} m_t + \tfrac{1}{4}[a_r, a_s]\,,\\
&[m_r, n_s] = i\epsilon_{rst} n_t, && [n_r, a_s] = i\delta_{rs} - \frac{i}{2}\{a_r, a_s\}\,,\\
&[m_r, a_s] = i\epsilon_{rst} a_t, && [n_r, a] = -\frac{i}{2}\{a_r, a\}\,,\\
&[m_r, a] = 0.
\end{aligned} \tag{9}$$

and satisfying of these relations is also sufficient for (1) – (4) to hold.

Finite-dimensional representations of the algebra (9) are easy to classify. By making use of trace properties one easily gets $n_r = 0$, $m_r = -i\epsilon_{rst}a_s a_t$, and $\{a_r, a_s\} = 2\delta_{rs}$, $\{a_r, a\} = 0$. Since eigenspaces of a^2 are invariant under a_r and a, we can reduce the problem to the case of $a^2 = 0$ or 1. In the first case what remains is a representation of the Clifford algebra of R^3, with just two irreducible representations $a_r = \sigma_r$ and $a_r = -\sigma_r$, where σ_r are the standard Pauli matrices. In the second case we get the Dirac algebra with only one irreducible representation. It follows that every covariant representation of the causal logic, with a finite-dimensional space of internal degrees of freedom, is completely reducible, and the only irreducible representations are those described by the Pauli or Dirac equations. No general theory of infinite-dimensional representations of the algebra (9) exists. A particular class of representations, with commuting velocity components, has been discussed in [4], where the reader can find more details (not without mistakes!).

REFERENCES

1. Cegła, W., and Jadczyk, A., *Rep. Math. Phys.* 9, 377 (1976).
2. Cegła, W., and Jadczyk, A., *Commun. Math. Phys.* 57, 213 (1977).
3. Cegła, W., and Jadczyk, A., 'States on Causal Logic', *Lett. Math. Phys.* 3, 109–112 (1979).
4. Borowiec, A., and Jadczyk, A., 'Relativistic Systems with a Conserved Probability Current', Preprint No 397, University of Wrocław, 1976.

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