Acta Applicandae Mathematicae **51:** 59–92, 1998.

Fundamental Geometric Structures for the Dirac Equation in General Relativity

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(Received: 13 December 1996)

Abstract. We present an axiomatic approach to Dirac's equation in General Relativity based on intrinsically covariant geometric structures. Structure groups and the related principal bundle formulation can be recovered by studying the automorphisms of the theory. Various aspects can be most neatly understood within this context, and a number of questions can be most properly addressed (specifically in view of the formulation of QFT on a curved background). In particular, we clarify the fact that the usual spinor structure can be weakened while retaining all essential physical aspects of the theory.

Mathematics Subject Classifications (1991): 15A66, 53A50, 53B35, 53C07, 53C15, 81R20, 81R25.

Key words: spinors, quantum mechanics on a curved background, connections, Dirac equation, Weyl equations.

Introduction

In physics literature, Dirac's equation is usually introduced and studied on flat Minkowski spacetime [IZ80, Ka61], while spinors have been extensively used within the context of classical General Relativity [HT85, PR84, PR88, Wa84]. Generalization of the quantum theory to curved spacetime is not so popular, perhaps because formulation of QFT on a curved background [BD82, Pr95] encounters severe difficulties and even paradoxes. On the other hand, there exists a rich mathematical literature about spinor structures and the Dirac equation on curved spacetime and general Riemannian manifolds, based essentially on the language of groups and principal bundles [BTu87, Bl81, BLM89, BTr87, Ge68, Ge70].

Despite the abundance of available literature, the nonexpert reader who wishes to understand the basic aspects of the relativistic physics of $\frac{1}{2}$-spin particles – and is not really interested in mathematical generalizations – will be puzzled by the not quite clear and, sometimes, misleading presentations found in the textbooks. Even the initiated may have some difficulties in stating clearly all the precise relations between the various objects appearing in the theory: which is to be viewed as fundamental and which as a derived object?; what does assuming a given object precisely imply from the algebraic and dynamical point of view?; why a charged spinor is not exactly a 'spinor with charge' (there is a subtle involvement of the group $\mathbb{Z}_2$)?, and so on.

This presentation is intended as a setting-up of the fundamental mathematical concepts needed for the Dirac equation in General Relativity. Our strategy is to deal with a set of 'minimal geometric data', namely we present a formulation containing no *distinguished* or *chosen* object devoid of a precise physical interpretation. This is to be contrasted with usual matricial formulations, which tend to mix different levels and different questions. Our language is essentially that of vector bundles and connections on vector bundles. The principal bundle approach is recovered *a-posteriori*, the symmetry group being a group of automorphisms of the assumed structures. Actually, we recognize that principal bundle techniques are invaluable for many purposes, but we also observe that several essential features of the theory of connections, which are commonly attributed to principal bundles, can be formulated with greater generality – and even simplicity – at a more basic level.

At the algebraic level, the fundamental objects are the complex vector bundle $\boldsymbol{W}$ of '4-spinors' over general relativistic spacetime $\boldsymbol{M}$, a Dirac map* γ: $T\boldsymbol{M} \to \mathrm{End}(\boldsymbol{W})$ and a Hermitian 2-form k on $\boldsymbol{W}$, with signature $(2,2)$, commuting with γ (this is essentially the 'Dirac adjoint' map usually denoted by $\psi \mapsto \bar{\psi}$). The group of automorphisms turns out to be a kind of complexified Spin group, which in mathematical works is often denoted by Spin^c (here is the involvement of the group $\mathbb{Z}_2$). Namely, this structure is weaker than the usually assumed spinor structure, but we argue that it is completely sufficient for describing all physical facts. Assuming a proper spinor structure amounts to fixing a 'charge conjugation' $\mathcal{C}$ or, equivalently, a symplectic form ε of a certain type on $\boldsymbol{W}$. At the purely algebraic level, there is no stringent motivation to regard any one of the three objects, k, $\mathcal{C}$ and ε, as more fundamental than the others; if one is fixed, the others are determined up to some factor. But a factor which may change from point to point is a physical field; the connections which preserve our Spin^c structure contain the electromagnetic potential in a natural way, with correct gauge transformations. Fixing $\mathcal{C}$ or ε yields a *global* 1-form A, which is too much. So, by assuming given $\mathcal{C}$ or ε, we would get an unnecessary extra-structure. Similarly, fixing a *positive* Hermitian metric on $\boldsymbol{W}$ is equivalent to fixing an observer, so this is an unnecessary extra-structure too. We also stress that the 2-spinor approach turns out to be completely equivalent to our weakened 4-spinor approach. In particular, note that in $\boldsymbol{W} = \boldsymbol{S} \oplus \overline{\boldsymbol{S}^{\star}}$, where $\boldsymbol{S}$ is the 2-spinor bundle, k is just a natural contraction.

A language which is not based on principal bundles in an essential way, besides being suitable for the kind of clarification we seek, may suggest gen-

* This yields what is also called a 'module of Clifford algebras'.

eralizations which, in the principal bundle context, are hardly attainable. For example, one could allow for k to become a dynamical rather than a fixed and static object. This would mean that the gauge group itself can be a dynamical variable and need not be constant throughout spacetime. In such a case, standard principal bundle techniques will apply only in some regions.

Similarly, it is easy to generalize our formulation in such a way as to allow discussion of spacetimes with a degenerate metric of nonconstant rank and no requirement for the existence of Spin^c structures [CJ96]. Such a formulation of the dynamics may prove to be necessary in those approaches that strive to include quantum fluctuations of the metric. The 'classical' approach is not suitable for this purpose.

Even if an expert may find that we presented no essentially new mathematical results, our consistent scheme is not at all a trivial consequence of known facts. The paper can be seen as a self-consistent alternative introduction to the Dirac equation of $\frac{1}{2}$-spin particles. The reader is supposed to have some familiarity only with the very basic concepts concerning (real and complex) vector bundles and connections, which are briefly recalled in the first section.

PART I: PRELIMINARIES

1. Some Essential Mathematics

In this section we summarize the main prerequisite concepts and notations.

1.1. TANGENT SPACE

The *tangent bundle* of a manifold $\boldsymbol{M}$ will be denoted by $T\boldsymbol{M} \to \boldsymbol{M}$. A local chart (x^λ) of $\boldsymbol{M}$ yields the local chart $(x^\lambda, \dot{x}^\lambda)$ of $T\boldsymbol{M}$, the local basis of vector fields (∂x_λ) and the dual local basis of forms $(\mathrm{d}x^\lambda)$. The *tangent prolongation* of a map f: $\boldsymbol{M} \to \boldsymbol{N}$ is the map Tf: $T\boldsymbol{M} \to T\boldsymbol{N}$ with coordinate expression $Tf = \partial_\lambda f^j d^\lambda \otimes (\partial_j \circ f)$.

A manifold $\boldsymbol{F}$ is said to be *fibred* over the *base space* $\boldsymbol{B}$ if it is equipped with a surjective map p: $\boldsymbol{F} \to \boldsymbol{B}$ whose rank equals the dimension of $\boldsymbol{B}$. A *bundle* is a fibred manifold which can be covered by local trivializations defined on open 'tubelike' subsets.

A chart (x^λ, y^j) of $\boldsymbol{F}$ is said to be *fibred* if the coordinates x^λ depend only on the base space. A fibred chart of $\boldsymbol{F}$ yields the local frame of vector fields $(\partial x_\lambda, \partial y_j)$ and the dual local frame of forms $(\mathrm{d}x^\lambda, \mathrm{d}y^j)$ on $\boldsymbol{F}$. Hence, we also obtain the chart $(x^\lambda, y^i; \dot{x}^\lambda, \dot{y}^i)$ of $T\boldsymbol{F}$.

The *vertical subbundle* $V\boldsymbol{F} \subset T\boldsymbol{F}$ is constituted by all vectors tangent to the fibres and is characterized by the equation $(\dot{x}^\lambda = 0)$. Thus, a vector field X is vertical iff $Tp(X) = 0$, i.e. iff its coordinate expression is $X = X^j \partial y_j$.

1.2. JET SPACE

The *jet space at* $x \in \boldsymbol{B}$ of the fibred manifold p: $\boldsymbol{F} \to \boldsymbol{B}$ is defined to be the set $J_x\boldsymbol{F}$ of all equivalence classes of sections s: $\boldsymbol{B} \to \boldsymbol{F}$ which have the same value $s(x)$ and the same derivatives $\partial_\lambda s^i(x)$ (this condition is independent of the particular chosen chart). The *jet space* $J\boldsymbol{F}$ is the union of all $J_x\boldsymbol{F}$ for $x \in \boldsymbol{B}$. We have the natural fibred chart $(x^\lambda, y^j, y^j_\lambda)$ of $J\boldsymbol{F}$, and the *jet prolongation* js: $\boldsymbol{B} \to J\boldsymbol{F}$ characterized by the coordinate expression $(y^j, y^j_\lambda) \circ js = (s^j, \partial_\lambda s^j)$. We can identify js with Ts: $T\boldsymbol{B} \to T\boldsymbol{F}$, which projects over $\mathbf{1}_{T\boldsymbol{B}}$. Accordingly, we can regard $J\boldsymbol{F}$ as the subbundle of $T^*\boldsymbol{B} \otimes_{\boldsymbol{F}} T\boldsymbol{F}$ whose elements are projectable over $\mathbf{1}_{T\boldsymbol{B}} \in T^*\boldsymbol{B} \otimes_{\boldsymbol{B}} T\boldsymbol{B}$.

If p': $\boldsymbol{F}' \to \boldsymbol{B}$ is another fibred manifold, then the *jet prolongation* of a fibred map f: $\boldsymbol{F} \to \boldsymbol{F}'$ is the fibred map Jf: $J\boldsymbol{F} \to J\boldsymbol{F}'$ with coordinate expresssion $y'^j_\lambda \circ Jf = \partial_\lambda f^j + y^h_\lambda \partial_h f^j$.

We have a natural isomorphism $JV\boldsymbol{F} \cong VJ\boldsymbol{F}$, which is immediately read as coordinate 'exchange' in the respective fibred coordinates $(x^\lambda, y^j, \dot{y}^j, y^j_\lambda, \dot{y}^j_\lambda)$ and $(x^\lambda, y^j, y^j_\lambda, \dot{y}^j, \dot{y}^j_\lambda)$. This fact allows the jet prolongation of any vertical vector field v: $\boldsymbol{F} \to V\boldsymbol{F}$ to a vertical vector field $v' \cong Jv : J\boldsymbol{F} \to VJ\boldsymbol{F}$, with coordinate expression $v' = v^j \partial y_j + (\partial_\lambda v^j + y^h_\lambda \partial_h v^j) \partial y^\lambda_j$. This construction can be generalized [MM83b] to the jet prolongation of any vector field on $\boldsymbol{F}$.

1.3. CONNECTIONS

There are several equivalent ways to define the concept of a (possibly nonlinear) connection on a general fibred manifold [Ga72, Ko84, MM83a, Mo91].

In general, we present a connection on a fibred manifold $\boldsymbol{F} \to \boldsymbol{B}$ as a section c: $\boldsymbol{F} \to J\boldsymbol{F}$ which, via the natural inclusion, can be seen as a *horizontal prolongation* c: $\boldsymbol{F} \to T^*\boldsymbol{B} \otimes_{\boldsymbol{F}} T\boldsymbol{F}$, whose coordinate expression is of the type $c = \mathrm{d}x^\lambda \otimes (\partial x_\lambda + {c_\lambda}^j \partial y_j)$, with ${c_\lambda}^j$: $\boldsymbol{F} \to \mathbb{R}$.

The associated *vertical projection* is ν_c: $\boldsymbol{F} \to T^*\boldsymbol{F} \otimes_{\boldsymbol{F}} V\boldsymbol{F}$, with coordinate expression $\nu_c = (\mathrm{d}y^j - {c_\lambda}^j \,\mathrm{d}x^\lambda) \otimes \partial y_j$.

The *covariant differential* of a section s: $\boldsymbol{B} \to \boldsymbol{F}$ is defined to be the section $\nabla[c]s := js - c \circ s = Ts \,\lrcorner\, \nu_c$: $\boldsymbol{B} \to T^*\boldsymbol{B} \otimes_{\boldsymbol{F}} T\boldsymbol{F}$, with coordinate expression $\nabla_\lambda s^j = \partial_\lambda s^j - {c_\lambda}^j \circ s$.

The *curvature tensor* of the connection c is defined to be the tensor field $R[c]$: $\boldsymbol{F} \to \wedge^2(T^*\boldsymbol{B}) \otimes_{\boldsymbol{F}} V\boldsymbol{F}$ characterized by $R[c](u,v) := \frac{1}{2}([u \,\lrcorner\, c,\, v \,\lrcorner\, c] - [u,v] \,\lrcorner\, c)$ for any two vector fields u, v: $\boldsymbol{B} \to \boldsymbol{F}$. Namely, the curvature tensor 'measures' how much the horizontal prolongation c differs from being a morphism of Lie algebras. Its coordinate expression is $R[c] = {R_{\lambda\mu}}^j \,\mathrm{d}x^\lambda \wedge d^\mu \otimes \partial y_j$, where ${R_{\lambda\mu}}^j = \partial_{[\lambda} c^{\,j}_{\mu]} + c^{\,h}_{[\lambda} \partial_h c^{\,j}_{\mu]}$.

If $\boldsymbol{F} \to \boldsymbol{B}$ is a vector bundle, then $J\boldsymbol{F} \to \boldsymbol{B}$ also turns out to be a vector bundle. A connection c: $\boldsymbol{F} \to J\boldsymbol{F}$ is then said to be *linear* if it is a linear morphism over $\boldsymbol{B}$. In linear fibred coordinates this means $c_\lambda{}^j = c^{\,j}_{\lambda\,k}\, y^k$ with $c^{\,j}_{\lambda\,k}\colon \boldsymbol{B} \to \mathbb{R}$. In the domain of a given linear fibred chart (a 'gauge') we have the endomorphisms $c_\lambda\colon \boldsymbol{B} \to \mathrm{End}(\boldsymbol{F})$ whose matrix expression is $(c^{\,j}_{\lambda\,k})$.* Local expressions of covariant derivatives can be conveniently expressed through these, e.g. $\nabla_\lambda s = \partial_\lambda s - c_\lambda s$ where $\partial_\lambda s$ denotes the covariant derivative of the trivial connection induced by the gauge (that whose coefficients vanish in the gauge). Similarly, the expression of the curvature tensor can be written $R_{\lambda\mu} = \partial_{[\lambda} c_{\mu]} - c_{[\lambda} c_{\mu]}$.

1.4. COMPLEX SPACES

If X is any set and $f\colon\; X \to \mathbb{C}$ then we denote by $\bar{f}$ the conjugate map, $\bar{f}(x) := \overline{f(x)}$.

Let $\boldsymbol{U}$ be an n-dimensional complex vector space. Then $\boldsymbol{U}$ is also a $2n$-dimensional real vector space. We shall denote by $\boldsymbol{U}^\star$ and $\boldsymbol{U}^*$ the complex and real dual spaces, respectively. Moreover we shall denote by $\overline{\boldsymbol{U}^\star}$ the antidual space, i.e. the space of all antilinear maps $\boldsymbol{U} \to \mathbb{C}$. We have the (conjugation) anti-isomorphism $\mathcal{K}\colon\; \boldsymbol{U}^\star \to \overline{\boldsymbol{U}^\star}\colon\; \lambda \mapsto \bar{\lambda}$, and the natural inclusion $\boldsymbol{U}^* \subset \boldsymbol{U}^\star \oplus \overline{\boldsymbol{U}^\star}$.

The *conjugate space* of $\boldsymbol{U}$ is defined to be $\overline{\boldsymbol{U}} := \boldsymbol{U}^{\star\overline{\star}} \cong \boldsymbol{U}^{\overline{\star}\star}$. Conjugation, denoted again by $\mathcal{K}$, is an anti-isomorphsim $\boldsymbol{U}^{\star\star} \cong \boldsymbol{U} \to \overline{\boldsymbol{U}}\colon\; u \mapsto \bar{u}$. We have

$$\overline{\boldsymbol{U}^\star} \cong \overline{\boldsymbol{U}}^\star \cong \boldsymbol{U}^{\overline{\star}}, \qquad \overline{\boldsymbol{U}^{\overline{\star}}} \cong \overline{\boldsymbol{U}}^{\overline{\star}} \cong \boldsymbol{U}^\star .$$

Let (ζ_a) be a basis of $\boldsymbol{U}$, $a = 1, \ldots, n$, and (z^a) the dual basis of $\boldsymbol{U}^\star$. Then we have the conjugate bases $(\bar{\zeta}_{a^\cdot}) := (\overline{\zeta_a})$ of $\overline{\boldsymbol{U}}$ and $(\bar{z}^{a^\cdot}) := (\overline{z^a})$ of $\overline{\boldsymbol{U}^\star}$. If $u = u^a \zeta_a$ and $\lambda = \lambda_a z^a$ then $\bar{u} = \bar{u}^{a^\cdot} \bar{\zeta}_{a^\cdot}$ and $\bar{\lambda} = \bar{\lambda}_{a^\cdot}\, \bar{z}^{a^\cdot}$ with $\bar{u}^{a^\cdot} = \overline{u^a}$, $\bar{\lambda}_{a^\cdot} = \overline{\lambda_a}$.

