QUANTUM MECHANICS OF A SPIN PARTICLE IN A CURVED SPACETIME WITH ABSOLUTE TIME

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We present a new covariant approach to the quantum mechanics of a charged 1/2-spin particle in given electromagnetic and gravitational fields. The background space is assumed to be a curved Galilean spacetime, that is a curved spacetime with absolute time. This setting is intended both as a suitable approximation for the case of low speeds and feeble gravitational fields, and as a guide for eventual extension to fully Einsteinian spacetime. Moreover, in the flat spacetime case one completely recovers the standard non-relativistic quantum mechanics.

This work is a generalization of [18], where the quantum mechanics of scalar particles was formulated within a similar approach.

1. Introduction

Recently Jadczyk and Modugno [17, 18] have proposed a new geometric formulation of quantum mechanics of a scalar charged particle, with given gravitational and electromagnetic classical fields, in the framework of a general relativistic Galilean spacetime. In this paper we extend that formulation to quantum mechanics of a particle with spin 1/2.

Our work is related to abundant literature on classical and quantum Galilean theory, starting from E. Cartan [1] (see also [4, 5, 2, 3], [6], [13], [21], [36, 37], [26, 27], [29], [32], [34], [22, 23, 24, 25, 19], [26, 27], [29], [32], [34], [36, 37], [39], [40, 41], [42]). Moreover our theory has evident relations, but also important differences, with geometric quantization (see [43]). Our touchstone is the standard quantum mechanics [38].

Our research is intended as a step toward a covariant formulation of quantum mechanics in the Einstein general relativistic background. In fact, such a full goal

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would demand the solutions of too many problems at the same time; so, it is worth splitting the research into steps by separating different kinds of difficulties.

We found that the Galilean general relativistic spacetime provides a suitable background for the start. Thus our current setting stands in between a non-relativistic and a fully relativistic formulation of quantum mechanics. It is mathematically self-consistent, while from the physical point of view it is intended both as a suitable approximation for the case of low speeds and feeble gravitational fields, and as a guide for eventual extension to fully Einsteinian spacetime. Actually, the assumptions of a classical spacetime with the absolute time and an Euclidean spacelike metric allows us to skip (temporarily) some difficulties related to the Lorentz metric, but we pay a price for that. Namely, we are forced to consider a weaker version of the Maxwell and Einstein equations. Nevertheless, what we learn in this weakened context seems to preserve its interest in view of future developments. Moreover, in the flat spacetime case one completely recovers the standard non-relativistic quantum mechanics along with a new understanding of known objects.

The mathematical language of the paper is that of the geometry of fibred manifolds, jets and non-linear connections. We do not deal explicitly with theoretical group representations: rather we directly obtain physical objects from our initial structures via functorial methods; of course, the resulting objects are automatically equivariant with respect to the action of the groups of automorphisms of the initial structures. The reader who is not completely acquainted with this language will find, besides intrinsic formulations, a full coordinate description of all results.

The main points of our theory can be summarized as follows.

First, we sketch the basic features of our background classical spacetime. Namely, we assume a 4-dimensional spacetime fibred over time and equipped with a spacelike Euclidean metric, a time preserving linear connection (the gravitational field) and a 2-form (the electromagnetic field). We can couple the gravitational and electromagnetic fields into a unique spacetime connection; this yields a number of 'total' geometric objects, including a cosymplectic 2-form which will play a key role. We postulate the closure of this form, thus obtaining a link between the above geometrical structures and the first Maxwell equation; moreover, we postulate a kind of 'reduced' Einstein and second Maxwell equations expressing the interaction of the above fields with their matter sources. The cosymplectic form yields a distinguished Lie algebra of functions which are called 'quantizable' in view of their role in the theory of quantum operators.

Then we develop the quantum theory starting from the quantum bundle, defined as a Hermitian bundle over spacetime; its fibres are either 1-dimensional (scalar case) or 2-dimensional (spin case). On the scalar quantum bundle we assume a Hermitian connection which, in a sense, is parametrized by all classical observers, and has some natural properties (it is 'universal' and its curvature is proportional to the cosymplectic form). In the spin case we postulate a 'Pauli map', which is an isometry between the bundle of spacelike vectors and the bundle of Hermitian endomorphisms of the quantum spin bundle; this, via a natural link with the scalar case, yields a Hermitian connection on the quantum spin bundle. This is our only primitive quantum structure; all other objects will be derived from it getting free from observers through a 'principle of projectability', which is our implementation of covariance. In particular, we obtain a distinguished Lagrangian which yields the generalized Pauli equation and conserved quantities. Quantum operators are obtained in three steps. First, we exhibit a distinguished algebra of quantum vector fields which preserves the quantum structures, and we study its relation with the algebra of quantizable functions. Next, we show the natural action of quantum vector fields, as 'almost-quantum operators', on 'quantum histories' (sections of the quantum bundle). Finally, we introduce the quantum Hilbert bundle over time and show how to obtain quantum operators from almost-quantum operators. To this end, we have to eliminate the time derivative; we accomplish this task by a geometric procedure which uses the quantum Euler–Lagrange operator.

The original features of the paper can be summarized as follows.

I. Time, both in the classical and quantum theory, is not merely a parameter, but it is an essential ingredient which deeply affects all involved structures. Actually we point out—in contrast to the approach which is usually implicit in geometric quantization—that the spacelike structures do not carry sufficient physical information for a covariant theory. Accordingly, we deal with a cosymplectic rather than symplectic form, with a spacetime rather than vertical (spacelike) connection, and so on. Also, jets are required for a manifestly covariant formulation; in particular, the jet space of spacetime plays the role of phase-space and replaces the more standard tangent space.

II. New connections are introduced and studied. These play a fundamental and unifying role. In particular, the coupling of the electromagnetic and gravitational fields is represented by a spacetime connection which works in classical field theory and mechanics as well as in quantum mechanics; on the other hand, all quantum structures are derived from the quantum connection. With regard to the latter, we observe that the notion of 'universality' of a connection allows us to skip the problem of polarizations, which is typical in geometric quantization (we do not need to know the constants of motion in order to develop the quantum theory). Furthermore, the quantum Euler–Lagrange operator is interpreted as a connection on the infinite-dimensional Hilbert bundle (whose definition uses the notion of smoothness introduced by A. Frölicher).