The (real) differential of a function $f\colon\; \boldsymbol{U} \to \mathbb{R}$ is a 1-form $\mathrm{d}f\colon\; \boldsymbol{U} \to \boldsymbol{U}^* \subset \boldsymbol{U}^\star \oplus \overline{\boldsymbol{U}^\star}$. In coordinates we write

$$\mathrm{d}f = \frac{\partial f}{\partial z^a}\, \mathrm{d}z^a + \frac{\partial f}{\partial \bar{z}^{a^\cdot}}\, \mathrm{d}\bar{z}^{a^\cdot},$$

namely, we formally consider z^a and $\bar{z}^{a^\cdot}$ as real independent coordinates (see for example [We80]).

Conjugation can be naturally extended to tensor products of the above spaces with any number of factors. If τ is a tensor then $\bar{\tau}$ has dotted indices in the place of undotted indices of τ, and vice-versa.

A tensor $w \in \boldsymbol{U} \otimes \overline{\boldsymbol{U}}$ is said to be Hermitian if $\bar{w} = w^{\mathrm{T}}$, where $^{\mathrm{T}}$ denotes transposition. In coordinates this means $\bar{w}^{ab^\cdot} = w^{b^\cdot a}$. We have the real decomposition $\boldsymbol{U} \otimes \overline{\boldsymbol{U}} = \boldsymbol{H} \oplus i\boldsymbol{H}$ into Hermitian and anti-Hermitian subspaces.

A Hermitian 2-form is a Hermitian tensor $h \in \overline{\boldsymbol{U}^\star} \otimes \boldsymbol{U}^\star$. The associated quadratic form $u \mapsto h(u, u)$ is real-valued. The concepts of signature and nondegeneracy of Hermitian 2-forms are introduced similarly to the case of real bilinear

* These are also the components, in the considered gauge, of the *connection forms* introduced in the principal bundle approach (see [CC91I, CC91II] for a detailed comparison).

forms. If h is nondegenerate then it yields the anti-isomorphisms $h^\flat\colon \boldsymbol{U} \to \boldsymbol{U}^\bigstar$, $h^\# := (h^\flat)^{-1}\colon \boldsymbol{U}^\bigstar \to \boldsymbol{U}$, where $h^\flat(u) := h(u, \_) = h_{a^{\cdot} b}\bar{u}^{a^{\cdot}}\zeta_b$, and the isomorphisms $\bar{h}^\flat\colon \boldsymbol{U} \to \boldsymbol{U}^\bigstar$, $\bar{h}^\#\colon \boldsymbol{U}^\bigstar \to \overline{\boldsymbol{U}}$. The Hermitian conjugate of $f \in \mathrm{End}(\boldsymbol{U}) \cong \boldsymbol{U} \otimes \boldsymbol{U}^\bigstar$ with respect to h is defined to be $f^\dagger := h^\# \circ f^\bigstar \circ h^\flat \in \mathrm{End}(\boldsymbol{U})$, where $f^\bigstar \in \mathrm{End}(\boldsymbol{U}^\bigstar)$ is the transpose of f.

2. Units of Measurement

We briefly present the ideas which allow a general and rigorous formulation of physical units [CJM95, JM92].

Observe that homogeneous units can be added and multiplied by real numbers; in some cases, however, no zero unit exists and only multiplication by positive real numbers is allowed. Then we are lead to define a *unit space* as a one-dimensional semivector space, i.e. as a semifield $\mathbb{U}$ associated with the semiring $\mathbb{R}^+$ (the axioms are analogous to those of vector spaces, with the only difference that $\mathbb{U}$ and $\mathbb{R}^+$ are additive semigroups and not groups). Moreover, a unit space is said to be *positive* if the multiplication by numbers cannot be extended to either $\mathbb{R}^+ \cup \{0\}$ or to $\mathbb{R}$. Thus, a unit space is a vector space, or a positive unit space, or a positive unit space extended by the zero element.

Several concepts and results of standard linear and multilinear algebra can be easily repeated for unit spaces. The main caution to be taken is to avoid formulations which involve the zero element.

In particular, we can define the tensor product (over $\mathbb{R}^+$) of unit spaces; the tensor product (over $\mathbb{R}^+$) of a unit space and a vector space naturally becomes a vector space. Also, we can define the $\mathbb{R}^+$-dual $\mathbb{U}^*$ of a unit space $\mathbb{U}$; then we obtain the natural identification $\mathbb{U} \otimes \mathbb{U}^* \cong \mathbb{R}^+$. Furthermore, we can define in a natural way the 'root' unit space $\mathbb{U}^{1/r}$ of $\mathbb{U}$, for any positive integer r.

In order to write formulas similar to the standard ones of physics, we use a 'number-like' notation for unit spaces. Namely, if $\boldsymbol{V}$ is a vector space and $u \in \mathbb{U}$, $v \in \boldsymbol{V}$, then we write $u\,v$ for $u \otimes v$; accordingly, we set $\mathbb{U}^2 := \mathbb{U} \otimes \mathbb{U}$ and the like. Moreover, if $\mathbb{U}$ is a unit space which does not contain 0, then we write $\mathbb{U}^{-1} = \mathbb{U}^*$ and denote by $1/u \in \mathbb{U}^{-1}$ the dual element of $u \in \mathbb{U}$.

We shall assume as fundamental unit spaces the oriented vector space $\mathbb{T}$ of *time units*, the positive space $\mathbb{L}$ of *lengths* and the positive space $\mathbb{M}$ of *masses*. Quantities possessing physical 'dimensions', like spacetime metric and electromagnetic field, will be described mathematically as 'scaled' fields, namely as sections of tensor bundles tensorialized by unit spaces.

We shall attach to each particle a *mass* m and a *charge* q, where

$$m \in \mathbb{M}, \qquad q \in \mathbb{Q} := \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}.$$

Moreover, we shall postulate two universal coupling constants,* namely the *speed of light* and the *Planck constant*

$$c \in \mathbb{T}^* \otimes \mathbb{L}, \qquad \hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}.$$

PART II: SPINOR ALGEBRA

First we summarize some of the main facts about the Dirac algebra [Ch54, Cr90, Gr78, HS84, St94]. Then we introduce complex spinor structures and spinor structures at the algebraic level and study some of their main properties.

3. Clifford Algebra of Minkowski Space

We assume $(\boldsymbol{V}, g)$ to be a Minkowski space, namely $\boldsymbol{V}$ to be a four-dimensional real vector space and $g \in \boldsymbol{V}^* \otimes \boldsymbol{V}^*$ a Lorentz metric with signature $(1, 3)$. The *Clifford algebra* $\boldsymbol{C}(\boldsymbol{V}, g)$, henceforth denoted simply by $\boldsymbol{C}$, is the associative algebra generated by $\boldsymbol{V}$ where the product of any $u, v \in \boldsymbol{V}$ is subjected to the condition

$$u\,v + v\,u = 2\,g(u, v)\,\mathbf{1}$$

(this is equivalent to $v\,v = g(v, v)\,\mathbf{1}$ $\forall v \in \boldsymbol{V}$). The Clifford algebra fulfills the following *universal property*: if $\boldsymbol{A}$ is an associative algebra with unity and γ: $\boldsymbol{V} \to \boldsymbol{A}$ is a *Clifford map*, namely a linear map such that

$$\gamma(v)\,\gamma(v) = g(v, v)\,\mathbf{1}, \quad v \in \boldsymbol{V},$$

then γ extends to a unique homomorphism $\hat{\gamma}$: $\boldsymbol{C} \to \boldsymbol{A}$. Namely, the image of $\hat{\gamma}$, together with the restriction of the algebra product of $\boldsymbol{A}$, turns out to be an algebra isomorphic to $\boldsymbol{C}$.

It can be proved that $\boldsymbol{C}$ is isomorphic, as a vector space, to the vector space underlying the exterior algebra $\wedge\boldsymbol{V}$. The isomorphism is characterized by the identification

$$\mathcal{A}(v_1\,\ldots\,v_p) \equiv v_1 \wedge \cdots \wedge v_p,$$

where $\mathcal{A}$ stands for the antisymmetrisazion operator defined by

$$\mathcal{A}(v_1\,\ldots\,v_p) = \frac{1}{p!}\sum_{\pi} \varepsilon(\pi)\,v_{\pi(1)}\ldots v_{\pi(p)};$$

the sum is extended to all permutations of the set $\{1, \ldots, p\}$ and $\varepsilon(\pi)$ denotes the permutation's sign. In other terms, we have two distinct algebras on the same

* We are not concerned with Newton's gravitational constant since we deal only with given background gravitational field.

underlying vector space. Any element of $\boldsymbol{C}$ can be uniquely expressed as a sum of terms, each of well-defined exterior degree. In particular

$$u\,v = u \wedge v + g(u,v), \quad u,v \in \boldsymbol{V}.$$

From this one sees that the Clifford algebra product does not preserve the exterior algebra degree, but only its parity. Namely $\boldsymbol{C}$ is $\mathbb{Z}_2$-graded.

Also, let $\phi \in \wedge^r\boldsymbol{V}$, $\theta \in \wedge^s\boldsymbol{V}$, $r \leqslant s$. Then the Clifford product $\phi\,\theta$ turns out to be a sum of terms of exterior degree $r{+}s,\ r{+}s{-}2, \dots, r{-}s$, from $\phi\wedge\theta$ to $(-1)^{r(r-1)/2}\, i_\phi\theta$. In particular, for any chosen orientation of $\boldsymbol{V}$ consider the unique g-unimodular positively oriented volume form η; then

$$\begin{aligned} &v\,\eta = -\eta\,v = {*v}, \quad v \in \boldsymbol{V},\\ &\eta\,\eta = -1. \end{aligned} \tag{3.1}$$

If γ: $\boldsymbol{V} \to \boldsymbol{A}$ is a Clifford map, then $\hat\gamma$, being an injective vector space morphism, transfers onto its image also the exterior algebra structure. In particular, for decomposable elements of exterior degree 2 we obtain

$$\gamma_u \wedge \gamma_v := \hat\gamma(u\wedge v) = \tfrac{1}{2}\,[\gamma_u, \gamma_v] := \tfrac{1}{2}\,(\gamma_u\,\gamma_v - \gamma_v\,\gamma_u).$$

Also, note that $\hat\gamma$ yields a homomorphism $\boldsymbol{C}^{[,]} \to \boldsymbol{A}^{[,]}$ of the commutator-induced Lie algebras.

4. Fundamental Groups in the Clifford Algebra

Let $\boldsymbol{C}^\times$ be the group of all invertible elements of $\boldsymbol{C}$. The *adjoint action* of $\boldsymbol{C}^\times$ on $\boldsymbol{C}$ is defined by

$$\mathrm{Ad}(\Lambda)(\Phi) := \Lambda\Phi\Lambda^{-1}, \quad \Lambda \in \boldsymbol{C}^\times, \quad \Phi \in \boldsymbol{C}.$$

The *Clifford group* $\mathrm{Cl} \equiv \mathrm{Cl}(\boldsymbol{V}, g)$ is defined to be the group of all invertible elements of $\boldsymbol{C}$ for which $\boldsymbol{V} \subset \boldsymbol{C}$ is stable under the adjoint action. It turns out [Cr90, Gr78] that Cl is (multiplicatively) generated by all $v \in \boldsymbol{V}$ such that $g(v,v) \neq 0$. Namely any element of Cl is of the form $\theta = v_1\,v_2 \dots v_n$, with $v_j \in \boldsymbol{V}$, and its inverse is given by $\theta^{-1} = 1/\nu(\theta)\,v_n \dots v_2\,v_1$ where $\nu(\theta) := g(v_1,v_1)\,g(v_2,v_2)\dots g(v_n,v_n)$. We have the subgroup $\mathrm{Cl}^\uparrow \subset \mathrm{Cl}$ characterized by $\nu(\theta) > 0$ (i.e. $\theta \in \mathrm{Cl}^\uparrow$ iff it has an *even* number of Clifford factors with negative square) and the subgroup $\mathrm{Cl}^+ \subset \mathrm{Cl}$ constituted by all even-degree elements. We also set $\mathrm{Cl}^{+\uparrow} := \mathrm{Cl}^+ \cap \mathrm{Cl}^\uparrow$.

We shall denote by $\mathrm{Spin} \subset \mathrm{Pin} \subset \mathrm{Cl}$ the Spin and Pin subgroups. Namely Pin is the group (multiplicatively) generated by all $v \in \boldsymbol{V}$ such that $g(v,v) = \pm 1$, and $\mathrm{Spin} := \mathrm{Pin}^+$ is the subgroup of Pin constituted by all even-degree elements. Then $\mathrm{Cl} = \mathbb{R}^+ \times \mathrm{Pin}$ and $\mathrm{Cl}^+ = \mathbb{R}^+ \times \mathrm{Spin}$. We set $\mathrm{Pin}^\uparrow := \mathrm{Pin} \cap \mathrm{Cl}^\uparrow$ and $\mathrm{Spin}^\uparrow := \mathrm{Spin} \cap \mathrm{Cl}^\uparrow = \mathrm{Pin} \cap \mathrm{Cl}^{+\uparrow}$. If $\theta = v_1\,v_2 \dots v_n \in \mathrm{Pin}$ then $\theta^{-1} = \pm v_n \dots v_2\,v_1$, and the plus sign holds if $\theta \in \mathrm{Pin}^\uparrow$.

We shall denote by $L(\boldsymbol{V}) := O(\boldsymbol{V}, g)$ the (full) Lorentz group and by $L^+(\boldsymbol{V}) := \mathrm{SO}(\boldsymbol{V}, g)$ its special (i.e. orientation preserving) subgroup. Let $v \in \boldsymbol{V}$ be such that $g(v, v) \neq 0$. Then one easily sees that $\mathrm{Ad}(v)$ is the negative of the reflection through the hyperplane perpendicular to v. From a well-known theorem [Cr90, Gr78] it then follows that the adjoint action restricted to Cl is a group epimorphism onto the Lorentz group. The restriction to Pin turns out to be a two-to-one epimorphism Pin $\to L(\boldsymbol{V})$, while the restriction to Spin turns out to be a two-to-one epimorphism Spin $\to L^+(\boldsymbol{V})$.

Furthermore, from the above recalled interpretation of $\mathrm{Ad}(v)$ as a reflection one sees that $\mathrm{Ad}(v)$ preserves time-orientation iff $g(v, v) > 0$. It follows that $\mathrm{Cl}^\uparrow$ is the subgroup of Cl which preserves time-orientation, and the same holds for the other 'up arrow' subgroups. In particular, $\mathrm{Spin}^\uparrow$ turns out to be the double covering of the special orthochronous Lorentz group $L^{+\uparrow}(\boldsymbol{V})$.

Clearly, all the above introduced groups are Lie groups. In general, we shall denote by $\mathfrak{L}(G)$ the Lie algebra of the Lie group G. We have (see [Cr90], Ch. 6)

$$\mathfrak{L}(\boldsymbol{C}^\times) = \boldsymbol{C}^{[,]},$$

$$\begin{aligned} \mathfrak{L}(\mathrm{Cl}) &= \mathfrak{L}(\mathrm{Cl}^+) = \mathfrak{L}(\mathrm{Cl}^\uparrow) = \mathfrak{L}(\mathrm{Cl}^{+\uparrow}) \\ &= \mathbb{R} \oplus \wedge^2 \boldsymbol{V} \subset \boldsymbol{C}^{[,]}, \end{aligned}$$

$$\begin{aligned} \mathfrak{L}(\mathrm{Pin}) &= \mathfrak{L}(\mathrm{Spin}) = \mathfrak{L}(\mathrm{Pin}^\uparrow) = \mathfrak{L}(\mathrm{Spin}^\uparrow) \\ &= \wedge^2 \boldsymbol{V} \subset \boldsymbol{C}^{[,]}. \end{aligned}$$

The double covering Pin $\to L(\boldsymbol{V})$ determines a Lie algebra isomorphism. If $\lambda \in \mathfrak{L}(\mathrm{Pin})$ and $\tilde{\lambda} \in \mathfrak{L}(L(\boldsymbol{V}))$ are corresponding elements and (e_μ) is a basis of $\boldsymbol{V}$ we have

$$\lambda = \tfrac{1}{4}\, \tilde{\lambda}^\mu_\rho\, g^{\nu\rho}\, e_\mu \wedge e_\nu. \tag{4.1}$$

5. Dirac Algebra

By $\boldsymbol{W}$ we shall denote a four-dimensional complex vector space. Let

$$\gamma\colon\ \boldsymbol{V} \to \mathrm{End}(\boldsymbol{W})\colon\ v \mapsto \gamma_v := \gamma(v)$$

be a Clifford map. Then $\boldsymbol{D}_\gamma := \hat{\gamma}(\boldsymbol{C}(\boldsymbol{V}, g)) \subset \mathrm{End}(\boldsymbol{W})$ is a real vector algebra called the *Dirac algebra* generated by γ. We have $\mathbb{C} \otimes \boldsymbol{D}_\gamma := \boldsymbol{D}_\gamma \oplus i\, \boldsymbol{D}_\gamma = \mathrm{End}(\boldsymbol{W})$.

The Dirac algebra has the canonical element $\gamma_\eta := \hat{\gamma}(\eta)$. Since $\gamma_\eta^2 = -1$, we have a splitting $\boldsymbol{W} = \boldsymbol{S} \oplus \boldsymbol{S}'$ into the direct sum of the (complex) eigenspaces of $i\gamma_\eta$ with eigenvalues ± 1. We call this the *chiral splitting*, and $\boldsymbol{S}$ and $\boldsymbol{S}'$ the *chiral subspaces* of $\boldsymbol{W}$.

PROPOSITION 5.1. *The Clifford map* γ *exchanges the chiral subspaces, i.e.* $\forall v \in \boldsymbol{V}$ *we have* $\gamma_v(\boldsymbol{S}) = \boldsymbol{S}'$, $\gamma_v(\boldsymbol{S}') = \boldsymbol{S}$. *If* $g(v,v) \neq 0$, *then the restrictions of* γ_v *to* $\boldsymbol{S}$ *and* $\boldsymbol{S}'$ *are isomorphisms.*

Proof. From (3.1) we have $\gamma_v\gamma_\eta = -\gamma_\eta\gamma_v$, hence $\gamma_v(1 - i\gamma_\eta) = (1 + i\gamma_\eta)\gamma_v$ and $\gamma_v(1 + i\gamma_\eta) = (1 - i\gamma_\eta)\gamma_v$. The second statement follows from $\gamma_v\gamma_v(\psi_{\boldsymbol{S}}) = g(v,v)\,\psi_{\boldsymbol{S}}$. □

From the above proposition, it follows immediately that the odd part of $\boldsymbol{D}_\gamma$ exchanges the chiral subspaces, while the even part leaves them invariant.

If (e_λ) is a basis of $\boldsymbol{V}$, then one sets $\gamma_\lambda := \gamma(e_\lambda)$. In physics texts, one usually takes an orthonormal and positively oriented basis, and sets $\gamma_5 := -i\gamma_0\gamma_1\gamma_2\gamma_3$. Then $\gamma_5 = -i\gamma_\eta$.

The natural extension $\hat{\gamma}$ of the Clifford map γ sends the subgroups of $\boldsymbol{C}^\times$ introduced in Section 4 to subgroups of the general linear group $\mathrm{Gl}(\boldsymbol{W})$ of all complex automorphsims of $\boldsymbol{W}$. Moreover, each of these restrictions of $\hat{\gamma}$ turns out to be a group isomorphism. Hence, when no confusion arises, we may just identify $\mathrm{Pin} \equiv \hat{\gamma}(\mathrm{Pin})$, $\mathrm{Spin} \equiv \hat{\gamma}(\mathrm{Spin})$ and so on.

Let $G \subset \boldsymbol{C}^\times$ be any of these subgroups. We have the natural 'complexified' extension $G^c \subset \mathrm{Gl}(\boldsymbol{W})$ constituted by all elements of G multiplied by a phase factor, namely

$$G^c := \mathrm{U}(1) \tilde{\times} \mathrm{G} := (\mathrm{U}(1) \times \mathrm{G})/\!\sim\, = (\mathrm{U}(1) \times \mathrm{G})/\mathbb{Z}_2,$$

where $\sim$ is the equivalence relation $(\lambda,\theta) \sim (\lambda',\theta') \Leftrightarrow \lambda\theta = \lambda'\theta'$. The last equality follows from considering the subgroup of $\mathrm{U}(1) \times \mathrm{G}$ generated by $(-1, -\mathbf{1})$, which is a normal subgroup isomorphic to $\mathbb{Z}_2$. Explicitly, we have*

$$\begin{aligned}
&\mathrm{Cl}^c := \mathrm{U}(1) \tilde{\times} \mathrm{Cl} := \mathbb{R}^+ \times \mathrm{U}(1) \tilde{\times} \mathrm{Pin} := \mathbb{R}^+ \times \mathrm{Pin}^{\mathrm{c}},\\
&\mathrm{Cl}^{+\,c} := \mathrm{U}(1) \tilde{\times} \mathrm{Cl}^+ := \mathbb{R}^+ \times \mathrm{U}(1) \tilde{\times} \mathrm{Spin} := \mathbb{R}^+ \times \mathrm{Spin}^{\mathrm{c}},\\
&\mathrm{Cl}^{\uparrow\,c} := \mathrm{U}(1) \tilde{\times} \mathrm{Cl}^\uparrow := \mathbb{R}^+ \times \mathrm{U}(1) \tilde{\times} \mathrm{Pin}^\uparrow := \mathbb{R}^+ \times \mathrm{Pin}^{\uparrow\,\mathrm{c}},\\
&\mathrm{Cl}^{+\uparrow\,c} := \mathrm{U}(1) \tilde{\times} \mathrm{Cl}^{+\uparrow} := \mathbb{R}^+ \times \mathrm{U}(1) \tilde{\times} \mathrm{Spin}^\uparrow := \mathbb{R}^+ \times \mathrm{Spin}^{\uparrow\,\mathrm{c}}.
\end{aligned}$$

We have the Lie algebras

$$\begin{aligned}
\mathfrak{L}(\mathrm{Cl}^c) &= \mathfrak{L}(\mathrm{Cl}^{+\,c}) = \mathfrak{L}(\mathrm{Cl}^{\uparrow\,c}) = \mathfrak{L}(\mathrm{Cl}^{+\uparrow\,c})\\
&= \mathbb{C} \oplus \hat{\gamma}(\wedge^2\boldsymbol{V}) \subset \mathrm{End}^{[,]}(\boldsymbol{W}),\\
\mathfrak{L}(\mathrm{Pin}^c) &= \mathfrak{L}(\mathrm{Spin}^c) = \mathfrak{L}(\mathrm{Pin}^{\uparrow\,c}) = \mathfrak{L}(\mathrm{Spin}^{\uparrow\,c})\\
&= i\,\mathbb{R} \oplus \hat{\gamma}(\wedge^2\boldsymbol{V}) \subset \mathrm{End}^{[,]}(\boldsymbol{W}), \qquad (5.1)\\
\mathfrak{L}(\mathrm{Pin}) &= \mathfrak{L}(\mathrm{Spin}) = \mathfrak{L}(\mathrm{Pin}^\uparrow) = \mathfrak{L}(\mathrm{Spin}^\uparrow)\\
&= \hat{\gamma}(\wedge^2\boldsymbol{V}) \subset \mathrm{End}^{[,]}(\boldsymbol{W}).
\end{aligned}$$

* Pin^c and its complexified subgroups are also called 'torogonal groups' [Cr90].

From the definitions, it follows that Cl^c is the group of *all* elements of $\mathrm{Gl}(\boldsymbol{W})$ for which $\gamma(\boldsymbol{V})$ is stable under the adjoint action. Then we introduce the map $\varphi\colon\ \mathrm{Cl}^c \to \mathrm{End}(\boldsymbol{V})$ defined by

$$\gamma(\varphi(\Lambda)v) = \mathrm{Ad}(\Lambda)(\gamma(v)) := \Lambda\gamma(v)\Lambda^{-1}, \quad v \in \boldsymbol{V},\ \Lambda \in \mathrm{Cl}^c.$$

Hence, φ is a group epimorphism $\mathrm{Cl}^c \to L(\boldsymbol{V})$, as well as its restriction to Pin^c. The restrictions of φ to $\mathrm{Cl}^{+\,c}$ and Spin^c turn out to be group epimorphisms onto $L^+(\boldsymbol{V})$. Then we see that the subgroup $L(\boldsymbol{V}) \times_\gamma \mathrm{Cl}^c \subset L(\boldsymbol{V}) \times \mathrm{Cl}^c$ constituted by all elements of the form $(\varphi(\Lambda), \Lambda)$ can be identified with Cl^c itself. Similarly $L^+(\boldsymbol{V}) \times_\gamma \mathrm{Cl}^{+\,c} \cong \mathrm{Cl}^{+\,c}$.

Note that γ can be seen as an element of $\boldsymbol{V}^* \otimes \mathrm{End}(\boldsymbol{W})$. Then the natural action of $(\tilde{\Lambda}, \Lambda) \in L(\boldsymbol{V}) \times \mathrm{Cl}^c$ on γ is given by $(\tilde{\Lambda}, \Lambda)^*(\gamma) = \Lambda^{-1}\,(\gamma \circ \tilde{\Lambda})\,\Lambda$. In particular, the action of $(\varphi(\Lambda), \Lambda) \cong \Lambda \in \mathrm{Cl}^c$ on γ is given by

$$\Lambda^*(\gamma) = \Lambda^{-1}\,(\gamma \circ \varphi(\Lambda))\,\Lambda = \Lambda^{-1}\Lambda\,\gamma\,\Lambda^{-1}\Lambda = \gamma.$$

Namely, we obtain

PROPOSITION 5.2.
(a) *The group of all automorphisms of* (g, γ) *is* Cl^c.
(b) *The group of all automorphisms of* (g, η, γ) *is* $\mathrm{Cl}^{+\,c}$.

Moreover, we have the time-orientation preserving subgroups $\mathrm{Cl}^{\uparrow\,c}$ and $\mathrm{Cl}^{+\uparrow\,c}$.

The isomorphism (4.1) between the Lie algebras of $\mathrm{Pin} \equiv \hat{\gamma}(\mathrm{Pin})$ and of $L(\boldsymbol{V})$ can be read as $\lambda = \frac{1}{4}\,\tilde{\lambda}^\mu_\rho\, g^{\nu\rho}\,\gamma_\mu \wedge \gamma_\nu$. In physics texts, this is usually written as

$$\lambda = -\tfrac{i}{4}\,\tilde{\lambda}^{\mu\nu}\,\sigma_{\mu\nu}, \quad \text{with } \sigma_{\mu\nu} := \tfrac{i}{2}\,[\gamma_\mu, \gamma_\nu] = i\gamma_\mu \wedge \gamma_\nu.$$

Let $\tilde{\Lambda} \in L(\boldsymbol{V})$ be a Lorentz transformation. Then $\gamma \circ \tilde{\Lambda}$ is a new Clifford map which has the same image and yields the same chiral splitting as γ. Namely, we have an equivalence relation between Clifford maps, where two Clifford maps are regarded as equivalent iff they are related by a Lorentz transformation. Moreover, $(\gamma \circ \tilde{\Lambda})_\eta = \pm\gamma_\eta$, where the plus sign holds iff $\tilde{\Lambda} \in L^+(\boldsymbol{V})$.

We have the following characterization of the family of Clifford maps with given canonical element.

PROPOSITION 5.3. *Up to a Lorentz transformation, any Clifford map* $\tilde{\gamma}$ *having canonical element* $\tilde{\gamma}_\eta = \gamma_\eta$ *is of the form*

$$\tilde{\gamma} = \cos(z)\,\gamma + \sin(z)\,\gamma\,\gamma_\eta, \quad z \in \mathbb{C}.$$

Proof. Since $\tilde{\gamma}_\eta = \gamma_\eta$, $\tilde{\gamma}$ anticommutes with γ_η, so its expansion in terms of $\hat{\gamma}$ contains only odd-degree terms. Namely, there exist two linear maps $A, B\colon\ \boldsymbol{V} \to \mathbb{C} \otimes \boldsymbol{V}$ such that

$$\tilde{\gamma} = \gamma \circ A + (\gamma \circ B)\gamma_\eta.$$

For each $v \in \boldsymbol{V}$, we obtain

$$\tilde{\gamma}_v\,\tilde{\gamma}_v = (\gamma_{Av}\,\gamma_{Av} + \gamma_{Bv}\,\gamma_{Bv}) + (\gamma_{Av}\,\gamma_{Bv} - \gamma_{Bv}\,\gamma_{Av})\,\gamma_\eta.$$

Since $\tilde{\gamma}$ is a Clifford map, the second term in the right-hand side has to be zero. Hence, $Av \propto Bv\ \forall\, v \in \boldsymbol{V}$. If $A = 0$, then obviously B has to be a Lorentz transformation, so our statement is true with $z = \pi/2$. If $A \neq 0$ then A must be injective for $\tilde{\gamma}$ to be injective. It follows $B = b\,A$ with $b \in \mathbb{C}$. We now have

$$g(v, v) = \tilde{\gamma}_v\,\tilde{\gamma}_v = (1 + b^2)(\gamma_{Av}\,\gamma_{Av}) = (1 + b^2)\,g(Av, Av).$$

Hence, $b \neq \pm i$ and $\sqrt{1 + b^2}\,A = \tilde{\Lambda}$, where $\tilde{\Lambda}$ is a Lorentz transformation. Letting $\cos z = 1/\sqrt{1 + b^2}$ and $\sin z = b/\sqrt{1 + b^2}$, we obtain the stated expression. □

Conversely, one easily sees that the expression of the above proposition for $\tilde{\gamma}$ yields a Clifford map for any $z \in \mathbb{C}$.

6. Algebraic Complex Spinor Structures

An algebraic *complex spinor structure* on the Minkowski space $(\boldsymbol{V}, g)$ is defined to be a complex two-dimensional vector space $\boldsymbol{W}$ together with a Clifford map $\gamma\colon\ \boldsymbol{V} \to \mathrm{End}(\boldsymbol{W})$ and a Hermitian form $k \in \overline{\boldsymbol{W}^\star} \otimes \boldsymbol{W}^\star$ fulfilling $k(\gamma_v\phi, \psi) = k(\phi, \gamma_v\psi)$ for all $v \in \boldsymbol{V}$ and $\phi, \psi \in \boldsymbol{W}$. Then we also have $k(\gamma_\eta\phi, \psi) = k(\phi, \gamma_\eta\psi)$. We shall denote by $k^\flat\colon\ \boldsymbol{W} \to \boldsymbol{W}^\star$ the anti-isomorphism given by $k^\flat(\phi) := k(\phi, \_)$, and set $k^\# := (k^\flat)^{-1}\colon\ \boldsymbol{W}^\star \to \boldsymbol{W}$. A similar notation will be used for other invertible 2-tensors.

Remark. In physics texts, usually, there is no explicit notation for k. Instead one finds the notation $\overline{\psi}$ which is our $k^\flat(\psi)$.

PROPOSITION 6.1. *Each chiral subspace is a maximal totally k-isotropic subspace.*

Proof. We have $k \circ ((\mathbf{1} + i\gamma_\eta) \times (\mathbf{1} + i\gamma_\eta)) = k \circ ((\mathbf{1} - i\gamma_\eta) \times (\mathbf{1} - i\gamma_\eta)) = 0$, so $\boldsymbol{S}$ and $\boldsymbol{S}'$ are totally isotropic. Then the index of k (see [Cr90], prop. 1.2.10) is $\dim \boldsymbol{S} = 2$. □

PROPOSITION 6.2. *The maps*

$$\boldsymbol{S} \to \boldsymbol{S}'^\star\colon\ s \mapsto k^\flat(s)_{|\boldsymbol{S}'}, \qquad \boldsymbol{S}' \to \boldsymbol{S}^\star\colon\ s' \mapsto k^\flat(s')_{|\boldsymbol{S}},$$

are anti-isomorphisms.

Proof. Since $\boldsymbol{S}$ and $\boldsymbol{S}'^\star$ have the same dimension, we have to show that from $k^\flat(s)_{|\boldsymbol{S}'} = 0$ it follows that $s = 0$. Let $k(s, s') = 0$ for each $s' \in \boldsymbol{S}'$. Since $k(s, t) = 0$ for each $t \in \boldsymbol{S}$ and $\boldsymbol{W} = \boldsymbol{S} \oplus \boldsymbol{S}'$, we have $k(s, \psi) = 0$ for each $\psi \in \boldsymbol{W}$. Because k is nondegenerate, we have $s = 0$. □

PROPOSITION 6.3. *Let* $\boldsymbol{U} \subset \boldsymbol{S}$ *be a one-dimensional subspace. Then there exists a unique one-dimensional subspace* $\boldsymbol{U}' \subset \boldsymbol{S}'$ *such that* $k(\boldsymbol{U}, \boldsymbol{U}') = 0$.

Proof. Uniqueness: Let $u \in \boldsymbol{U}$, $u' \in \boldsymbol{S}'$, $k(u, u') = 0$. Then (see the above proposition) there is no $t' \in \boldsymbol{S}'$ linearly independent from u' such that $k(u, t')$, so $\boldsymbol{U}'$ is unique and is the space generated by u'.