III. We obtain a generalized Pauli equation and quantum operators in the curved case. Actually, a quantization procedure (a way of obtaining quantum operators from classical observables) was not the primary goal of our approach; however, as a matter of fact, we get a quantization just as a free consequence of geometric results arising naturally in our discussion. We have obtained natural algebras of quantizable functions and quantum vector fields, which yield quantum operators, in two steps: first by considering sections of the quantum bundle over spacetime (almost-quantum operators), and then sections of the Hilbert bundle over time. In particular, we are able to skip the problems of ordering, and achieve the quantum operator corresponding to energy. Note also that, differently from other geometrical approaches to quantum mechanics, no new quantum example is required (all non-relativistic examples of standard quantum mechanics hold automatically in our formulation).

IV. Incidentally, several results are obtained within the covariant approach to classical mechanics on a curved Galilean background. In particular, the study of the first and second order spacetime connections and the cosymplectic form, and a compact formulation of the link between the (non-relativistic) metric and spacetime connection. Moreover we draw conclusions which are not a common belief: classical mechanics cannot be covariantly formulated through a Lagrangian or Hamiltonian approach; only an approach based on a non-linear connection is suitable for that (the Hamiltonian language, however, has an important role in the correspondence principle for quantum mechanics).

V. Finally, we introduce a new mathematically rigorous treatment of physical quantities which makes our approach manifestly independent of the choice of measurement units. Incidentally, these methods may also raise a pedagogical interest.

Remark: Throughout this paper we shall consider smooth manifolds and maps. For the sake of simplicity we shall always refer to global maps. In some situations, however, one should more properly refer to *sheaves* of local maps. The reader who is interested in such a refinement will have no difficulty in reformulating our statements accordingly.

2. Preliminaries

2.1. Recalls on fibred manifolds

In this section we summarize the main concepts and notations of differential geometry which we shall use throughout the paper.

2.1.1. *Tangent space*

Let M be a manifold. We denote the $\mathbb{R}$-Lie algebra of functions $f\colon M \to \mathbb{R}$ by $\mathcal{F}M$, the *tangent bundle* of M by $TM \to M$ and the $\mathbb{R}$-Lie algebra of vector fields $X\colon M \to TM$ by $\mathcal{T}M$. A local chart (x^λ) of M induces a local chart $(x^\lambda, \dot{x}^\lambda)$ of TM, a local basis of vector fields $(\partial_\lambda) := (\partial x_\lambda)$ and a dual local basis of forms $(d^\lambda) := (dx^\lambda)$. The *tangent prolongation* of a map $f\colon M \to N$ is the map $Tf\colon TM \to TN$ with coordinate expression $Tf = \partial_\lambda f^i d^\lambda \otimes (\partial_i \circ f)$.

2.1.2. *Fibred manifolds*

A manifold F is said to be *fibred* over the *base space* B if it is equipped with a surjective map $p\colon F \to B$ whose rank equals the dimension of B. A fibred manifold can be covered by local trivializations defined on open subsets $F' \subset F$. Thus the concept of a fibred manifold is more general than that of a *bundle* (which can be covered by local trivializations defined on open subsets of the type $F' = p^{-1}(U)$, where $U \subset B$ is an open subset).

A chart (x^λ, y^i) of F is said to be *fibred* if the coordinates x^λ depend only on the base space. A fibred chart of F induces the local frame of vector fields $(\partial_\lambda, \partial_i)$ and the dual local frame of forms (d^λ, d^i) on F. Hence, we obtain also the chart $(x^\lambda, y^i; \dot{x}^\lambda, \dot{y}^i)$

of TF, the local frame of vector fields $(\partial_\lambda, \partial_i; \dot\partial_\lambda, \dot\partial_j)$ and the dual local frame of forms $(d^\lambda, d^i, \dot d^\lambda; \dot d^i)$.

We have a natural projection $TF \to TB$. A vector field X: $F \to TF$ is said to be *projectable* if it admits a projection $\underline{X}$: $B \to TB$ on the base space, i.e. if its coordinate expression is of the type $X = X^\lambda \partial_\lambda + X^i \partial_i$, with $X^\lambda \in \mathcal{F}B$.

The *vertical subbundle* $VF \subset TF$ of F is constituted by all vectors tangent to the fibres and is characterized by the equation $(\dot x^\lambda = 0)$. Thus, a vector field X is vertical iff it is projectable over 0, i.e. iff $X^\lambda = 0$. The subset $\mathcal{V}F \subset \mathcal{T}F$ of all vertical vector fields is an ideal.

We have a natural projection $T^*F \to V^*F$, yielding the vertical restrictions of forms which we shall indicate by a check (ˇ). Thus, for example, $(\check d^i)$ is a local frame of the vector bundle $VF \to F$.

2.1.3. Jet space

The *jet space at* $x \in B$ of $F \to B$ is defined to be the set $J_{1x}F$ of all equivalence classes of sections s: $B \to F$ which have the same value of $s(x)$ and the same derivatives $\partial_\lambda s^i(x)$. The *jet space* J_1F is the union of all $J_{1x}F$ for $x \in B$. We have the natural fibred charts $(x^\lambda, y^i, y^i_\lambda)$ of J_1F, and the *jet prolongation* j_1s: $B \to J_1F$ characterized by the coordinate expression $(y^i, y^i_\lambda) \circ j_1 s = (s^i, \partial_\lambda s^i)$. We can identify $j_1 s$ with Ts: $TB \to TF$, which projects over 1_B. Accordingly, we can regard J_1F as a subbundle of $T^*B \otimes_F TF$ whose elements are projectable over 1_B. This inclusion is a map[1]

$$\text{д}: J_1F \to T^*B \underset{F}{\otimes} TF,$$

with coordinate expression $\text{д} = d^\lambda \otimes \text{д}_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_\lambda \partial_j)$. We also have the complementary map ϑ: $J_F \to T^*F \otimes_F VF$, with coordinate expression $\vartheta = \vartheta^j \otimes \partial_j = (d^j - y^j_\lambda d^\lambda) \otimes \partial_j$.

The vertical bundle of J_1F over the base space F turns out to be

$$V_F J_1 F = J_1 F \underset{F}{\times} (T^*B \underset{F}{\otimes} VF).$$

2.1.4. Connections

Connections will play an essential role in our approach. There are several equivalent ways to define the concept of a (possibly non-linear) connection (see [8, 20, 30, 33]).

In general, we present a connection on a fibred manifold $F \to B$ as a section c: $F \to J_1F$ which, via the natural inclusion д, can be seen as a *horizontal prolongation* c: $F \to T^*B \otimes_F TF$, whose coordinate expression is of the type $c = d^\lambda \otimes (\partial_\lambda + c_\lambda{}^j \partial_j)$, with $c_\lambda{}^j \in \mathcal{F}F$. The associated *vertical projection* is ν_c: $F \to T^*F \otimes_F VF$, with coordinate expression $\nu_c = (d^j - c_\lambda{}^j d^\lambda) \otimes \partial_j$.