Existence: Let $u \in \boldsymbol{U}$ and take any two linearly independent elements $s', t' \in \boldsymbol{S}'$. Then $u' := k(u, t')\, s' - k(u, s')\, t'$ fulfills $k(u, u') = 0$. □

The subspaces $\boldsymbol{U}$, $\boldsymbol{U}'$ of the above proposition are said to be mutually k*-conjugated*.

PROPOSITION 6.4.

(a) *The group of all automorphisms of* (g, γ, k) *is* $\mathrm{Pin}^{\uparrow c}$.
(b) *The group of all automorphisms of* (g, η, γ, k) *is* $\mathrm{Spin}^{\uparrow c}$.

Proof. Let $\vartheta := \mu\theta_1 \ldots \theta_n \in \mathrm{Cl}^c$, with $\theta_j \in \gamma(\boldsymbol{V})$, $\mu \in \mathbb{C}^\times$. Then

$$k \circ (\vartheta \times \vartheta) = \bar{\mu}\mu\, k \circ ((\theta_1 \ldots \theta_n) \times (\theta_1 \ldots \theta_n)) = \pm\bar{\mu}\mu\, k,$$

where the plus sign holds iff $\theta_1 \ldots \theta_n \in \mathrm{Pin}^\uparrow$. □

We see that there is a strict relation between k and time-orientation. We shall clarify this point in Section 10.

We shall be involved with the following description of the family of all complex spinor structures for given k and γ_η.

PROPOSITION 6.5. *Up to a Lorentz transformation, any Clifford map* $\tilde{\gamma}$ *having canonical element* $\tilde{\gamma}_\eta = \gamma_\eta$ *and such that* $(\tilde{\gamma}, k)$ *determines an algebraic weak spinor structure is of the form*

$$\tilde{\gamma} = \pm \cosh(y)\, \gamma + i\, \sinh(y)\, \gamma\, \gamma_\eta, \quad y \in \mathbb{R}.$$

Proof. Inserting the expression for $\tilde{\gamma}$ given in Proposition 5.3 into the equation $k\circ(\tilde{\gamma}\times\mathbf{1}) = k\circ(\mathbf{1}\times\tilde{\gamma})$, we obtain the condition $\sin(\bar{z} + z)/2 = 0$, i.e. $z = n\pi + iy$ with $n \in \mathbb{N}$. For even n obtain the stated expression with the plus sign, while for odd n we get an overall minus sign which, in the second term, can be included in y. □

Note that the Clifford maps fulfilling the hypotheses of Proposition 6.5 are divided into two disconnected classes, according to the sign of the first term in the expression of $\tilde{\gamma}$.

PROPOSITION 6.6. *Two Clifford maps* γ^y *and* $\gamma^{y'}$, *fulfilling the hypotheses of Proposition* 6.5, *are in the same class iff for any timelike* $u \in \boldsymbol{V}$ *one has*

$$\gamma^y_u \gamma^{y'}_u + \gamma^{y'}_u \gamma^y_u > 0.$$

Proof. Let $v \in \boldsymbol{V}$. We obtain $\gamma^y_v \gamma^{y'}_v + \gamma^{y'}_v \gamma^y_v = \pm 2 \cosh(y + y')\, g(v, v)$, where the plus sign holds iff γ^y and $\gamma^{y'}$ are in the same class. □

We then see that the two classes of Clifford maps can be put in one-to-one correspondence with time orientations of $\boldsymbol{V}$. Thus, we shall call them *time-orientation classes*. In Section 10, we shall see that there is a distinguished way of choosing this correspondence.

7. Special Bases

Let (ζ_1, ζ_2) be a basis of $\boldsymbol{S}$. Then there exists a unique basis (ζ_3, ζ_4) of $\boldsymbol{S}'$ such that ζ_4 is k-conjugated to ζ_1, ζ_3 is k-conjugated to ζ_2, and $k(\zeta_1, \zeta_3) = k(\zeta_2, \zeta_4) = -1$. In other terms, $(k^\flat(\zeta_1), k^\flat(\zeta_2))$ is minus the antidual basis of (ζ_3, ζ_4). We call (ζ_a), $a = 1, \ldots, 4$, a *k-normal basis* of $\boldsymbol{W}$ (not orthonormal!), and we indicate by (z^a) the dual basis. Then we have the coordinate expressions

$$k = -\bar{z}^1 \otimes z^3 - \bar{z}^3 \otimes z^1 - \bar{z}^2 \otimes z^4 - \bar{z}^4 \otimes z^2,$$
$$\gamma_\eta = i\,(\zeta_1 \otimes z^1 + \zeta_2 \otimes z^2 - \zeta_3 \otimes z^3 - \zeta_4 \otimes z^4). \tag{7.1}$$

We see that k turns out to have signature $(2, 2)$.

Using a basis of W we can prove the *existence* of an algebraic complex spinor structure. First, any basis determines a Hermitian 2-form k via the expression (7.1). Now, chose an orthonormal basis (e_λ) of $\boldsymbol{V}$ and consider the map $\gamma\colon \boldsymbol{V} \to \operatorname{End}(\boldsymbol{W})$ given by

$$\gamma_0 := \gamma(e_0) := -\zeta_1 \otimes z^3 - \zeta_2 \otimes z^4 - \zeta_3 \otimes z^1 - \zeta_4 \otimes z^2,$$
$$\gamma_1 := \gamma(e_1) := -\zeta_1 \otimes z^4 - \zeta_2 \otimes z^3 + \zeta_3 \otimes z^2 + \zeta_4 \otimes z^1,$$
$$\gamma_2 := \gamma(e_2) := i(\zeta_1 \otimes z^4 - \zeta_2 \otimes z^3 - \zeta_3 \otimes z^2 + \zeta_4 \otimes z^1), \tag{7.2}$$
$$\gamma_3 := \gamma(e_3) := -\zeta_1 \otimes z^3 + \zeta_2 \otimes z^4 + \zeta_3 \otimes z^1 - \zeta_4 \otimes z^2,$$

namely the matrix expression of γ is

$$(\gamma_0) = \begin{pmatrix} 0 & -(\sigma_0) \\ -(\sigma_0) & 0 \end{pmatrix}, \qquad (\gamma_j) = \begin{pmatrix} 0 & -(\sigma_j) \\ (\sigma_j) & 0 \end{pmatrix}, \tag{7.3}$$

where (σ_λ) denotes the λth Pauli matrix. Then, it is easy to check that (k, γ) constitute an algebraic complex spinor structure. The expression (7.2) or (7.3) is called the *Weyl representation*, and (ζ_a) is called a *Weyl basis*. Since the $\mathrm{Pin}^{\uparrow c}$ group preserves the given algebraic complex spinor structure, any of its elements sends (e_λ) and (ζ_a) to new bases (e'_λ) and (ζ'_a) in which (γ, k) is again expressed by the Weyl representation.

LEMMA 7.1. *Let (e_λ) be a positively-oriented orthonormal basis of* $\boldsymbol{V}$*, and let (ζ_a), $(\tilde{\zeta}_a)$ be k-normal bases of* $\boldsymbol{W}$*. Let γ and $\tilde{\gamma}$ be the Clifford maps whose expressions, in these bases, are given by* (7.2)*. Then γ and $\tilde{\gamma}$ are in the same time-orientation class* (*Propositions* 6.5 *and* 6.6)*.*

Proof. Since $\mathrm{Gl}(2, \mathbb{C})$ is connected, there is a continuous curve, valued in the space of all bases of ***V***, joining (ζ_a) and $(\tilde{\zeta}_a)$. This yields a continuous curve joining γ and $\tilde{\gamma}$. Since the two time-orientation classes are disconnected, γ and $\tilde{\gamma}$ are in the same class. □

By a straightforward calculation one checks:

PROPOSITION 7.1. *Any algebraic complex spinor structure can be expressed, in suitable bases of* ***V*** *and* ***W****, by* (7.1) *and* (7.2)*. More precisely, let* $(\tilde{\zeta}_a)$ *be any* k*-normal basis of* ***W*** *and* (e_λ) *a positively-oriented orthonormal basis of* ***V****. Then there is a unique* $y \in \mathbb{R}$ *such that the* k*-normal basis*

$$\zeta_1 = \mathrm{e}^{-y/2}\tilde{\zeta}_1, \qquad \zeta_2 = \mathrm{e}^{-y/2}\tilde{\zeta}_2, \qquad \zeta_3 = \mathrm{e}^{y/2}\tilde{\zeta}_3, \qquad \zeta_4 = \mathrm{e}^{y/2}\tilde{\zeta}_4$$

is a Weyl basis associated with (e_λ).

We can recover the familiar *Dirac representation* as follows. Since $(\gamma_0)^2 = 1$, γ_0 determines a splitting of ***W*** into subspaces with eigenvalues ± 1, projections $p_0^{\pm} := \frac{1}{2}(\mathbf{1} \pm \gamma_0)$. Applying these projections to the elements of a Weyl basis we find a basis adapted to this splitting, called a *Dirac basis*:

$$\begin{aligned}
\zeta_1' &:= \sqrt{2}\, p_0^+(\zeta_1) = -\sqrt{2}\, p_0^+(\zeta_1) = \tfrac{1}{\sqrt{2}}(\zeta_1 - \zeta_3),\\
\zeta_2' &:= \sqrt{2}\, p_0^+(\zeta_2) = -\sqrt{2}\, p_0^+(\zeta_4) = \tfrac{1}{\sqrt{2}}(\zeta_2 - \zeta_4),\\
\zeta_3' &:= \sqrt{2}\, p_0^-(\zeta_1) = \sqrt{2}\, p_0^-(\zeta_3) = \tfrac{1}{\sqrt{2}}(\zeta_1 + \zeta_3),\\
\zeta_4' &:= \sqrt{2}\, p_0^-(\zeta_2) = \sqrt{2}\, p_0^-(\zeta_4) = \tfrac{1}{\sqrt{2}}(\zeta_2 + \zeta_4).
\end{aligned}$$

In this basis the matrix expression of γ is

$$(\gamma_0) = \begin{pmatrix} (\sigma_0) & 0 \\ 0 & -(\sigma_0) \end{pmatrix}, \qquad (\gamma_j) = \begin{pmatrix} 0 & -(\sigma_j) \\ (\sigma_j) & 0 \end{pmatrix}.$$

8. Algebraic Spinor Structures

In this section we consider a given algebraic complex spinor structure. A k-*conjugation* is defined to be an antilinear map $\mathcal{C}$: ***W*** $\to$ ***W*** fulfilling

$$\mathcal{C}^2 = \mathbf{1}, \tag{8.1}$$

$$k(\mathcal{C}\psi, \psi) = 0, \quad \forall \psi \in \boldsymbol{W}, \tag{8.2}$$

$$\mathcal{C} \circ \gamma_\eta = \gamma_\eta \circ \mathcal{C}. \tag{8.3}$$

From (8.3), it follows $\mathcal{C}(\boldsymbol{S}) = \boldsymbol{S}'$, $\mathcal{C}(\boldsymbol{S}') = \boldsymbol{S}$. Then, because of (8.2), $\mathcal{C}$ sends any one-dimensional subspace of ***S*** or ***S***$'$ into its k-conjugated subspace (Proposition 6.3). Note also that, given (8.1), (8.2) is equivalent to $k \circ (\mathcal{C} \times \mathcal{C}) = -\bar{k}$.

We obtain the following characterization of the family of all k-conjugations.

LEMMA 8.1. *Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be k-conjugations. Then*

$$\tilde{\mathcal{C}} = \mathrm{e}^{-it}\,(\cosh x\,\mathcal{C} + i\,\sinh x\,\mathcal{C}\circ\gamma_\eta)$$

with $t, x \in \mathbb{R}$.

Proof. Write $\mathcal{C} = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{A} := \tfrac{1}{2}\,\mathcal{C}\circ(1 - i\gamma_\eta) = \tfrac{1}{2}\,(\mathcal{C} + i\,\mathcal{C}\gamma_\eta),$$
$$\mathcal{B} := \tfrac{1}{2}\,\mathcal{C}\circ(1 + i\gamma_\eta) = \tfrac{1}{2}\,(\mathcal{C} - i\,\mathcal{C}\gamma_\eta)$$

(roughly speaking, $\mathcal{A}$ and $\mathcal{B}$ can be seen as the restrictions of $\mathcal{C}$ to $\boldsymbol{S}$ and $\boldsymbol{S}'$, respectively). Then $\mathcal{B}\mathcal{A}_{|\boldsymbol{S}} = \mathbf{1}_{\boldsymbol{S}}$, $\mathcal{A}\mathcal{B}_{|\boldsymbol{S}'} = \mathbf{1}_{\boldsymbol{S}'}$. Because of (8.2), $\mathcal{A}$ and $\tilde{\mathcal{A}}$ both send any $s \in \boldsymbol{S}$ in the space conjugate to s. It follows that $\tilde{\mathcal{A}} = a\,\mathcal{A}$ with $a \in \mathbb{C}^\times$. Similarly $\tilde{\mathcal{B}} = b\,\mathcal{B}$ with $b \in \mathbb{C}^\times$. We have $\tilde{\mathcal{A}}\tilde{\mathcal{B}} = a\bar{b}\,\mathcal{A}\mathcal{B} = \mathbf{1}_{\boldsymbol{S}}$, hence $a\bar{b} = 1$. Setting $a = \mathrm{e}^{x-it}$ we obtain the stated result. □

A *charge conjugation* is defined to be a k-conjugation fulfilling

$$\mathcal{C}\circ\gamma_v + \gamma_v\circ\mathcal{C} = 0, \quad \forall v \in \boldsymbol{V}.$$

An algebraic complex spinor structure together with a charge conjugation will be called a (real) *algebraic spinor structure*. Using Lemma 8.1, by a straightforward calculation we obtain:

PROPOSITION 8.1. *Let $\mathcal{C}$ be a charge conjugation and $\tilde{\mathcal{C}}$ a k-conjugation. Expressing $\tilde{\mathcal{C}}$ through $\mathcal{C}$ as in Lemma* 8.1 *gives, for any $v \in \boldsymbol{V}$,*

$$\tilde{\mathcal{C}}\circ\gamma_v\circ\tilde{\mathcal{C}} = -(1 + 2\,\sinh^2 x)\,\gamma_v + 2i\,\sinh x\cosh x\,\gamma_v\gamma_\eta.$$

Hence, charge conjugation is unique up to a phase factor.

The expression of a charge conjugation in a Weyl basis turns out to be

$$\mathcal{C} = \mathrm{e}^{-it}(-\zeta_1\otimes\bar{z}^4 + \zeta_2\otimes\bar{z}^3 + \zeta_3\otimes\bar{z}^2 - \zeta_4\otimes\bar{z}^1) \tag{8.4}$$

with $t \in [0, 2\pi)$.

PROPOSITION 8.2. *The group of all automorphisms of $(g, \eta, \gamma, k, \mathcal{C})$ is* $\mathrm{Spin}^\uparrow$.

Proof. Let $\vartheta := \mathrm{e}^{it}\theta_1\ldots\theta_n \in \mathrm{Pin}^{\uparrow c}$, with $\theta_j \in \gamma(\boldsymbol{V})$ (Proposition 6.4). We have $\theta_j\mathcal{C} = -\mathcal{C}\theta_j$, then

$$\vartheta^*(\mathcal{C}) = \vartheta^{-1}\circ\mathcal{C}\circ\vartheta = \mathrm{e}^{-2it}\,\theta_n\ldots\theta_1\mathcal{C}\theta_1\ldots\theta_n = (-1)^n\,\mathrm{e}^{-2it}\,\mathcal{C}.$$

Hence, $\vartheta^*(\mathcal{C}) = \mathcal{C}$ if either n is even and $t = m\,\pi$ (so $\vartheta \in \mathrm{Spin}$), or n is odd and $t = (2m+1)\pi/2$. □

A given Weyl basis yields a spinor structure by letting $t = 0$ in Equation (8.4). Conversely, given a spinor structure, any Weyl basis $(\tilde{\zeta}_a)$ can be transformed to a $\mathcal{C}$-normal Weyl basis (ζ_a) according to $\zeta_a = \mathrm{e}^{it/2}\,\tilde{\zeta}_a$. More generally, let $P \in \mathrm{Pin}^{\uparrow}$. Then $\Lambda := \mathrm{e}^{it/2}\,P \in \mathrm{Pin}^{\uparrow\,c}$ applied to the given Weyl basis yields a $\mathcal{C}$-normal Weyl basis. We obtain

$$\mathcal{C} = -\zeta_1 \otimes \bar{z}^4 + \zeta_2 \otimes \bar{z}^3 + \zeta_3 \otimes \bar{z}^2 - \zeta_4 \otimes \bar{z}^1.$$

From (7.2), it follows $\det P = 1\ \forall P \in \mathrm{Pin}$, hence $t = -(i/2)\log\det\Lambda$.

Using Proposition 6.5, we can see that γ_η, k and $\mathcal{C}$ determine γ up to a Lorentz transformation. Namely:

PROPOSITION 8.3. *Let $\mathcal{C}$ be a charge conjugation. Let $\tilde{\gamma}$: $\boldsymbol{V} \to \mathrm{End}(\boldsymbol{W})$ be a Clifford map such that $\tilde{\gamma}_\eta = \gamma_\eta$ and together with k and $\mathcal{C}$ determines an algebraic spinor structure. Then $\tilde{\gamma}$ coincides with γ up to a Lorentz transformation.*

Since $\mathcal{C}$ is an antilinear involution, its *real* eigenspaces with eigenvalues ± 1 determine a splitting of $\boldsymbol{W}$, with projections $p^{\pm}_{\mathcal{C}} = \frac{1}{2}(1 \pm \mathcal{C})$. Multiplication by i and the action of $\gamma(\boldsymbol{V})$ exchange these subspaces, so the action of $i\gamma(\boldsymbol{V})$ leaves them invariant. The algebra generated by $i\gamma(\boldsymbol{V})$ can be seen as the Dirac algebra of the Minkowski space $i\boldsymbol{V}$, with signature $(3, 1)$; its representation in $\boldsymbol{W}$, called the *Majorana representation*, is therefore the sum of two real representations.

Remark. Signs in the axioms and basic formulas involving the fundamental objects of a (either complex or real) spinor structure can be changed while retaining essentially the same theory. So, for example, it is easily checked that $k'(\phi, \psi) := k(\gamma_\eta\phi, \psi)$ is again a Hermitian form with signature $(2, 2)$, fulfilling $k'(\gamma(u)\phi, \psi) = -k'(\phi, \gamma(u)\psi)$. Moreover, to assign either k or k' is equivalent. Similarly $\mathcal{C}$ could be replaced with $\mathcal{C}\gamma_\eta$, which commutes with γ_v and whose square is $-\mathbf{1}$. Note also that $(\mathbf{1}, i, \mathcal{C}\gamma_\eta, i\mathcal{C}\gamma_\eta)$ generates a representation of the quaternion algebra.

9. Symplectic Forms and 2-Spinor Approach

By a *chiral* 2-*form* we shall mean a 2-form $\varepsilon \in \wedge^2\boldsymbol{W}^\star$ fulfilling

$$\varepsilon(\gamma_v\phi, \psi) = \varepsilon(\phi, \gamma_v\psi).$$

Then we obtain

$$\varepsilon(\gamma_\eta\phi, \psi) = \varepsilon(\phi, \gamma_\eta\psi).$$

PROPOSITION 9.1. *A nonzero chiral 2-form is a symplectic form (i.e. it is nondegenerate) and the chiral splitting turns out to be a symplectic splitting, namely $\varepsilon = \varepsilon_{\boldsymbol{S}} + \varepsilon_{\boldsymbol{S}'}$, where $\varepsilon_{\boldsymbol{S}}$ and $\varepsilon_{\boldsymbol{S}'}$ can be viewed as symplectic forms on $\boldsymbol{S}$ and $\boldsymbol{S}'$.*

Moreover, let $\tilde{\varepsilon}$ be another chiral 2-form. Then $\tilde{\varepsilon} = b\,\varepsilon,\ b \in \mathbb{C}$.

Proof. Write $p_{\boldsymbol{S}} := \frac{1}{2}(\mathbf{1} - i\gamma_\eta)$. Then $\varepsilon \circ (p_{\boldsymbol{S}} \times \mathbf{1}) = \varepsilon \circ (\mathbf{1} \times p_{\boldsymbol{S}})$. Since $p_{\boldsymbol{S}}$ is a projection, we obtain $\varepsilon_{\boldsymbol{S}} := \varepsilon \circ (p_{\boldsymbol{S}} \times p_{\boldsymbol{S}}) = \varepsilon \circ (p_{\boldsymbol{S}} \times \mathbf{1})$, and similar for $p_{\boldsymbol{S}'} := \frac{1}{2}(\mathbf{1} + i\gamma_\eta)$. Thus, $\varepsilon = \varepsilon_{\boldsymbol{S}} + \varepsilon_{\boldsymbol{S}'}$. Moreover, we have $\gamma\, p_{\boldsymbol{S}} = p_{\boldsymbol{S}'}\, \gamma, \quad p_S\, \gamma = \gamma\, p_{\boldsymbol{S}'}$, hence

$$\varepsilon_S \circ (\gamma \times \mathbf{1}) = \varepsilon_{\boldsymbol{S}'} \circ (\mathbf{1} \times \gamma), \quad \varepsilon_{\boldsymbol{S}'} \circ (\gamma \times \mathbf{1}) = \varepsilon_{\boldsymbol{S}} \circ (\mathbf{1} \times \gamma).$$

This implies that either $\varepsilon = 0$ or both $\varepsilon_{\boldsymbol{S}}$ and $\varepsilon_{\boldsymbol{S}'}$ are nonzero. Now observe that any $\tilde{\varepsilon} \in \wedge^2 \boldsymbol{S}^\star \oplus \wedge^2 \boldsymbol{S}'^\star$ can be written as $\tilde{\varepsilon} = a\, \varepsilon_{\boldsymbol{S}} + b\, \varepsilon_{\boldsymbol{S}'}$ with $a, b \in \mathbb{C}$. We get

$$\tilde{\varepsilon} \circ (\gamma \times \mathbf{1}) = a\, \varepsilon_{\boldsymbol{S}'} \circ (\mathbf{1} \times \gamma) + b\, \varepsilon_{\boldsymbol{S}} \circ (\mathbf{1} \times \gamma).$$

Then $\tilde{\varepsilon} \circ (\gamma \times \mathbf{1}) = \tilde{\varepsilon} \circ (\mathbf{1} \times \gamma)$ iff $a = b$. □

Hence, the the set of chiral 2-forms is a (complex) one-dimensional subspace $\boldsymbol{Q}^\star \subset \wedge^2 \boldsymbol{W}^\star$ (the existence of nonzero chiral forms follows from formula (9.1) below). Its dual space $\boldsymbol{Q}$ can be identified with a one-dimensional subspace of $\wedge^2 \boldsymbol{W}$, i.e. the space of all $\omega \in \wedge^2 \boldsymbol{W}$ such that $\omega \circ (\gamma^\star_v \times \mathbf{1}) = \omega(\mathbf{1} \times \gamma^\star_v),\ v \in \boldsymbol{V}$, where $\gamma^\star\colon\ \boldsymbol{V} \to \mathrm{End}(\boldsymbol{W}^\star)$ is the dual (transpose) representation. The identification is defined in such a way to recover the standard conventions [HT85, PR84] about chiral symplectic forms. Namely, we write $\omega^\star \equiv \varepsilon$ iff $\frac{1}{4}\langle \varepsilon, \omega \rangle := \frac{1}{8}\, i_\omega \varepsilon = 1$, where $i_\omega \varepsilon$ is the standard exterior algebra contraction [Go69]. If $\omega^\star = \varepsilon$, then we set $\varepsilon^\# := \omega^\# = -(\varepsilon^\flat)^{-1}$, where $\omega^\#(\lambda) := \omega(\lambda, \_)$, $\lambda \in \boldsymbol{W}^\star$. We also write $\varepsilon^{ab} := \omega^{ab}$, and obtain

$$\varepsilon_{ab}\varepsilon^{ac} = \delta^c_b, \qquad \varepsilon_{ab}\varepsilon^{ab} = \delta^b_b = 4, \qquad \varepsilon^\#(\lambda) = \varepsilon^{ab}\lambda_a \zeta_b.$$

On a Weyl basis, the expressions of mutually dual elements $\omega \in \boldsymbol{Q}$ and $\varepsilon \in \boldsymbol{Q}^\star$ are

$$\begin{aligned} \omega &= \tfrac{2}{b}\,(\zeta_1 \wedge \zeta_2 + \zeta_3 \wedge \zeta_4) = \tfrac{1}{b}\,(\zeta_1 \otimes \zeta_2 - \zeta_2 \otimes \zeta_1 + \zeta_3 \otimes \zeta_4 - \zeta_4 \otimes \zeta_3), \\ \varepsilon &= 2\, b\,(z^1 \wedge z^2 + z^3 \wedge z^4) = b\,(z^1 \otimes z^2 - z^2 \otimes z^1 + z^3 \otimes z^4 - z^4 \otimes z^3), \end{aligned}$$

with $b \in \mathbb{C}^\times$.

The Hermitian 2-form k yields a Hermitian 2-form $k_\otimes$ on the tensor algebra $\otimes \boldsymbol{W}$. We define $k_{\boldsymbol{Q}}$ to be $\frac{1}{4}\, k_\otimes$ restricted to $\boldsymbol{Q}$, then $k_{\boldsymbol{Q}}$ turns out to be a Hermitian metric. The corresponding quadratic form $k^\diamondsuit_{\boldsymbol{Q}}$ has the coordinate expression

$$k^\diamondsuit_{\boldsymbol{Q}}(\omega) = \tfrac{1}{4}\, k_{a^{\cdot}\, b}\, k_{c^{\cdot}\, d}\, \bar{\omega}^{a^{\cdot}\, c^{\cdot}}\, \omega^{bd}.$$

Similarly, we have the inverse Hermitian metric on $\boldsymbol{Q}^\star$. In a Weyl basis, the expressions of $k_{\boldsymbol{Q}}$-unimodular (or k-*normalized*) elements $\omega \in \boldsymbol{Q}$ and $\varepsilon \in \boldsymbol{Q}^\star$ are

$$\begin{aligned} \omega &= \mathrm{e}^{it'}\,(\zeta_1 \otimes \zeta_2 - \zeta_2 \otimes \zeta_1 + \zeta_3 \otimes \zeta_4 - \zeta_4 \otimes \zeta_3), \\ \varepsilon &= \mathrm{e}^{it}\,(z^1 \otimes z^2 - z^2 \otimes z^1 + z^3 \otimes z^4 - z^4 \otimes z^3) \end{aligned}$$

with $t', t \in \mathbb{R}$. Note that, in the above formulas, $\omega^\star = \varepsilon$ iff $t' = -t$. Then we see that ω is k-normalized iff $\omega^\star$ is such.

THEOREM 9.1. *There is a one-to-one correspondence between charge conjugations and k-normalized chiral forms, given by*

$$\mathcal{C} = -i\gamma_\eta \circ k^\# \circ \varepsilon^\flat, \qquad \varepsilon = i\, k \circ (\mathcal{C}\gamma_\eta \times \mathbf{1}).$$

Proof. Let ε be a k-normalized chiral form. From $\gamma_v \circ k^\# = k^\# \circ \gamma_v^\bigstar$ and $\varepsilon^\flat \circ \gamma_v = \gamma_v^\bigstar \circ \varepsilon^\flat$ it follows that γ_v commutes with $k^\# \circ \varepsilon^\flat$.

Set $\mathcal{C}_\varepsilon := -i\gamma_\eta \circ k^\# \circ \varepsilon^\flat$. Taking into account the antilinearity of $k^\flat$ we have $k^\flat \circ \mathcal{C}_\varepsilon = i\,\varepsilon^\flat \circ \gamma_\eta$. Then we obtain

$$\begin{aligned}
&(\mathcal{C}_\varepsilon)^2 = (-i\gamma_\eta)\circ(i\gamma_\eta)\circ k^\# \circ \varepsilon^\flat \circ k^\# \circ \varepsilon^\flat = -k^\# \circ \varepsilon^\flat \circ k^\# \circ \varepsilon^\flat = \mathbf{1},\\
&k(\mathcal{C}_\varepsilon\psi, \psi) = \langle k^\flat \circ \mathcal{C}_\varepsilon(\psi), \psi\rangle = i\,\langle \varepsilon^\flat \circ \gamma_\eta(\psi), \psi\rangle = i\,\varepsilon(\gamma_\eta\psi, \psi) = 0,\\
&\mathcal{C}_\varepsilon \circ \gamma_v = -i\gamma_\eta \circ k^\# \circ \varepsilon^\flat \circ \gamma_v = -i\gamma_\eta \circ \gamma_v \circ k^\# \circ \varepsilon^\flat\\
&\qquad = \gamma_v \circ (i\gamma_\eta \circ k^\# \circ \varepsilon^\flat) = -\gamma_v \circ \mathcal{C}_\varepsilon.
\end{aligned}$$

Hence, $\mathcal{C}_\varepsilon$ is a charge conjugation.

Conversely, let $\mathcal{C}$ be a charge conjugation and set $\varepsilon_\mathcal{C} := i\,k \circ (\mathcal{C}\gamma_\eta \times \mathbf{1})$. We have $k^\# \circ \varepsilon^\flat_\mathcal{C} = -i\,\mathcal{C}\gamma_\eta$. Then

$$\begin{aligned}
&\varepsilon_\mathcal{C}(\phi, \psi) = i\,k(\mathcal{C}\gamma_\eta\phi, \psi) = -i\,\bar{k}(\phi, \mathcal{C}\gamma_\eta\psi) = -i\,k(\mathcal{C}\gamma_\eta\psi, \phi) = -\varepsilon_\mathcal{C}(\psi, \phi),\\
&\varepsilon_\mathcal{C}(\gamma_v\phi, \psi) = i\,k(\mathcal{C}\gamma_\eta\gamma_v\phi, \psi) = i\,k(\gamma_v\mathcal{C}\gamma_\eta\phi, \psi)\\
&\qquad = i\,k(\mathcal{C}\gamma_\eta\phi, \gamma_v\psi) = \varepsilon(\phi, \gamma_v\psi),\\
&k^\# \circ \varepsilon^\flat_\mathcal{C} \circ k^\# \circ \varepsilon^\flat_\mathcal{C} = (-i\,\mathcal{C}\gamma_\eta)\circ(-i\,\mathcal{C}\gamma_\eta) = -\mathbf{1}.
\end{aligned}$$

Hence, $\varepsilon_\mathcal{C}$ is a k-normalized chiral form. □

If ε and $\mathcal{C}$ are corresponding objects as in the above theorem, then one finds that $\varepsilon \circ (\mathcal{C} \times \mathcal{C}) = \bar{\varepsilon}$. Moreover, $\mathrm{e}^{it}\varepsilon$ corresponds to $\mathrm{e}^{-it}\mathcal{C}$. A Weyl basis is $\mathcal{C}$-normal iff

$$\varepsilon = 2(z^1 \wedge z^2 + z^3 \wedge z^4) = z^1 \otimes z^2 - z^2 \otimes z^1 + z^3 \otimes z^4 - z^4 \otimes z^3, \tag{9.1}$$

namely, $\varepsilon_{\boldsymbol{S}} = \varepsilon_{AB}\, z^A \wedge z^B$ where ε_{AB} is exactly the antisymmetric Ricci matrix.

Next we present the basic ingredients for a comparison between 4-spinor and 2-spinor approaches to the Dirac equation [BTu87, HT85, PR84, PR88, Wa84]; essentially, the 2-spinor approach consists in identifying $\boldsymbol{S}'$ with some space associated with $\boldsymbol{S}$. The most convenient choice turns out to be $\boldsymbol{S}' \cong \overline{\boldsymbol{S}^\bigstar}$, where identification is via the map $k^\flat$ (Proposition 6.2), but note that one can take $\boldsymbol{S}' \cong \overline{\boldsymbol{S}}$ when a charge conjugation is chosen. A somewhat different approach is given in [Tr94], where $\boldsymbol{W} \equiv \boldsymbol{S} \oplus \boldsymbol{S}$.

Every chiral 2-form ε determines a bilinear form g_ε on the fibres of $\boldsymbol{S} \otimes \overline{\boldsymbol{S}}$, defined, for decomposable elements, by

$$g_\varepsilon(x \otimes \bar{y}, u \otimes \bar{v}) := \tfrac{1}{2}\,\varepsilon(x, u)\bar{\varepsilon}(\bar{y}, \bar{v}).$$

Then g_ε restricted to the Hermitian real subbundle $\boldsymbol{H}$ (Section 1.4) turns out to be a Lorentz metric; decomposable elements of $\boldsymbol{S}\otimes\overline{\boldsymbol{S}}$ are g_ε-isotropic.

The identification $\boldsymbol{S}'\cong\overline{\boldsymbol{S}^\star}$ leads to considering the linear map

$$\begin{aligned}\chi_\varepsilon &: \boldsymbol{S}\otimes\overline{\boldsymbol{S}} \to (\boldsymbol{S}\otimes\overline{\boldsymbol{S}})\oplus(\overline{\boldsymbol{S}^\star}\otimes\boldsymbol{S}^\star)\subset\boldsymbol{W}\otimes\boldsymbol{W}^\star\equiv\mathrm{End}(\boldsymbol{W})\\ &: u\otimes\bar{v}\mapsto u\otimes\bar{v}+\bar{\varepsilon}^\flat(\bar{v})\otimes\varepsilon^\flat(u).\end{aligned}$$

Then χ_ε restricted to $\boldsymbol{H}$ turns out to be a Clifford map. Note now that g_ε and χ_ε do not change when ε is multiplied by a phase factor. Since the assigned complex spinor structure determines the distinguished family of k-unimodular chiral 2-forms, we actually have a distinguished Lorentz metric $g' := g_\varepsilon$ and a distinguished Clifford map $\chi := \chi_\varepsilon$.