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[1] д is the Cyrillic character corresponding to Latin d.

The *covariant differential* of a section $s\colon \boldsymbol{B} \to \boldsymbol{F}$ is defined to be the section $\nabla[c]s := j_1 s - c \circ s = Ts \lrcorner \nu_c\colon \boldsymbol{B} \to T^*\boldsymbol{B} \otimes_{\boldsymbol{F}} T\boldsymbol{F}$, with coordinate expression $\nabla_\lambda s^i = \partial_\lambda s^i - c_\lambda^j \circ s$.

The *curvature tensor* of the connection c is defined to be the tensor field $R[c]\colon \boldsymbol{F} \to \wedge^2(T^*\boldsymbol{B})\otimes_{\boldsymbol{F}} V\boldsymbol{F}$ characterized by $R[c](u,v) := \frac{1}{2}([u\lrcorner c,\, v\lrcorner c] - [u,v]\lrcorner c)$ for any two vector fields $u, v\colon \boldsymbol{B} \to \boldsymbol{F}$. Namely the curvature tensor 'measures' how much the horizontal prolongation c differs from being a morphism of Lie algebras. Its coordinate expression is $R[c] = R_{\lambda\mu}{}^j d^\lambda \wedge d^\mu \otimes \partial_j$, where $R_{\lambda\mu}{}^j = \partial_{[\lambda} c_{\mu]}^j - c_{[\lambda}^h \partial_h c_{\mu]}^j$.

2.1.5. Vertical space of a vector bundle

If $p\colon \boldsymbol{F} \to \boldsymbol{B}$ is a vector bundle, then one has the natural identification $V\boldsymbol{F} \equiv \boldsymbol{F} \times_{\boldsymbol{B}} \boldsymbol{F}$. This fact yields some important consequences. First, any section $s\colon \boldsymbol{B} \to \boldsymbol{F}$ can be regarded as the *basic* vertical vector field $\boldsymbol{F} \to V\boldsymbol{F}\colon \varphi \mapsto (\varphi, s(p(\varphi)))$. Hence, if $v\colon \boldsymbol{F} \to T\boldsymbol{F}$ is a linear vector field, projectable over $\underline{v}\colon \boldsymbol{B} \to T\boldsymbol{B}$, then the Lie bracket $[v, s]$ is a basic vertical vector field, i.e. it determines the section $v.s\colon \boldsymbol{B} \to \boldsymbol{F}$ with coordinate expression $(v.s)^j = v^\lambda \partial_\lambda s^j - v^j_k s^k$. Moreover, any linear map $f\colon \boldsymbol{F} \to \boldsymbol{F}$ fibred over $\boldsymbol{B}$ can be regarded as the vertical vector field $\boldsymbol{F} \to V\boldsymbol{F}\colon \varphi \mapsto (\varphi, f(\varphi))$. In particular, the *Liouville* vector field[2] is defined to be the vertical vector field $и\colon \boldsymbol{F} \to V\boldsymbol{F}\colon \varphi \mapsto (\varphi, \varphi)$ associated with $\mathbf{1}_{\boldsymbol{F}}$.

2.2. Units of measurement

Our theory is to be manifestly invariant with respect to any choice of measurement units; this is just an aspect of the general covariance. In order to treat measurement units in a rigorous way, we need a few technical concepts.

We observe that homogeneous units can be added and multiplied by real numbers; however, in some cases, no zero unit exists and only multiplication by positive real numbers is allowed. These facts lead us to define algebraically a *semi-vector space* as a semi-field $\mathbb{U}$ associated with the semi-ring $\mathbb{R}^+$ (the axioms are analogous to those of vector spaces, with the only difference that $\mathbb{U}$ and $\mathbb{R}^+$ are additive semi-groups and not groups). Moreover, a semi-vector space is said to be *positive* if the multiplication by numbers can be extended neither to $\mathbb{R}^+ \cup \{0\}$ nor to $\mathbb{R}$. Each vector space is also a semi-vector space; moreover, a vector space and a basis yield a positive semi-vector space. Thus, a semi-vector space is a vector space, or a positive semi-vector space, or a positive semi-vector space extended by the zero element.

Several concepts and results of standard linear and multi-linear algebra related to vector spaces can be easily reproduced for semi-vector spaces and positive semi-vector spaces (including linear and multi-linear maps, bases, dimension, tensor products and duality, with respect to $\mathbb{R}^+$). The main precaution to be taken is to avoid formulations which involve the zero element.

In particular, we can define the tensor product (over $\mathbb{R}^+$) of semi-vector spaces; the tensor product (over $\mathbb{R}^+$) of a semi-vector space and a vector space becomes naturally

[2] $и$ is the Cyrillic character corresponding to Latin i.

also a vector space. Consider an oriented 1-dimensional vector space $\mathbb{U}$ and the associated positive sub semi-space $\mathbb{U}^+$; if $\mathbb{V}$ is another vector space, then $\mathbb{U}^+ \otimes \mathbb{V} = \mathbb{U} \otimes \mathbb{V}$ and, in particular, $\mathbb{U}^+ \otimes \mathbb{R} = \mathbb{U} \otimes \mathbb{R}$. Moreover, we can define the $\mathbb{R}^+$-dual $\mathbb{U}^*$ of a semi-vector space $\mathbb{U}$; if $\mathbb{U}$ is a positive 1-dimensional semi-vector space, then we obtain the natural identification $\mathbb{U} \otimes \mathbb{U}^* \cong \mathbb{R}^+$. Furthermore, if $\mathbb{U}$ is a positive 1-dimensional semi-vector space, then we can easily define the 'root' (positive 1-dimensional semi-vector) space $\mathbb{U}^{1/r}$ of $\mathbb{U}$, for any positive integer r.

DEFINITION 2.1. A unit space is a *1-dimensional semi-vector space.* ♣

In order to write formulae which resemble the standard ones used by physicists, we adopt a 'number-wise' notation for unit spaces. Namely, if $\mathbb{U}$ and $\mathbb{V}$ are semi-vector spaces and $u \in \mathbb{U}$, $v \in \mathbb{V}$, then we write $uv \equiv u \otimes v$; accordingly, we set $\mathbb{U}^2 := \mathbb{U} \otimes \mathbb{U}$ and the like. Moreover, if $\mathbb{U}$ is a unit space which does not contain 0, then we write $\mathbb{U}^{-1} = \mathbb{U}^*$ and denote by $1/u \in \mathbb{U}^{-1}$ the dual element of $u \in \mathbb{U}$.