The *Pauli basis* (σ_α) of $\boldsymbol{H}$ associated with a symplectic basis (ζ_A) of $\boldsymbol{S}$ is defined by $\sigma_\alpha := {\sigma_\alpha}^{AB^\cdot}\zeta_A\otimes\bar{\zeta}_{B^\cdot}$, where the $({\sigma_\alpha}^{AB^\cdot})$'s are the Pauli matrices. One checks easily that (σ_α) is an orthonormal basis. Let (e_α) be the orthonormal basis of $\boldsymbol{V}$ given by $\gamma(e_\alpha)=\chi(\sigma_\alpha)$. Then a direct calculation shows that the k-normal basis (ζ_a) determined by (ζ_A), namely $\zeta_3 := -\bar{z}^1$ and $\zeta_4 := -\bar{z}^2$, is a Weyl basis associated with (e_α). We obtain

PROPOSITION 9.2. *There exists a unique isometry* $\tau\colon\ \boldsymbol{V}\to\boldsymbol{H}$ *such that* $\chi\circ\tau=\gamma$.

Namely, we have $\sigma_\alpha=\tau(e_\alpha)$.

The equivalence of 2-spinor and 4-spinor approaches then follows: assignment of an algebraic complex spinor structure on $(\boldsymbol{V},g)$ is equivalent to that of a two-dimensional complex vector space $\boldsymbol{S}$ together with a Hermitian metric on $\wedge^2\boldsymbol{S}$ and an isometry $\tau\colon\ \boldsymbol{V}\to\boldsymbol{H}\subset\boldsymbol{S}\otimes\overline{\boldsymbol{S}}$. The components of τ are the so-called *Infeld–van der Waerden symbols* [PR84].

10. Observers and Hermitian Metrics

In physics texts, where spinors and the Dirac algebra are often treated in terms of matrices, the operations of transposition and Hermitian conjugation are commonly used. In this section we see how those operations are not canonical, but related to the choice of an observer.

By taking an adapted Weyl basis (Proposition 7.1), one proves:

PROPOSITION 10.1. *Let* $u\in\boldsymbol{V}$ *be a unit timelike vector. Then* $h := k\circ(\gamma_u\times\mathbf{1})\in\overline{\boldsymbol{W}^\star}\otimes\boldsymbol{W}^\star$*, i.e.* $h(\phi,\psi) := k(\gamma_u\phi,\psi)=k(\phi,\gamma_u\psi)$*, is a Hermitian metric, either positive or negative definite.*

The Hermitian metric h of the above proposition fulfills

$$h(\boldsymbol{S},\boldsymbol{S}')=0,$$

$$\begin{aligned}h(\gamma_v\phi,\psi) &= -h(\phi,\gamma_v\psi)+2g(u,v)\,h(\phi,\gamma_u\psi)=\\ &= h(\phi,\gamma_u\gamma_v\gamma_u\psi)\end{aligned}$$

for all $v \in \boldsymbol{V}$ (the latter formula says that the Hermitian conjugate of γ_v with respect to h is $\gamma_v^\dagger = \gamma_u \gamma_v \gamma_u$). Moreover, for any charge conjugation $\mathcal{C}$, we have

$$h \circ (\mathcal{C} \times \mathcal{C}) = \bar{h}.$$

Clearly, all u belonging to a given half of the timelike cone yield a Hermitian metric of the same signature. Hence, we obtain a distinguished time-orientation of $\boldsymbol{V}$, given by the requirement that future-pointing timelike unit vectors (called 'observers') yield a positive-definite Hermitian metric.

Given an observer, a Weyl basis such that $\gamma(u) = \gamma_0$ (Section 7) turns out to be orthonormal with respect to the induced h. The corresponding Dirac basis is orthonormal with respect to both h and k. Then the eigenspaces of γ_u relative to eigenvalues ± 1 can be characterized by the restriction of k to coincide with $\pm h$. Moreover, the Hermitian forms k and h are mutually normalized, namely we have

$$k^\# \circ h^\flat = h^\# \circ k^\flat = \gamma_u, \tag{10.1}$$

i.e. $k^{a \cdot b} h_{a \cdot c} = h^{a \cdot b} k_{a \cdot c} = (\gamma_u)^b{}_c$. Conversely, one can prove the following characterization of the family of all observer-induced Hermitian metrics.

PROPOSITION 10.2. *Let h be a (positive definite) k-normalized Hermitian metric on $\boldsymbol{W}$ such that $\forall v \in \boldsymbol{V}$ one has $\gamma_v^\dagger \in \gamma(\boldsymbol{V})$. Then there is a unique observer u such that $h = k \circ (\gamma_u \times \mathbf{1})$; this is given by formula* (10.1).

Remark. Hermitian conjugation in standard physics texts, indicated by †, is taken relatively to the Hermitian metric h associated with the element e_0 of a given orthonormal basis. Ordinary transposition, indicated by $^\mathrm{T}$, corresponds to taking complex space conjugation (not charge conjugation) together with Hermitian conjugation relatively to h. Charge conjugation is usually written (see [IZ80], p. 85) as $\psi^c = \eta_c C \bar{\psi}^\mathrm{T}$, where η_c is a phase factor. The Dirac's adjoint $\bar{\psi}$ corresponds in our notation to $k^\flat(\psi)$, then $\bar{\psi}^\mathrm{T}$ corresponds to $\bar{h}^\# \circ k^\flat(\psi) = \bar{\gamma}_0(\psi) = \mathcal{K} \circ \gamma_0(\psi)$, where $\mathcal{K}$: $\boldsymbol{W} \to \overline{\boldsymbol{W}}$ denotes complex space conjugation (Section 1.4). So we can write the above usual definition for charge conjugation as $\mathcal{C}(\psi) \equiv \psi^c = \eta_c C \circ \mathcal{K} \circ \gamma_0(\psi)$, i.e. $C = \bar{\eta}_c \mathcal{C} \circ \gamma_0 \circ \mathcal{K}$, a *linear* map.

PART III: DIRAC EQUATION

11. Spinor Structures in General Relativity

Henceforth, we shall consider a spacetime $(\boldsymbol{M}, g)$, namely $\boldsymbol{M}$ is assumed to be an oriented connected Hausdorff manifold and

$$g\colon\ \boldsymbol{M} \to \mathbb{L}^2 \otimes T^*\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} T^*\boldsymbol{M}$$

a 'scaled' Lorentz metric (Section 2), of signature $(1,3)$. We also assume $(\boldsymbol{M}, g)$ to be time-oriented. The metric connection on $T\boldsymbol{M}$ will be indicated by Γ.

A *complex spinor structure* on $(\boldsymbol{M}, g)$ is a smooth algebraic complex spinor structure on the fibres. Namely we assume a complex vector bundle $\pi_{\boldsymbol{W}} : \boldsymbol{W} \to \boldsymbol{M}$ with four-dimensional fibres, a Clifford map γ: $\mathbb{L}^* \otimes T\boldsymbol{M} \to \mathrm{End}(\boldsymbol{W})$ over $\boldsymbol{M}$ and a scaled Hermitian 2-form k: $\boldsymbol{M} \to \mathbb{L}^{-3} \otimes \overline{\boldsymbol{W}^\star} \otimes_{\boldsymbol{M}} \boldsymbol{W}^\star$, smoothly determining an algebraic complex spinor structure on each fibre. The Clifford map γ can be viewed as a scaled *soldering form* from $T\boldsymbol{M}$ to the bundle of endomorphisms of $\boldsymbol{W}$.

Note that the role which, in the algebraic setting, was of the Minkowski space $\boldsymbol{V}$, is now played by the vector bundle $\boldsymbol{V} := \mathbb{L}^* \otimes T\boldsymbol{M} \to \boldsymbol{M}$, on which the spacetime scaled metric g can be viewed as a true Lorentz metric. Moreover, since the Hermitian 2-form k is now scaled, k-normal frames (ζ_a) are frames of $\mathbb{L}^{3/2} \otimes \boldsymbol{W}$, so when a section ψ: $\boldsymbol{M} \to \boldsymbol{W}$ is expressed as $\psi^a \zeta_a$, its components ψ^a: $\boldsymbol{M} \to \mathbb{C} \otimes \mathbb{L}^{-3/2}$. We write $k = k_{a^\cdot b} \bar{z}^{a^\cdot} \otimes z^b$ with $k_{a^\cdot b}$: $\boldsymbol{M} \to \mathbb{C}$.

The task of reformulating the statements about algebraic spinor structures in the General Relativistic context presents no difficulty. In particular, note that we have a chiral bundle splitting $\boldsymbol{W} = \boldsymbol{S} \oplus_{\boldsymbol{M}} \boldsymbol{S}'$ and a Hermitian line bundle $\boldsymbol{Q} \subset \mathbb{L}^3 \otimes \wedge^2 \boldsymbol{W} \to \boldsymbol{M}$.

In order to use the whole algebraic stuff we only need one further result.

PROPOSITION 11.1. *Given a complex spinor structure on* $(\boldsymbol{M}, g)$, *there exists a Weyl frame in a neighbourhood of any* $p \in \boldsymbol{M}$.

Moreover, let (e_α) *be a given orthonormal frame of* $\mathbb{L}^* \otimes T\boldsymbol{M}$ *in a neighbourhood of* p. *Then there exists a Weyl frame* (ζ_a) *associated with* (e_α).

Proof. In a neighbourhood of $p \in \boldsymbol{M}$, we choose a frame of $\boldsymbol{S}$ and get the induced k-normal frame $(\tilde{\zeta}_a)$ of $\boldsymbol{W}$. Moreover, we choose a positively oriented orthonormal frame of $\mathbb{L}^* \otimes T\boldsymbol{M}$. Pointwise we can use the same argument as in the proof of Proposition 7.1. The function y: $\boldsymbol{M} \to \mathbb{R}$ determined by the argument is obviously smooth, hence the induced Weyl frame (ζ_a) is smooth. □

In other terms, $(\mathbb{L}^* \otimes T\boldsymbol{M}) \times_{\boldsymbol{M}} (\mathbb{L}^{3/2} \otimes \boldsymbol{W}) \to \boldsymbol{M}$ can be seen as a vector bundle associated with the principal bundle of Weyl frames, whose structure group is $\mathrm{Pin}^{+\uparrow}(1,3)$. By analogy with the results on complex spinor structures on Riemannian manifolds (e.g. see [BLM89], appendix D), one may expect the existence of a complex spinor structure to require less strict topological conditions than a spinor structure.

In the next sections we shall develop Dirac's theory on a curved background, assuming a complex spinor structure on $\boldsymbol{M}$ as the basic geometric setting. In general, we shall not assume a global spinor structure (namely a global charge conjugation).

A couple $((e_\alpha), (\zeta_a))$ of frames, respectively of $\mathbb{L}^* \otimes T\boldsymbol{M}$ and $\mathbb{L}^{3/2} \otimes \boldsymbol{W}$, such that γ and k have constant expression (then also the components of the metric tensor g turn out to be constant), will be called a *complex spinor gauge*. The frame (e_α) will be called the *classical gauge*, and (ζ_a) the *quantum gauge*. Proposition 11.1 can be easily generalized to cover complex spinor gauges corresponding to matrix representations of the Dirac algebra other than Weyl's.

Accordingly, a section Λ: $\boldsymbol{M} \to \mathrm{Spin}^{\uparrow c}$ will be called a *gauge transformation*, acting on the classical gauge as $e_\alpha \mapsto \gamma^{-1}(\Lambda \circ \gamma_\alpha \circ \Lambda^{-1})$.

From Proposition 11.1 we have:

COROLLARY 11.1. *There exists a charge conjugation in a neighbourhood of any* $p \in \boldsymbol{M}$.

12. Complex Spinor Connections

In this section we assume a given complex spinor structure on $(\boldsymbol{M}, g)$. By a simple coordinate calculation we prove:

LEMMA 12.1. *Let* Ч, Ч′ *be linear connections on* $\boldsymbol{W} \to \boldsymbol{M}$ *which yield the same connection on* End$(\boldsymbol{W}) \equiv \boldsymbol{W} \otimes_{\boldsymbol{M}} \boldsymbol{W}^\star \to \boldsymbol{M}$. *Then* $Ч' - Ч = \alpha \otimes \mathbf{1}_{\boldsymbol{W}}$, *where* α: $\boldsymbol{M} \to \mathbb{C} \otimes T^*\boldsymbol{M}$ *is a complex* 1*-form.*

Note that the Clifford map γ can be seen as a tensor field

$$\gamma\colon\ \boldsymbol{M} \to \mathbb{L} \otimes T^*\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{W} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{W}^\star.$$

Then a covariant derivative of γ is naturally defined in terms of the metric spacetime connection Γ and of a connection Ч on $\boldsymbol{W} \to \boldsymbol{M}$.

PROPOSITION 12.1. *Let* Ч *be a linear connection on* $\boldsymbol{W} \to \boldsymbol{M}$ *such that* $\nabla\gamma = 0$. *Then the coordinate expression of* Ч, *in the linear bundle charts induced by a complex spinor gauge, is* $Ч_{\lambda}{}^{a}{}_{b} = \alpha_\lambda\, \delta^a{}_b + \frac{1}{4}\Gamma_\lambda^{\rho\sigma}(\gamma_\rho \wedge \gamma_\sigma)^a{}_b$, *with* α_λ: $\boldsymbol{M} \to \mathbb{C}$.

Proof. Write

$$Ч_\lambda = ч_\lambda \mathbf{1} + ч_\lambda^\rho \gamma_\rho + ч_\lambda^{\rho\sigma}\gamma_\rho \wedge \gamma_\sigma + \breve{ч}_\lambda^\rho \gamma_\rho\gamma_\eta + \breve{ч}_\lambda \gamma_\eta$$

with $ч_\lambda, ч_\lambda^\rho, ч_\lambda^{\rho\sigma}, \breve{ч}_\lambda^\rho, \breve{ч}_\lambda$: $\boldsymbol{M} \to \mathbb{C}$, and $ч_\lambda^{\rho\sigma}$ is antisymmetric in the upper indices. For any $\theta : \boldsymbol{M} \to \mathrm{End}(\boldsymbol{W})$ we have $\nabla_\lambda\theta = \partial_\lambda\theta + \theta Ч_\lambda - Ч_\lambda\theta$.

Since $\hat{\gamma}$ can be seen as the wedge extension of γ, we have $\nabla\hat{\gamma} = 0$. Hence, from $\nabla\eta = 0$ it follows $\nabla\gamma_\eta = 0$, namely $2(ч_\lambda^\rho \gamma_\eta\gamma_\rho + \breve{ч}_\lambda^\rho\gamma_\rho) = 0$ which implies $ч_\lambda^\rho = \breve{ч}_\lambda^\rho = 0$.

In a complex spinor gauge $\partial_\lambda\gamma = 0$, so the condition $\nabla\gamma = 0$ can be expressed as $\nabla_\lambda[Ч]\gamma_\beta = -\Gamma_{\lambda}{}^{\alpha}{}_{\beta}\gamma_\alpha$. Then we get $\Gamma_{\lambda}{}^{\alpha}{}_{\beta}\gamma_\alpha = Ч_\lambda\gamma_\beta - \gamma_\beta Ч_\lambda = 2\breve{ч}_\lambda\gamma_\eta\gamma_\beta + 2ч_\lambda^{\rho\sigma}(g_{\beta\sigma}\gamma_\rho - g_{\beta\rho}\gamma_\sigma)$. Hence, $\breve{ч}_\lambda = 0$ and $\Gamma_{\lambda}{}^{\alpha}{}_{\beta}\gamma_\alpha = 2ч_\lambda^{\rho\sigma}(g_{\beta\sigma}\gamma_\rho - g_{\beta\rho}\gamma_\sigma)$. The latter relation yields $ч_\lambda^{\alpha\rho} = \frac{1}{4}\Gamma_{\lambda}{}^{\alpha\rho}$. No condition is to be imposed on $ч_\lambda$, which is renamed α_λ. □

Namely, a connection Ч such that $\nabla\gamma = 0$ yields, on the domain of a gauge, a complex 1-form α, which can be characterized as the difference between Ч and the trivial connection determined by the gauge.