In our theory we shall assume the following fundamental unit spaces: the oriented vector space $\mathbb{T}$ of *time units,* the positive space $\mathbb{M}$ of *masses* and the positive space $\mathbb{L}$ of *lengths.* A time unit of measurement is denoted by $u_0 \in \mathbb{T}^+$ or $u^0 \in \mathbb{T}^{+*}$. We also set $u^{00} := u^0 \otimes u^0$ and the like. For any $v \in \mathbb{T}$, $w \in \mathbb{T}^*$, according to our conventions, we shall often write $u^0 v$, $u_0 w \in \mathbb{R}$.

Throughout this paper we shall be often concerned with *scaled* tensor fields, i.e. with sections of tensor bundles originated by spacetime and tensorialized with unit spaces. It is physically relevant the fact that fundamental tensor fields such as the metric, the electromagnetic field and others are scaled.

We shall attach to each particle a *mass* m, a *charge* q and a *magnetic constant* μ, where

$$m \in \mathbb{M}, \qquad q \in \mathbb{Q} := \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}; \qquad \mu \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}.$$

Moreover, we shall postulate two universal coupling constants, namely the *Newton gravitational constant* and the *Planck constant*

$$\kappa \in \mathbb{T}^{*2} \otimes \mathbb{L}^3 \otimes \mathbb{M}^*, \qquad \hbar \in (\mathbb{T}^+)^* \otimes \mathbb{L}^2 \otimes \mathbb{M}.$$

As it is well known, in the Galilean framework we miss the speed of light c, which cannot be interpreted in this context. Of course, this is a weak feature of the Galilean theory.

3. Quantum mechanics of a scalar particle

This section is a summary of the main ideas involved in the scalar case, especially those that are needed for the subsequent generalization to the quantum mechanics of a particle with spin. We shall skip certain details concerning results which, later, will be stated in the more general spin case. For further details and complete proofs the reader should refer to [18].

3.1. Classical spacetime

We introduce classical spacetime and the related fundamental structures that are needed as a background for the quantum theory; further details can be found in [18].

POSTULATE C1. *Classical spacetime* is assumed to be a 4-dimensional oriented fibred manifold $t: \boldsymbol{E} \to \boldsymbol{T}$, where the base space $\boldsymbol{T}$ (*time*) is a 1-dimensional oriented affine space associated with the vector space $\mathbb{T}$. ♣

We shall not assume any distinguished splitting of the spacetime into space and time (there is no distinguished observer). Actually our theory is observer-independent, namely it fulfils the general relativity principle in the 'Galilean' sense (with the absolute time).

We shall use fibred spacetime charts, denoted by $(x^\lambda) := (x^0, y^i)$, where the coordinate x^0 is defined through the time unit $u^0 \in \mathbb{T}$ (see Section 2.2) and a time origin $\tau_0 \in \boldsymbol{T}$ by $x^0(e) := u^0(t(e) - \tau_0)$.

We have the scaled *time form* $dt: \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$, with the coordinate expression $dt = u_0 \otimes dx^0$.

Each fibre $\boldsymbol{E}_\tau$ of $\boldsymbol{E}$ represents the 'space at a given time' $\tau \in \boldsymbol{T}$; by analogy with Einstein relativity we say that the vertical space $V\boldsymbol{E}$ is constituted by all 'spacelike' vectors on $\boldsymbol{E}$ (while we are not allowed to use the term 'timelike' in the present context).

POSTULATE C2. The fibres of $\boldsymbol{E}$ are assumed to be scaled Riemannian manifolds, i.e. spacetime is assumed to be equipped with a *scaled vertical Riemannian metric* $g: \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes_{\boldsymbol{E}} V^*\boldsymbol{E})$. ♣

The coordinate expression of the metric is $g = g_{hj} \check{d}y^h \otimes \check{d}y^j$ (we indicate by a check ($\check{}$) vertical (i.e. spacelike) restrictions). We stress that, differently form the Einstein case, we do not have a full spacetime metric: this is a weak feature of the Galilean theory. The metric yields vertical 'index-lowering' and 'index-raising' isomorphisms, $g^\flat: V\boldsymbol{E} \to \mathbb{L}^2 \otimes V^*\boldsymbol{E}$ and $g^\#: \mathbb{L}^2 \otimes V^*\boldsymbol{E} \to V\boldsymbol{E}$, but no similar isomorphisms between $T\boldsymbol{E}$ and $T^*\boldsymbol{E}$.

The metric and the time-form, along with the chosen orientation, yield the scaled *spacetime* and *spacelike volume forms*:

$$v: \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^3) \otimes \wedge^4 T^*\boldsymbol{E}, \qquad \eta: \boldsymbol{E} \to \mathbb{L}^3 \otimes \wedge^3 V^*\boldsymbol{E},$$

with coordinate expressions

$$v = \sqrt{|g|} u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 := \sqrt{|g|} u_0 \otimes \omega,$$
$$\eta = \sqrt{|g|} \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3 := \sqrt{|g|} \check{\omega}_0,$$

where for brevity we set

$$d^\lambda := dx^\lambda, \qquad \omega := d^0 \wedge d^1 \wedge d^2 \wedge d^3, \qquad \omega_0 := \partial_0 \lrcorner\, \omega = d^1 \wedge d^2 \wedge d^3.$$

The *phase space* of our theory is the jet bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$; its induced fibred coordinates are denoted by (x^0, y^j, y^j_0). From the general theory of jet spaces (Subsection 2.1.3) we recall that $J_1\boldsymbol{E}$ can be regarded as a subbundle of $\mathbb{T}^* \otimes T\boldsymbol{E}$ over $\boldsymbol{E}$, via the natural map д which has the coordinate expression д $= u^0 \otimes (\partial_0 + y_0^j \partial_j)$. Then $J_1\boldsymbol{E}$ is constituted by all tensors v for which the time component $v_0^{;0} = 1$. In other words, having chosen a time unit u_0, the phase space $J_1\boldsymbol{E}$ can be identified with the affine subbundle of $T\boldsymbol{E}$ constituted by vectors v with the time component $v^0 = 1$. We stress that the tangent space is insufficient to represent the phase space of a theory which is explicitly independent of the units of measurement.

The classical *particle motion* is defined to be a section $s\colon \boldsymbol{T} \to \boldsymbol{E}$; its (observer-independent) *velocity* is the jet prolongation $j_1 s\colon \boldsymbol{T} \to J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$, with the coordinate expression

$$j_1 s = u^0 \otimes ((\partial_0 \circ s) + \partial_0 s^j (\partial_j \circ s)).$$

Thus the jet space $J_1\boldsymbol{E}$ can be seen as the space of all 4-velocities of the particle. We stress that 4-velocity v has no norm $\|v\|$, and that its physical dimension is given just by $\mathbb{T}^*$ and not by $\mathbb{T}^* \otimes \mathbb{L}$.