On the domain of a given gauge, we indicate by $Ч_\lambda \colon \boldsymbol{M} \to \mathrm{End}(\boldsymbol{W})$ the endomorphisms whose matrix expression is $(Ч^{a}_{\lambda b})$, so the above expression for Ч can be written (see Section 1.3) $Ч_\lambda = \alpha_\lambda + \frac{1}{4}\Gamma^{\rho\sigma}_{\lambda}\gamma_\rho \wedge \gamma_\sigma$. Note how $Ч_\lambda\, \mathrm{d}x^\lambda$ can be seen as a 'connection form' valued in the Lie algebra of the complex spinor group (5.1), a fact clearly related to the principal bundle approach. We have

$$
\begin{aligned}
&k \circ (Ч_\lambda \times \mathbf{1}) = -k \circ (\mathbf{1} \times Ч_\lambda),\\
&k \circ (\gamma^\lambda Ч_\lambda \times \mathbf{1}) = -k \circ (\mathbf{1} \times Ч_\lambda \gamma^\lambda).
\end{aligned} \tag{12.1}
$$

Given a gauge chart over an open domain $\boldsymbol{X} \subset \boldsymbol{M}$ and a complex 1-form $\alpha\colon \boldsymbol{X} \to \mathbb{C} \otimes T\boldsymbol{X}$, we obtain a connection on $\boldsymbol{W}_{\boldsymbol{X}} := \pi_{\boldsymbol{W}}^{-1}(\boldsymbol{X}) \to \boldsymbol{X}$ by requiring its expression in the chart to be that of Proposition 12.1.

PROPOSITION 12.2. *Consider two gauge charts over domains* $\boldsymbol{X}, \boldsymbol{X}' \subset \boldsymbol{M}$, *together with complex* 1*-forms* α *and* α'. *Then the two connections* Ч *and* Ч$'$, *respectively determined on* $\boldsymbol{X}$ *and* $\boldsymbol{X}'$ *by the expression of Proposition* 12.1, *coincide on* $\boldsymbol{X} \cap \boldsymbol{X}'$ *iff*

$$
\alpha' - \alpha = -\tfrac{1}{4}\,\mathrm{d}(\log\det\Lambda) = -\frac{i}{2}\,\mathrm{d}t,
$$

where $\Lambda\colon \boldsymbol{M} \to \mathrm{Pin}^c$ *is the gauge transformation and* $\det\Lambda := \mathrm{e}^{2it}$.

Proof. The condition that Ч and Ч$'$ coincide can be expressed in terms of the induced endomorphisms $Ч_\lambda$ and $Ч'_\lambda$ as $Ч'_\lambda = Ч_\lambda + \Lambda\partial_\lambda\Lambda'$, where $\Lambda' := \Lambda^{-1}$, that is

$$
\alpha'_\lambda + \tfrac{1}{4}\Gamma'^{\rho\sigma}_{\lambda}\gamma'_\rho\gamma'_\sigma = \alpha_\lambda + \tfrac{1}{4}\Gamma^{\rho\sigma}_{\lambda}\gamma_\rho\gamma_\sigma + \Lambda\partial_\lambda\Lambda'.
$$

Taking the trace of this equation we get the stated relation between α and α'. The traceless part turns out to be satisfied, after some calculations, as a consequence of

$$
\begin{aligned}
&\gamma'_\rho := \tilde{\Lambda}^{\alpha}_{\rho}\gamma_\alpha = \Lambda\gamma_\rho\Lambda',\\
&\Gamma'^{\rho\sigma}_{\lambda} = \tilde{\Lambda}'^{\rho}_{\alpha}\tilde{\Lambda}'^{\sigma}_{\beta}\Gamma^{\alpha\beta}_{\lambda} + g^{\alpha\beta}\tilde{\Lambda}'^{\sigma}_{\beta}\partial_\lambda\tilde{\Lambda}'^{\rho}_{\alpha},
\end{aligned}
$$

where $\tilde{\Lambda} := \phi(\Lambda)$ is the Lorentz transformation induced by Λ (Section 5). □

Now consider a gauge atlas of $\boldsymbol{W}$ together with a family of complex 1-forms, one for each chart. We obtain a family of local connections on $\boldsymbol{W}$. These yield a global connection iff any two 1-forms of the family fulfill the condition of the above lemma on the intersection (if not empty) of their domains. Conversely, a connection such that $\nabla\gamma = 0$ determines a family of local 1-forms with that property; hence it yields a *global* real 1-form, which on the domain of each gauge is given by $\alpha + \bar{\alpha}\colon \boldsymbol{M} \to T^*\boldsymbol{M}$. This can be characterized as follows:

PROPOSITION 12.3. *Let* Ч *be a linear connection on* $\boldsymbol{W} \to \boldsymbol{M}$ *such that* $\nabla\gamma = 0$. *We have*

$$\nabla k = (\alpha + \bar{\alpha}) \otimes k.$$

A linear connection Ч on $\boldsymbol{W} \to \boldsymbol{M}$ such that $\nabla\gamma = 0$ and $\nabla k = 0$ will be called a *complex spinor connection*. Then Ч determines, on the domain of each gauge, an imaginary 1-form α (or equivalently a real 1-form a, $\alpha = ia$).

PROPOSITION 12.4. *Let* Ч *be a complex spinor connection. The connection determined by* Ч *on the tensor algebra of* $\boldsymbol{W}$ *reduces to connections on the Hermitian line bundles* $\boldsymbol{Q}$ *and* $\boldsymbol{Q}^\star$. *In particular, the covariant derivative of a local* k*-normalized chiral form* ε: $\boldsymbol{M} \to \boldsymbol{Q}^\star$ *and the corresponding charge conjugation* $\mathcal{C}$ *have, on the domain of a given gauge, the expression*

$$\nabla\varepsilon = i(2a + \mathrm{d}t) \otimes \varepsilon,$$
$$\nabla\mathcal{C} = -i(2a + \mathrm{d}t) \otimes \mathcal{C},$$

with $t := 1/2i \log\det\Lambda$, *where* Λ *is a gauge transformation which makes the considered gauge* ε*-symplectic.*

Conversely:

THEOREM 12.1. *Every Hermitian connection on* $\boldsymbol{Q} \to \boldsymbol{M}$ *determines a unique complex spinor connection.*

Proof. Consider a gauge atlas of $(\mathbb{L}^* \otimes T\boldsymbol{M}) \times_{\boldsymbol{M}} (\mathbb{L}^{3/2} \otimes \boldsymbol{W})$. On the domain of each gauge consider the normal frame of $\boldsymbol{Q}^\star$ induced by the gauge, namely the k-normalized chiral form ε whose expression is given by (9.1). The Hermitian connection on $\boldsymbol{Q} \to \boldsymbol{M}$ determines an imaginary 1-form ia by $\nabla_\lambda\varepsilon = 2ia_\lambda\,\varepsilon$. On the intersection of two gauges, the induced 1-forms are related by $a' - a = -\frac{1}{2}\,\mathrm{d}t = (i/4)\,\mathrm{d}(\log\det\Lambda)$, where Λ is the related gauge transformation. Now the coordinate formula of Proposition 12.1 determines a complex spinor connection on the domain of each gauge. From Proposition 12.2, it follows that the connections determined by any two gauge charts coincide on the intersection, namely we have a global connection. □

Note that any two gauge-related forms a and a' fulfill $\mathrm{d}a = \mathrm{d}a'$, namely for any complex spinor connection, the 2-form $\mathrm{d}a$: $\boldsymbol{M} \to \wedge^2 T^*\boldsymbol{M}$ is globally defined. The curvature tensor (see also [GP82]) of a complex spinor connection Ч turns ot to be

$$R[Ч] = 2i\,\mathrm{d}a \otimes \mathbf{1} + \tfrac{1}{4}\,\langle R[\Gamma], \gamma \wedge \gamma\rangle,$$

that is

$$R[Ч]_{\lambda\mu} = i(\partial_\lambda a_\mu - \partial_\mu a_\lambda) + \tfrac{1}{4} R[\Gamma]^{\rho\sigma}_{\lambda\mu}\,\gamma_\rho \wedge \gamma_\sigma,$$

where $R[\Gamma]$ is the curvature tensor of the metric connection Γ. In terms of the principal bundle approach to connections, $R[Ч]_{\lambda\mu}$ are the components of the curvature 2-form valued in the Lie algebra of the Spin^c group.

PROPOSITION 12.5. *A complex spinor connection* $Ч$ *together with a spinor structure (charge conjugation* $\mathcal{C}$ *or, equivalently,* k*-normalized chiral form* ε*) determine an imaginary* 1*-from* ia*. We have* $\nabla\varepsilon = 2ia \otimes \varepsilon,\ \nabla\mathcal{C} = -2ia \otimes \mathcal{C}$*.*

Conversely, given a spinor structure, any imaginary 1-from ia determines a complex spinor connection. The complex spinor connection $Ч^{\mathcal{C}}$ determined by $-ia$ is called the *charge conjugated connection* of $Ч$. It can be characterized through its covariant derivative by

$$\nabla^{\mathcal{C}}_{\lambda}(\mathcal{C}\psi) = \mathcal{C}\nabla_{\lambda}\psi \quad \Leftrightarrow \quad \nabla^{\mathcal{C}}_{\lambda}\psi = \mathcal{C}\nabla_{\lambda}(\mathcal{C}\psi) = \nabla_{\lambda}\psi + (\mathcal{C}\nabla_{\lambda}\mathcal{C})\psi. \tag{12.2}$$

Note that

$$\nabla^{\mathcal{C}}\theta = \nabla\theta, \quad \forall\theta\colon\ \boldsymbol{M} \to \mathrm{End}(\boldsymbol{W}). \tag{12.3}$$

For each complex spinor connection we introduce the *nabla slash* operator, acting on sections $\psi\colon\ \boldsymbol{M} \to \boldsymbol{W}$, given by

$$\nabla\!\!\!/\,\psi := \langle g^{\#}\gamma, \nabla\psi\rangle = g^{\lambda\mu}\gamma_{\lambda}\nabla_{\mu}\psi := \gamma^{\lambda}\nabla_{\lambda}\psi.$$

13. Dirac Equation

We use the results of the previous sections to formulate the quantum relativistic theory of one particle with mass $m \in \mathbb{M}$ and charge $q \in \mathbb{Q} = \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$ (Section 2), subjected to given electromagnetic and gravitational fields. We assume $(\boldsymbol{M}, g)$ to be a spacetime, endowed with a complex spinor structure (Section 11) and a complex spinor connection (Section 12). Moreover, we assume the particle's *quantum history* to be a section $\psi\colon\ \boldsymbol{M} \to \boldsymbol{W}$ obeying the *Dirac equation* $i\nabla\!\!\!/\,\psi - (mc/\hbar)\,\psi = 0$.

It should be clear, from the discussion of Section 12, that we are going to interpret the real 1-form a, determined locally by a complex spinor connection, as electromagnetic 4-potential (see also [GP75, IW33]). More precisely, we set

$$a = \frac{q}{c\hbar}\,A, \quad A\colon\ \boldsymbol{M} \to \mathbb{T} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{-1/2} \otimes T^*\boldsymbol{M}.$$

Then the (local) coordinate expression of the Dirac equation is

$$ig^{\lambda\mu}\gamma_{\lambda}{}^{a}{}_{b}\left(\partial_{\mu}\psi^{b} - i\,\frac{q}{c\hbar}\,A_{\mu}\,\psi^{b} - \tfrac{1}{4}\Gamma_{\mu}^{\rho\sigma}(\gamma_{\rho} \wedge \gamma_{\sigma})^{b}{}_{c}\,\psi^{c}\right) - \frac{mc}{\hbar}\,\psi^{a} = 0.$$

By applying $k^{\flat}$ to the Dirac equation, and taking into account the properties of k, one finds the equivalent equation satisfied by $k^{\flat}\psi$

$$i\nabla_{\lambda}(k^{\flat}\psi) \circ \gamma^{\lambda} + \frac{mc}{\hbar}\,(k^{\flat}\psi) = 0. \tag{13.1}$$

By applying the operator $i\nabla\!\!\!/ + mc/\hbar$ to the Dirac equation, we obtain the general relativistic form of the *Klein–Gordon equation* $\nabla\!\!\!/^2\psi + (m^2c^2/\hbar^2)\,\psi = 0$. By expanding the *spinor Laplacian* $\nabla\!\!\!/^2\psi$, this equation can written as the following (generalized) form of *Lichnerowicz's equation* [BTu87, Cr90, Li64]

$$g^{\lambda\mu}\nabla^2_{\lambda\mu}\psi - \frac{iq}{2c\hbar}\langle F, \gamma\wedge\gamma\rangle\,\psi - \tfrac{1}{4}\,R\psi + \frac{m^2c^2}{\hbar^2}\,\psi = 0,$$

where $F := 2\,dA : \boldsymbol{M} \to \mathbb{T}\otimes\mathbb{L}^{-3/2}\otimes\mathbb{M}^{-1/2}\otimes \wedge^2 T^*\boldsymbol{M}$ is the *electromagnetic field* and R is the scalar curvature of the spacetime connection.

14. Lagrangian Formulation

We shall use known general results on Lagrangian field theories [Ga74, GS73, MM83b, MV96, Tr67].

For brevity, we set $\xi := \mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3$. Then $\eta = \sqrt{|g|}\,\xi$, $|g| := \det(g)$.

The *Dirac Lagrangian density* is defined to be the 4-form $\mathcal{L} := \ell\xi\colon\ J\boldsymbol{W} \to \wedge^4 T^*\boldsymbol{M}$ where $\ell\colon\ J\boldsymbol{W} \to \mathbb{R}$ is given by

$$\ell[\psi] := \ell\circ j\psi := \left[\frac{i}{2}(k(\psi,\nabla\!\!\!/\psi) - k(\nabla\!\!\!/\psi,\psi)) - \frac{mc}{\hbar}\,k(\psi,\psi)\right]\sqrt{|g|}$$

for any section $\psi\colon\ \boldsymbol{M} \to \boldsymbol{W}$ ($j\psi$ denotes the first jet prolongation, see Section 1.2). Its coordinate expression is

$$\ell \;=\; k_{a^{\cdot}b}\left[\frac{i}{2}\,\bar{z}^{a^{\cdot}}\gamma^{\lambda b}{}_{c}(z^c_\lambda - \text{Ч}_{\lambda}{}^{c}{}_{d}z^d) - \right.$$
$$\left. -\,\frac{i}{2}\,\bar{\gamma}^{\lambda a^{\cdot}}{}_{c^{\cdot}}\,(\bar{z}^{c^{\cdot}}_\lambda - \bar{\text{Ч}}_{\lambda}{}^{c^{\cdot}}{}_{d^{\cdot}}\,\bar{z}^{d^{\cdot}})z^b - \frac{mc}{\hbar}\,\bar{z}^{a^{\cdot}}z^b\right]\sqrt{|g|}\,.$$

The Euler–Lagrange operator associated with $\mathcal{L}$ is a fibred morphism*

$$\mathcal{E}\colon\ J\boldsymbol{W} \to \wedge^4 T^*\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{W}^*,$$

where $\boldsymbol{W}^* \subset \boldsymbol{W}^\star \oplus \overline{\boldsymbol{W}^\star}$ is the *real* dual of $\boldsymbol{W}$. $\mathcal{E}$ can be calculated by formally treating z^a and $\bar{z}^{a^{\cdot}}$ as independent real coordinates. Taking into account Equation (12.1) and the identity $\partial_\lambda(\sqrt{|g|}\;g^{\lambda\mu}\gamma_\mu) = \sqrt{|g|}\;(\text{Ч}_\lambda\gamma^\lambda - \gamma^\lambda\text{Ч}_\lambda)$, which can be derived by a coordinate calculation, we obtain

$$\mathcal{E}\circ j\psi = \eta\otimes\left[k^\flat\left(i\nabla\!\!\!/\psi - \frac{mc}{\hbar}\,\psi\right) + \bar{k}^\flat\left(i\nabla\!\!\!/\psi - \frac{mc}{\hbar}\,\psi\right)\right].$$

Then the Dirac equation is the Euler–Lagrange equation of $\mathcal{L}$, namely $\mathcal{E}\circ j\psi = 0$.

The Poincaré–Cartan form is a morphism

$$\Theta\colon\ J\boldsymbol{W} \to \wedge^4 T^*\boldsymbol{W}.$$

* In a general field theory the Euler–Lagrange operator is second or higher order.

Setting $\xi_\lambda := \partial_\lambda \lrcorner\, \xi$, we have the coordinate expression

$$\begin{aligned}\Theta &= \mathcal{L} + \frac{\partial \ell}{\partial z^a_\lambda}\,(\mathrm{d}z^a \wedge \xi_\lambda) + \frac{\partial \ell}{\partial \bar{z}^{a\cdot}_\lambda}\,(\mathrm{d}\bar{z}^{a\cdot} \wedge \xi_\lambda) \\ &= k_{a\cdot\, b}\Big[\Big(-i\bar{z}^{a\cdot}\,(\gamma^\lambda \text{Ч}_\lambda)^b{}_c\, z^c - \frac{mc}{\hbar}\,\bar{z}^{a\cdot}\, z^b\Big)\,\xi\, + \\ &\qquad + \frac{i}{2}\,\gamma^{\lambda b}{}_c\,(\bar{z}^{a\cdot}\,\mathrm{d}z^c - z^c\,\mathrm{d}\bar{z}^{a\cdot}\,) \wedge \xi_\lambda\Big]\sqrt{|g|}\,.\end{aligned}$$

We now recall very briefly the relation between the Poincaré–Cartan form and conserved currents (Noether's theorem).

A section ψ: $\boldsymbol{M} \to \boldsymbol{W}$ is said to be *critical* if $\mathcal{E} \circ j\psi = 0$ (namely if ψ satisfies the Dirac equation). A projectable vector field w: $\boldsymbol{W} \to T\boldsymbol{W}$ is called a (infinitesimal) *symmetry* of Θ if $(j\psi)^* L_{w'}\Theta = 0$ for all critical sections, where w': $J\boldsymbol{W} \to TJ\boldsymbol{W}$ is the natural jet prolongation of w (Section 1). Then $w \lrcorner\, \Theta$: $J\boldsymbol{W} \to \wedge^3 T^*\boldsymbol{W}$ is a *current* 3*-form*, namely it fulfills $(j\psi)^* d(w \lrcorner\, \Theta) = 0$ for all critical sections.

In order to find a distinguished symmetry of Θ we consider the action of the group $U(1)$ on $\boldsymbol{W}$, given by

$$\mathbb{R} \times \boldsymbol{W} \to \boldsymbol{W}\colon\ (t, \zeta) \mapsto \mathrm{e}^{-it}\zeta.$$

This action can be naturally prolonged to an action on the jet space $J\boldsymbol{W}$. We then have two one-parameter groups, which are generated, respectively, by the vector fields w: $\boldsymbol{W} \to T\boldsymbol{W}$ and w': $J\boldsymbol{W} \to TJ\boldsymbol{W}$, whose coordinate expressions are

$$w = -iz^a\,\zeta_a, \qquad w' = -i(z^a\,\partial z_a + z^a_\lambda\,\partial z^\lambda_a).$$

Moreover, w' turns out to be the natural jet prolongation of w.

It is immediate to check that $\mathcal{L}$ and Θ are invariant with respect to the above $U(1)$-action. This also implies $L_{w'}\mathcal{L} = L_{w'}\Theta = 0$, so w' is an infinitesimal symmetry. Since w is a vertical vector field, the current 3-form $w \lrcorner\, \Theta$ is valued in $\wedge^3 T^*\boldsymbol{M}$. Its coordinate expression is

$$w \lrcorner\, \Theta = \sqrt{|g|}\ k_{a\cdot\, b}\bar{z}^{a\cdot}\,\gamma^{\lambda b}{}_c\, z^c \xi_\lambda.$$

For all ψ, we consider the evaluation

$$\mathcal{J}[\psi] := (j\psi)^*(w \lrcorner\, \Theta) = k(\psi, \gamma^\lambda \psi)\eta_\lambda,$$

where $\eta_\lambda := \partial_\lambda \lrcorner\, \eta = \sqrt{|g|}\ \xi_\lambda$. The component of this current in the direction of an observer u turns out to be

$$k(\psi, \gamma_u \psi) = h(\psi, \psi)\colon\ \boldsymbol{M} \to \mathbb{L}^{-3},$$

where h is the Hermitian metric induced by u (Section 10). The fact that this is a positive function allows the interpretation of $\mathcal{J}[\psi]$ as a *probability current*.

Finally, note that contraction with the contravariant volume form yields the scaled vector field

$$\boldsymbol{j}[\psi] := *\mathcal{J}[\psi]\colon\ \boldsymbol{M} \to \mathbb{L}^{-4} \otimes T\boldsymbol{M}$$

called the *vector current* of ψ, with coordinate expression

$$\boldsymbol{j}[\psi] = k(\psi, g^{\#}\gamma\psi) = k(\psi, \gamma^{\lambda}\psi)\,\partial_{\lambda}.$$

Because of the Noether theorem, the divergence of $\boldsymbol{j}[\psi]$ vanishes for ψ critical.

15. Neutrino

We recall that the Clifford map γ and the 2-form k *exchange* the chiral subbundles $\boldsymbol{S}$ and $\boldsymbol{S}'$ (Propositions 5.1 and 6.2), namely

$$\gamma_v(\boldsymbol{S}) = \boldsymbol{S}', \qquad \gamma_v(\boldsymbol{S}') = \boldsymbol{S},$$
$$k^{\flat}(\boldsymbol{S}) = \boldsymbol{S}'^{\star}, \qquad k^{\flat}(\boldsymbol{S}') = \boldsymbol{S}^{\star}.$$

Moreover it is easy to see, from the coordinate expression, that the complex spinor connection reduces to connections on $\boldsymbol{S}$ and $\boldsymbol{S}'$. Namely, let ψ be the quantum history of a particle of mass m, and consider its chiral decomposition $\psi = \psi_{\boldsymbol{S}} + \psi_{\boldsymbol{S}'}$. Then we have

$$\nabla_{\lambda}\psi_{\boldsymbol{S}} := \nabla^{\boldsymbol{S}}_{\lambda}\psi_{\boldsymbol{S}} : \boldsymbol{M} \to \boldsymbol{S}, \qquad \nabla_{\lambda}\psi_{\boldsymbol{S}'} := \nabla^{\boldsymbol{S}'}_{\lambda}\psi_{\boldsymbol{S}'} : \boldsymbol{M} \to \boldsymbol{S}'.$$

Hence, the term $i\gamma^{\lambda}\nabla_{\lambda}\psi$, in the Dirac equation for ψ, exchanges the chiral components of ψ, while the mass term obviously leaves them invariant. This situation is reversed when we consider the equivalent Equation (13.1), namely the Dirac equation composed with $k^{\flat}$, which we write as

$$ik^{\flat}\gamma^{\lambda}(\nabla_{\lambda}\psi) + \frac{mc}{\hbar}\,(k^{\flat}\psi) = 0.$$

Here the first term splits into the direct sum of the chiral components, while the mass term mixes them. Actually in a Weyl frame we have

$$(k^{\flat} \circ \gamma^0) = \begin{pmatrix} -(\bar{\sigma}_0) & 0 \\ 0 & -(\bar{\sigma}_0) \end{pmatrix}, \qquad (k^{\flat} \circ \gamma^j) = \begin{pmatrix} -(\bar{\sigma}_j) & 0 \\ 0 & (\bar{\sigma}_j) \end{pmatrix}.$$

It is now clear that for $m = 0$ the Dirac equation splits into the sum of decoupled components, one for each chiral subbundle. These components turn out to be the equations for the neutrino and the anti-neutrino.

We can recover the usual formulation of the Weyl equation by applying γ^0 to the Dirac equation, thus obtaining $\nabla_0\psi + \gamma^0\gamma^j\,\nabla_j\psi = 0$.

Recall that the observer $u \equiv e_0$ locally determines (Section 10) a Hermitian metric h, which restricts to Hermitian metrics on $\boldsymbol{S}$ and $\boldsymbol{S}'$. Through h, the Pauli frames (Section 9) of $\overline{\boldsymbol{S}^\star} \otimes \boldsymbol{S}^\star$ and $\overline{\boldsymbol{S}'^\star} \otimes \boldsymbol{S}'^\star$ yield a frame $(\hat{\sigma}_\alpha)$ of the h-Hermitian subbundle of End$(\boldsymbol{S})$, and a frame $(\hat{\sigma}'_\alpha)$ of the h-Hermitian subbundle of End$(\boldsymbol{S}')$. We have

$$\hat{\sigma}_\alpha = \sigma_{\alpha}{}^{A}{}_{B} \zeta_A \otimes z^B, \qquad \hat{\sigma}'_\alpha = \sigma_{\alpha}{}^{A'}{}_{B'} \zeta_{A'} \otimes z^{A'},$$

where $(\sigma_{\alpha}{}^{A}{}_{B})$ are the Pauli matrices, $1 \leqslant A, B \leqslant 2$, $3 \leqslant A', B' \leqslant 4$. We obtain

$$\gamma_0\gamma_j = -\hat{\sigma}_j \oplus \hat{\sigma}'_j, \qquad i\gamma_0\gamma_j\gamma_\eta = \hat{\sigma}_j \oplus \hat{\sigma}'_j.$$

Hence, the massless Dirac equation can be written in the form of two decoupled equations

$$(\nabla_0 - g^{jk}\hat{\sigma}_j \nabla^{\boldsymbol{S}}_k)\psi_{\boldsymbol{S}} = 0, \qquad (\nabla_0 + g^{jk}\hat{\sigma}'_j \nabla^{\boldsymbol{S}'}_k)\psi_{\boldsymbol{S}'} = 0.$$

16. Some Operators on Sections

A general geometric theory of Hilbert bundles and quantum operators for $\frac{1}{2}$-spin particles was presented in [CJM95] for the case when the background is assumed to be a curved Galileian spacetime. That setting was based essentially on the assumptions of absolute time and of a distinguished fibred Hermitian metric h, so it is not readily translatable to the present context: Einstein spacetime and a family of Hermitian metrics, one for each observer (Section 10). In this section we shall introduce some important operators on quantum histories, and see a few properties of theirs whose discussion does not need the Hilbert structure. A general theory of Hilbert bundles and quantum operators on Einstein spacetime is deferred to future work.

Given a (local) charge conjugation $\mathcal{C}$, the section $\mathcal{C}\psi$ represents the *antiparticle* of ψ. It fulfills the Dirac equation for a particle with opposite charge, namely

$$i\gamma^\lambda(\nabla_\lambda(\mathcal{C}\psi) + (\mathcal{C}\nabla_\lambda\mathcal{C})(\mathcal{C}\psi)) - \frac{mc}{\hbar}(\mathcal{C}\psi) = 0,$$

or (see (12.2))

$$i\gamma^\lambda(\nabla^{\mathcal{C}}_\lambda(\mathcal{C}\psi)) - \frac{mc}{\hbar}(\mathcal{C}\psi) = 0.$$

More generally, consider the group bundle $\boldsymbol{G} \to \boldsymbol{M}$ generated by Pin^c and local charge conjugations, and let $\mathcal{X}\colon \boldsymbol{M} \to \boldsymbol{G}$ be a local section. The quantum history $\mathcal{X}\psi$ fulfills the equation

$$(\mathcal{X}\, i\gamma^\lambda \mathcal{X}^{-1})\, \nabla^{\mathcal{X}}_\lambda (\mathcal{X}\psi) - \frac{mc}{\hbar}(\mathcal{X}\psi) = 0, \tag{16.1}$$

where $\nabla^{\mathcal{X}}_\lambda(\mathcal{X}\psi) := \mathcal{X}\nabla_\lambda\psi$, that is

$$\nabla^{\mathcal{X}}_\lambda\psi = \nabla_\lambda\psi + (\mathcal{X}\nabla_\lambda\mathcal{X}^{-1})\psi = \nabla_\lambda\psi - (\nabla_\lambda\mathcal{X})\mathcal{X}^{-1}\psi.$$

Equivalently, Equation (16.1) can be written

$$(\mathcal{X}\, i\gamma^\lambda\mathcal{X}^{-1})\,\nabla_\lambda(\mathcal{X}\psi) - \frac{mc}{\hbar}\,(\mathcal{X}\psi) = -(\mathcal{X}\, i\gamma^\lambda\nabla_\lambda\mathcal{X}^{-1})\,(\mathcal{X}\psi).$$

For a given observer u, we consider the related *parity* and *time inversion* operators

$$\mathcal{P} := \gamma_u\colon\ \boldsymbol{M} \to \boldsymbol{W} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{W}^\star, \qquad \mathcal{T} := \gamma_\eta\gamma_u\mathcal{C}\colon\ \boldsymbol{M} \to \boldsymbol{W} \underset{\boldsymbol{M}}{\otimes} \overline{\boldsymbol{W}^\star}.$$

We obtain

$$\begin{aligned}
&\mathcal{P}^2 = \mathbf{1}, \qquad \mathcal{T}^2 = -\mathbf{1},\\
&\mathcal{P}\mathcal{T} = \mathcal{T}\mathcal{P} = -\gamma_\eta\mathcal{C},\\
&\mathcal{P}\mathcal{C} = \mathcal{C}\mathcal{P} = \gamma_u\mathcal{C},\\
&\mathcal{C}\mathcal{T} = -\mathcal{T}\mathcal{C} = \gamma_u\gamma_\eta,\\
&(\mathcal{C}\mathcal{T})^2 = -(\mathcal{P}\mathcal{C})^2 = -(\mathcal{P}\mathcal{T})^2 = \mathbf{1},\\
&\mathcal{P}\mathcal{C}\mathcal{T} = \gamma_\eta.
\end{aligned}$$

For ψ obeying the Dirac equation, using Equation (16.1) we can write the equation obeyed by $\mathcal{P}\psi$, $\mathcal{T}\psi$, $\mathcal{P}\mathcal{C}\psi$ and so on. Choosing $\gamma_0 = \gamma_u$, the operators γ^λ transform according to

$$\begin{aligned}
&\mathcal{P}\gamma^\lambda\mathcal{P}^{-1} = \mathcal{T}\gamma^\lambda\mathcal{T}^{-1} = 2\delta^\lambda_0\gamma^0 - \gamma^\lambda,\\
&\mathcal{P}\mathcal{T}\gamma^\lambda(\mathcal{P}\mathcal{T})^{-1} = \gamma^\lambda,\\
&\mathcal{P}\mathcal{C}\gamma^\lambda(\mathcal{P}\mathcal{C})^{-1} = \mathcal{C}\mathcal{T}\gamma^\lambda(\mathcal{C}\mathcal{T})^{-1} = -2\delta^\lambda_0\gamma^0 + \gamma^\lambda,\\
&\mathcal{P}\mathcal{C}\mathcal{T}\gamma^\lambda(\mathcal{P}\mathcal{C}\mathcal{T})^{-1} = -\gamma^\lambda.
\end{aligned}$$

Recalling (12.3), we obtain

$$\begin{aligned}
&i(2\delta^\lambda_0\gamma^0 - \gamma^\lambda)\nabla_\lambda(\mathcal{P}\psi) - \frac{mc}{\hbar}(\mathcal{P}\psi) = i(2\delta^\lambda_0\gamma^0 - \gamma^\lambda)(\nabla_\lambda\gamma_0)\gamma_0\,(\mathcal{P}\psi),\\
&i(-2\delta^\lambda_0\gamma^0 + \gamma^\lambda)\nabla_\lambda(\mathcal{T}\psi) - \frac{mc}{\hbar}(\mathcal{T}\psi)\\
&\quad = i(-2\delta^\lambda_0\gamma^0 + \gamma^\lambda)\gamma_\eta\nabla_\lambda(\gamma_0\mathcal{C})\gamma_0\gamma_\eta\mathcal{C}\,(\mathcal{T}\psi),\\
&i\gamma^\lambda\nabla_\lambda(\mathcal{P}\mathcal{T}\psi) - \frac{mc}{\hbar}(\mathcal{P}\mathcal{T}\psi) = i\gamma^\lambda(\nabla_\lambda\mathcal{C})\mathcal{C}(\mathcal{P}\mathcal{T}\psi),\\
&i(2\delta^\lambda_0\gamma^0 - \gamma^\lambda)\nabla_\lambda(\mathcal{P}\mathcal{C}\psi) - \frac{mc}{\hbar}(\mathcal{P}\mathcal{C}\psi)\\
&\quad = i(2\delta^\lambda_0\gamma^0 - \gamma^\lambda)\nabla_\lambda(\gamma_0\mathcal{C})\mathcal{C}\gamma_0\,(\mathcal{P}\mathcal{C}\psi),
\end{aligned}$$

$$
\begin{aligned}
& i(-2\delta_0^\lambda\gamma^0+\gamma^\lambda)\nabla_\lambda(\mathcal{CT}\psi)-\frac{mc}{\hbar}(\mathcal{CT}\psi) \\
& \quad = i(-2\delta_0^\lambda\gamma^0+\gamma^\lambda)\nabla_\lambda(\gamma_0)\gamma_0\,(\mathcal{CT}\psi), \\
& i\gamma^\lambda\nabla_\lambda(\mathcal{PCT}\psi)+\frac{mc}{\hbar}(\mathcal{PCT}\psi)=0.
\end{aligned}
$$

Finally, we introduce spin operators. Let again u be an observer; let v be a vector field such that $g(u,v)=0$. The operator

$$S_v := \tfrac{i}{2}\,\gamma_u\gamma_v\gamma_\eta$$

is called the *spin operator in the v-direction*. For any orthonormal frame (e_λ) we have the spin operators (see also Section 15)

$$S_j := \tfrac{i}{2}\,\gamma_0\gamma_j\gamma_\eta = (\tfrac{1}{2}\,\hat{\sigma}_j)\oplus(\tfrac{1}{2}\,\hat{\sigma}'_j).$$

Acknowledgements

This research has been supported by Italian MURST (national and local funds). Thanks are due to Andrzej Trautman and Marco Modugno for stimulating discussion.

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