An *observer* is defined to be a section $o\colon \boldsymbol{E} \to J_1\boldsymbol{E}$, i.e. a field of particle velocities. Incidentally note that an observer can be regarded as a (possibly non-linear) connection on $\boldsymbol{E} \to \boldsymbol{T}$ (Subsection 2.1.4).

Differently from the Einstein case, the metric g does not characterize a unique spacetime connection; in order to fully appreciate the question, we need to examine spacetime connections in some detail. We first remark that there is a natural bijection between dt-preserving torsion-free linear connections on the tangent bundle $T\boldsymbol{E} \to \boldsymbol{E}$ and torsion-free affine connections on the jet bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$, i.e. respectively

$$K\colon T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E}, \qquad \Gamma\colon J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E}.$$

The coordinate expressions of such connections are

$$K = d^\lambda \otimes (\partial_\lambda + (K_\lambda{}^j{}_h \dot{y}^h + K_\lambda{}^j{}_0 \dot{x}^0)\dot{\partial}_j), \qquad \Gamma = d^\lambda \otimes (\partial_\lambda + (\Gamma_\lambda{}^j{}_h y_0^h + \Gamma_\lambda{}^j{}_0)\partial_j^0),$$

with

$$K_\lambda{}^j{}_\mu = K_\mu{}^j{}_\lambda = \Gamma_\lambda{}^j{}_\mu = \Gamma_\mu{}^j{}_\lambda.$$

Then a *spacetime connection* is defined to be any of such equivalent connections. One deals preferably with K in classical field theory, and with Γ in classical and quantum particle mechanics.

A spacetime connection yields, by vertical restriction, a linear connection

$$K'\colon V\boldsymbol{E} \to T^*\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} TV\boldsymbol{E}$$

on the bundle $V\boldsymbol{E} \to \boldsymbol{E}$, with the coordinate expression $K' = \check{d}^\lambda \otimes (\partial_\lambda + K_\lambda{}^j{}_h \dot{y}^h \dot{\partial}_j)$. This connection will play a central role in the classical and quantum theory of spin. A further vertical restriction gives the *vertical connection*

$$\check{K}\colon V\boldsymbol{E} \to V^*\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} V_{\boldsymbol{E}}V\boldsymbol{E}$$

(which, more properly, is a family of connections: for each $\tau \in \boldsymbol{T}$, $\check{K}_\tau$ is a connection on the manifold $\boldsymbol{E}_\tau := t^{-1}(\tau)$). Its coordinate expression is $\check{K} = \check{d}^h \otimes (\partial_h + K_h{}^j{}_k \dot{y}^k \dot{\partial}_j)$.

A spacetime connection is said to be *metrical* if it preserves the vertical metric, i.e. if $\nabla[K']g = 0$. If K is metrical, then $\check{K}$ is exactly the Riemannian connection on the spacetime fibres; however, if $\check{K}$ is the Riemannian connection, then K is not necessarily metrical, since $\nabla[K']g$ involves the covariant derivatives of g also along non-spacelike directions.

Recalling (Subsection 2.1.3) that

$$V_E J_1 \boldsymbol{E} = J_1 \boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\mathbb{T}^* \otimes V\boldsymbol{E}),$$

the vertical-valued 1-form associated with a spacetime connection Γ can be seen as a map

$$\nu_\Gamma \colon J_1\boldsymbol{E} \to \mathbb{T}^* \otimes (T^* J_1 \boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} V\boldsymbol{E})$$

with the coordinate expression $\nu_\Gamma = (d^j_0 - (\Gamma^{\ j}_{\lambda\ h} y^h_0 + \Gamma^{\ j}_{\lambda\ 0}) d^\lambda) \otimes \partial_j$.

A spacetime connection yields the following two important objects: the (non-linear) connection

$$\gamma := д \lrcorner \Gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$$

on the fibred manifold $J_1\boldsymbol{E} \to \boldsymbol{T}$ and the scaled 2-form[3]

$$\Omega := \nu_\Gamma \,\bar{\wedge}\, \vartheta \colon J_1\boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{L}^2) \otimes \wedge^2 T^* J_1\boldsymbol{E}$$

on the manifold $J_1\boldsymbol{E}$ (here $\bar{\wedge}$ indicates exterior product followed by a metric contraction and $\vartheta \colon J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} V\boldsymbol{E}$ is the complementary map of $д$ introduced in Subsection 2.1.3). These are called the *second order connection* and the *cosymplectic form* associated with Γ. Their coordinate expressions are

$$\gamma = u^0 \otimes (\partial_0 + y^j_0 \partial_j + \gamma^j \partial^0_j), \qquad \Omega = g_{jk} u^0 \otimes (d^j_0 - \gamma^j d^0 - \Gamma^j_h \vartheta^h) \wedge \vartheta^k,$$

where

$$\gamma^j := \Gamma^{\ j}_{h\ k} y^h_0 y^k_0 + 2\Gamma^{\ j}_{h\ 0} y^h_0 + \Gamma^{\ j}_{0\ 0}, \qquad \Gamma^j_h := (\Gamma^{\ j}_{\lambda\ h} y^h_0 + \Gamma^{\ j}_{\lambda\ 0}) d^\lambda.$$

These objects fulfil the equality $\gamma \lrcorner \Omega = 0$, and it can be seen that they characterize Γ itself.

For any motion s the map

$$\nabla[\gamma] j_1 s := j_2 s - \gamma \circ j_1 s \colon \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$$

is called the (observer-independent) *acceleration* of s. Moreover,

$$dt \wedge \Omega \wedge \Omega \wedge \Omega \colon J_1\boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^6) \otimes \wedge^7 T^* J_1\boldsymbol{E}$$

is a scaled volume form on $J_1\boldsymbol{E}$. Also, if $o \colon \boldsymbol{E} \to J_1\boldsymbol{E}$ is any observer, we have the *observed scaled 2-form*

$$\Phi := 2o^*\Omega \colon \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{L}^2) \otimes \wedge^2 T^*\boldsymbol{E},$$

which, in a coordinate system adapted to o (namely $y^j_0 \circ o = 0$), has the expression $\Phi = -2u^0 \otimes (\Gamma_{0j0} d^0 \wedge d^j + \Gamma_{hj0} d^h \wedge d^j)$.

[3]Janiška has proved that this form is essentially the unique natural object of this kind in the present framework.

From coordinate expressions it can be proved that, for a given observer, a spacetime connection is characterized by $\nabla[K']g$ and Φ. Namely these objects can be seen, in a sense, as the symmetric and antisymmetric parts of Γ with respect to a splitting determined by o. This is the keypoint for understanding how to characterize distinguished spacetime connections. In fact, a complex theorem proved in [18] states that the condition that Ω is closed, i.e.

$$d\Omega = 0, \tag{1}$$

is equivalent to the couple of conditions that K is metrical and, for every observer, Φ is closed; a connection that satisfies this equation is then determined by g and a local potential of Φ, that is a 1-form

$$a\colon \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{L}^2) \otimes T^*\boldsymbol{E}$$

such that $\Phi = 2da$. Then a distinguished spacetime connection obeying eq. (1) is determined, similarly to the Einsteinian case, by ten scalar potentials: here, these are the six components of g and the four components of a.

POSTULATE C3. We assume that the *gravitational* and *electromagnetic fields* are represented, respectively, by a spacetime connection $\Gamma^\natural$ and by a scaled 2-form

$$F : \boldsymbol{E} \to (\mathbb{L} \otimes \mathbb{M})^{1/2} \otimes \wedge^2 T^*\boldsymbol{E}. \qquad \clubsuit$$

The gravitational and electromagnetic fields can be coupled in a natural way through any constant $c \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}$. Namely, consider a *total cosymplectic form*

$$\Omega_c := \Omega^\natural + \tfrac{1}{2} c F,$$

where $\Omega^\natural$ is the cosymplectic form of $\Gamma^\natural$. Then one sees that Ω_c characterizes, in a natural way, a spacetime connection; namely there is a unique spacetime connection Γ_c such that $\Omega_c = \nu_{\Gamma_c} \,\overline{\wedge}\, \vartheta$ (that is, Ω_c is exactly the cosymplectic form associated with Γ_c). Actually, we can write $\Gamma_c = \Gamma^\natural + \Gamma^e_c$, where

$$\Gamma^e_c : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}.$$

We have the coordinate expressions:

$$(\Gamma_c)_h{}^j{}_k = \Gamma^\natural{}_h{}^j{}_k, \qquad (\Gamma_c)_0{}^j{}_k = \Gamma^\natural{}_0{}^j{}_k + \tfrac{1}{2} u_0 c F^j{}_k, \qquad (\Gamma_c)_0{}^j{}_0 = \Gamma^\natural{}_0{}^j{}_0 + u_0 c F^j{}_0.$$

Furthermore, the second order connection $\gamma_c := \text{д} \lrcorner \Gamma_c$ associated with Γ_c fulfils the condition $\gamma_c \lrcorner \Omega_c = 0$ and splits as $\gamma = \gamma^\natural + \gamma^e_c$, where

$$\gamma^e_c\colon J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes V\boldsymbol{E}$$

has the coordinate expression $\gamma^e_c = c(F^j{}_0 + F^j{}_h y^0_h) u^0 \otimes \partial^0_j$.

POSTULATE C4. We assume that the total connection Γ_c obeys the *first field equation* $d\Omega_c = 0$ for all c. $\clubsuit$

The closure of Ω_c implies that it is locally exact, but we cannot exhibit any distinguished potential. Clearly, this postulate is equivalent to the couple of conditions

$d\Omega^\natural = 0$ and $dF = 0$ (first Maxwell equation). Also the observed cosymplectic form splits as $\Phi = \Phi^\natural + cF$. Hence, a local potential a of Φ contributes both to the gravitational and electromagnetic fields, and it reduces to the usual electromagnetic potential in the flat spacetime case.

In [18] two possible natural choices for the coupling constant c have been taken into account (in the spin theory we shall consider a third possibility). The first choice, which yields the classical mechanics of a given charged particle,[4] is $c = q/m$, where $q \in \mathbb{Q} := \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$ and $m \in \mathbb{M}$ are the charge and the mass of the particle. We obtain the (classical) equation of motion of the particle which can be expressed as $\nabla[\gamma_c]j_1 s = 0$ with $c = q/m$. Then γ_c^e turns out to be just the *Lorentz force*.

The second choice is $c = \sqrt{\kappa}$, where κ is the Newton's gravitational constant. This choice allows us to couple Γ_c with matter sources. Namely:

POSTULATE C5. We postulate the *second field equations*:

$$r^\natural = \mathrm{T}, \qquad \mathrm{div}^\natural F = \rho\, dt,$$

where $r^\natural$ is the Ricci tensor of $K^\natural$; T is the timelike energy tensor, which involves κ and contains matter and electromagnetic terms; $\mathrm{div}^\natural$ is the spacelike divergence operator; ρ is the charge density of matter. ♣

These equations yield the following synthetic formula:

$$r_{\sqrt{\kappa}} = \mathrm{T}_{\sqrt{\kappa}},$$

where $r_{\sqrt{\kappa}}$ is the Ricci tensor of $K_{\sqrt{\kappa}}$, and $\mathrm{T}_{\sqrt{\kappa}} := \mathrm{T} + \sqrt{\kappa}\,\rho\, dt \otimes dt$.

We remark that these equations are weaker than the usual Maxwell–Einstein equations. In fact, because the metric is only spacelike, $r^\natural$ and $\mathrm{div}^\natural F$ carry less information than the corresponding objects do in the Einstein case. Thus they can be covariantly coupled only with the timelike components of the energy tensor and of the current.

Note also that the second field equations do not enter directly the quantum mechanics of one particle, which is formulated with *given* background fields. One deals with them only when considering specific examples of spacetime.

3.2. Scalar quantum mechanics

In the framework of the above described spacetime geometry we can now formulate the quantum mechanics of a particle with given mass m and charge q, subjected to given gravitational and electromagnetic fields. We shall deal with the total objects $\Gamma_{q/m}$, $\Omega_{q/m}$, $\gamma_{q/m}\ldots$ induced by the coupling constant $c := q/m$ (Section 3.1). For the sake of simplicity, these will be usually denoted simply by Γ, Ω, $\gamma, \ldots$

First we introduce the bundle which 'carries quantum kinematics'. We stress that, differently from standard geometric quantization, this bundle is over spacetime.

POSTULATE Q1. The *scalar quantum bundle* is assumed to be a (complex) line-bundle $\pi_Q\colon \boldsymbol{Q} \to \boldsymbol{E}$ over spacetime, endowed with a Hermitian metric h_Q. ♣

[4]The same choice for a coupling constant yields the fundamental object of the quantum theory, the quantum connection (see Section 3.2).

We shall denote by b an $h_{\boldsymbol{Q}}$-normalized frame of $\boldsymbol{Q}$, and by z the corresponding chart on the fibres of $\boldsymbol{Q}$. The induced frame of $V\boldsymbol{Q} \to \boldsymbol{Q}$ will be denoted by ∂z. Quantum histories are described by *quantum sections* $\Psi \colon \boldsymbol{E} \to \boldsymbol{Q}$, written locally as $\Psi = \psi b$ with $\psi := z \circ \Psi$. In view of the Hilbert scalar product it is also useful to regard a quantum section as a *quantum density*

$$\Psi^\eta := \Psi \otimes \sqrt{\eta} \colon \boldsymbol{E} \to \boldsymbol{Q}^\eta := \mathbb{L}^{3/2} \otimes \boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \sqrt{\wedge^3 V^* \boldsymbol{E}}.$$

The *Planck constant* (Section 2.2) is defined as an element

$$\hbar \in (\mathbb{T}^+)^* \otimes \mathbb{L}^2 \otimes \mathbb{M}.$$

Next we introduce the *quantum connection*, which is the main object of the quantum theory. Consider a general Hermitian linear connection Ч on the pullback bundle $\boldsymbol{Q}^\uparrow := J_1\boldsymbol{E} \times_{\boldsymbol{E}} \boldsymbol{Q} \to J_1\boldsymbol{E}$; it can be seen as a section[5]

$$Ч \colon \boldsymbol{Q}^\uparrow \to T^* J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} T\boldsymbol{Q}$$

with the coordinate expression

$$Ч = d^\lambda \otimes (\partial_\lambda + iЧ_\lambda\, z\, \partial z) + d_0^j \otimes (\partial_j^0 + iЧ_j^0\, z\, \partial z),$$

where $Ч_\lambda, Ч_j^0 \colon J_1\boldsymbol{E} \to \mathbb{R}$.

The coordinate condition $Ч_j^0 = 0$ for Ч can be formulated in a geometric way in the framework of systems of connections, by saying that Ч is a ‘universal’ connection. Very briefly, one proves the following fact (see [18, 8, 30] for details): if $\{\xi[o]\}$ is a *system of connections* of the bundle $\boldsymbol{Q} \to \boldsymbol{E}$, parametrized by the family of observers $\{o\}$, then there exists a unique connection Ч of the bundle $\boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$ such that, for each observer o, the pullback $o^*Ч$ equals $\xi[o]$. This connection Ч is said to be *universal*, and is characterized in coordinates by the condition $Ч_\lambda = \xi_\lambda$, $Ч_j^0 = 0$. Conversely, a connection Ч of the bundle $\boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$ such that $Ч_j^0 = 0$ is the universal connection of a system of connections $\{\xi[o]\}$ on the bundle $\boldsymbol{Q} \to \boldsymbol{E}$.

POSTULATE Q2. We assume that the *quantum connection* Ч is a Hermitian linear universal connection whose curvature is proportional to the classical total cosymplectic form, according to the formula

$$R[Ч] = i\frac{m}{\hbar}\Omega_{q/m} \otimes \mathbf{1}_{\boldsymbol{Q}} \colon \boldsymbol{Q}^\uparrow \to \wedge^2 T^* J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q},$$

where $\mathbf{1}_{\boldsymbol{Q}} = z\, b$ is the identity of $\boldsymbol{Q}$. ♣

Then the quantum connection satisfies $Ч_j^0 = 0$. Because of the curvature requirement, the expression of the other components of Ч turns out to be of the type

$$Ч_0 = -u_0\frac{H}{\hbar}, \qquad Ч_j = \frac{p_j}{\hbar},$$

[5] Ч is the Cyrillic character which is usually transliterated as Ch.

where

$$H = u^{00}(\tfrac{1}{2}mg_{jk}y^j_0 y^k_0 - ma_0)\colon\ J_1\boldsymbol{E} \to (\mathbb{T}^*)^2 \otimes \mathbb{L}^2 \otimes \mathbb{M},$$
$$p = p_j \check{d}^j = u^0(mg_{jk}y^k_0 + ma_j)\check{d}^j\colon\ J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M} \otimes V^*\boldsymbol{E}$$

are the classical Hamiltonian and momentum associated with the frame of reference attached to the chosen chart, given a suitable gauge of the total potential a of Φ.

We stress that the two simple assumptions, of the quantum bundle to be over spacetime and of the quantum connection to be universal, enable us to avoid the intricate problems related to polarizations, which are typical in geometric quantization.

If Ψ is a quantum section, then we have the *quantum covariant differential*

$$\nabla[\mathrm{Ч}]\Psi\colon\ J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q},$$

with the coordinate expression

$$\nabla\Psi = \left(\left(\partial_0\psi + \frac{i}{\hbar}u_0 H\psi\right)d^0 + \left(\partial_j\psi - \frac{i}{\hbar}p_j\psi\right)d^j\right) \otimes b.$$

Essentially, the quantum connection is the only structure assumed for the quantum mechanics of a scalar particle; all other quantum objects, including the quantum Lagrangian and quantum operators, can be derived from it. But note that the quantum connection 'lives' on the pull-back bundle $\boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$. This fact can be expressed by saying that Ч is 'parametrized' by all observers (given an observer, one obtains by pull-back an object living on $\boldsymbol{Q}$). However, physically significant objects should live on $\boldsymbol{Q}$, i.e. quantum theory should be observer-independent. This problem can be solved by means of a *principle of projectability*. Namely, each time we are looking for a physical object on $\boldsymbol{Q}$, we happen to meet two analogous distinguished objects on $\boldsymbol{Q}^\uparrow$, and we are able to show that there is a unique linear combination of them which projects on $\boldsymbol{Q}$. Then we assume that this combination is the searched physical object. This procedure works pretty well in all cases and yields an effective heuristic method. Thus it can be regarded as a new way of implementing the principle of general relativity in the framework of quantum mechanics.

The principle of projectability enables us to exhibit a distinguished *quantum Lagrangian*:[6]

$$\mathcal{L}\colon\ J_1\boldsymbol{Q} \to \mathbb{L}^3 \otimes \wedge^4 T^*\boldsymbol{E},$$

with the coordinate expression

$$-\mathcal{L}[\Psi] = \tfrac{1}{2}\Big(i(\overline{\psi}\partial_0\psi - \psi\partial_0\overline{\psi}) - u_0\frac{\hbar}{m}g^{jk}\partial_j\psi\partial_k\overline{\psi} +$$
$$+ig^{jk}a_j(\psi\partial_k\overline{\psi} - \overline{\psi}\partial_k\psi) + u^0\frac{m}{\hbar}\psi\overline{\psi}(2a_0 - a^j a_j)\Big)\sqrt{|g|}\omega.$$

[6]Here we do not write down the procedure explicity, since it will be repeated later in the more general case of a particle with spin.

The quantum Lagrangian yields the *quantum 4-momentum*

$$\mathfrak{p}\colon J_1\boldsymbol{Q} \to \mathbb{T}^* \otimes T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q},$$

with the coordinate expression

$$\mathfrak{p}[\Psi] = u^0\Big(\psi\partial_0 - g^{hj}\Big(iu_0\frac{\hbar}{m}\partial_j\psi + a_j\psi\Big)\partial_h\Big) \otimes b.$$

The Euler–Lagrange equation associated with the quantum Lagrangian turns out to be the *generalized Schrödinger equation*

$$iu_0\partial_0\psi + iu_0\frac{\partial_0\sqrt{|g|}}{2\sqrt{|g|}}\psi + \frac{m}{\hbar}a_0\psi + \frac{\hbar}{2m}g^{jk}\Big(u_0\partial_j - i\frac{m}{\hbar}a_j\Big)\Big(u_0\partial_k - i\frac{m}{\hbar}a_k\Big)\psi = 0,$$

which can be also obtained, in a coordinate-free way, from the quantum covariant differentials of Ψ and $\mathfrak{p}$ via the principle of projectability.

The invariance of the quantum Lagrangian with respect to the group $U(1)$ yields a conserved *probability 4-current* $j\colon J_1\boldsymbol{Q} \to \mathbb{L}^3 \otimes \wedge^3 T^*\boldsymbol{E}$, with the coordinate expression

$$j[\Psi] = \sqrt{|g|}\Big(\bar{\psi}\psi\,\omega_0 - \Big(u_0\frac{i\hbar}{2m}g^{hk}(\bar{\psi}\partial_k\psi - \psi\partial_k\bar{\psi}) - a^h\bar{\psi}\psi\Big)\,\omega_h\Big),$$

where $\omega_\lambda := \partial_\lambda \lrcorner\, \omega$.

3.3 Phase quantum operators

In this section we describe the correspondence between classical functions and quantum operators. This is achieved by a new approach which is only roughly comparable to the usual one based on symplectic geometry. Actually, our phase space $J_1\boldsymbol{E}$ is odd-dimensional, thus there is no symplectic structure on it. Instead, we have the cosymplectic form Ω which yields the linear morphism over $J_1\boldsymbol{E}$:

$$\Omega^\flat\colon TJ_1\boldsymbol{E} \to T^*J_1\boldsymbol{E}\colon v \mapsto \frac{m}{\hbar}\Omega(v).$$

This is not an isomorphism. In fact, from $\gamma \lrcorner\, \Omega = 0$ it follows that $\Omega^\flat$ vanishes on any $v \in TJ_1\boldsymbol{E}$ which is in the image of $\gamma\colon \boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$. However, consider the vector subbundle over $J_1\boldsymbol{E}$:

$$T^*_\gamma J_1\boldsymbol{E} := \{\phi \in T^*J_1\boldsymbol{E} : \gamma \lrcorner\, \phi = 0\};$$

let $\tau\colon J_1\boldsymbol{E} \to \mathbb{T}$ be *any* smooth map (called a *time scale*), and $T_\tau J_1\boldsymbol{E}$ the subbundle of $TJ_1\boldsymbol{E}$ whose elements have time component equal to τ, namely

$$T_\tau J_1\boldsymbol{E} := \{v \in TJ_1\boldsymbol{E} : v^0 = \tau(\pi(v))\},$$

where $\pi\colon TJ_1\boldsymbol{E} \to J_1\boldsymbol{E}$ is the natural tangent bundle projection. Then one sees easily that $\Omega^\flat$ is an isomorphism $T_\tau J_1\boldsymbol{E} \to T^*_\gamma J_1\boldsymbol{E}$.

Now, with any function $f\colon J_1\boldsymbol{E} \to \mathbb{R}$ we can associate a 1-form:

$$d_\gamma f := df - \gamma \lrcorner\, df\colon J_1\boldsymbol{E} \to T^*_\gamma J_1\boldsymbol{E},$$

and, for any time scale $\tau\colon J_1\boldsymbol{E} \to \mathbb{T}$, the vector field

$$f^{\#}_{\tau} := \Omega^{\#}_{\tau}(d_{\gamma}f)\colon\ J_1\boldsymbol{E} \to T_{\tau}J_1\boldsymbol{E},$$

where $\Omega^{\#}_{\tau} := (\Omega^{\flat})^{-1}$. In particular, by taking $\tau = 0$ we can define the generalized Poisson bracket

$$\{f_1, f_2\} := \frac{m}{\hbar}\,\Omega((f_1)^{\#}_0, (f_2)^{\#}_0),$$

which has the property

$$\{f_1, f_2\}^{\#}_0 = [(f_1)^{\#}_0, (f_2)^{\#}_0].$$

In the quantum theory we shall be involved with projectable Hamiltonian lifts. Now, one can prove that the vector field $f^{\#}_{\tau}$ is projectable over a vector field $\boldsymbol{E} \to T\boldsymbol{E}$ iff f is, with respect to the fibres of $J_1\boldsymbol{E} \to \boldsymbol{E}$, a polynomial of degree 2, whose second derivative equals $(m/\hbar)\tau g$. Namely, the coordinate expression of f must be of the type

$$f = u^{00} f'' \frac{m}{2\hbar} g_{jk} y^j_0 y^k_0 + f_j y^j_0 + f_\circ$$

with $f_j, f_\circ\colon \boldsymbol{E} \to \mathbb{R}$, $f''\colon \boldsymbol{E} \to \mathbb{T}$, and τ must be equal to f''. Functions of this kind will be called *quantizable phase functions*. The classical time, position, momentum, Hamiltonian and Lagrangian functions turn out to be of this kind.

If for any quantizable phase function we choose $\tau = f''$, we obtain the vector field

$$f^{\#} := \Omega^{\#}_{f''}\colon\ J_1\boldsymbol{E} \to T_{f''}J_1\boldsymbol{E}.$$

Its projection

$$X[f]\colon \boldsymbol{E} \to T\boldsymbol{E},$$

with the coordinate expression

$$X[f] = u^0 f'' \partial_0 - u_0 \frac{\hbar}{m} g^{jk} f_k \partial_j,$$

is called the *tangent lift* of f.

Let now f_1 and f_2 be quantizable phase functions, and set

$$[f_1, f_2] := \{f_1, f_2\} + (f_1''\gamma).f_2 - (f_2''\gamma).f_1.$$

Then, after long computations, one proves that the previous formula defines a Lie bracket. This coincides with the usual Poisson bracket in the particular case when the involved quantizable functions are affine ($f_1'' = f_2'' = 0$). We shall indicate by $\mathcal{A}^{\mathrm{P}}$ the Lie algebra of phase quantizable functions, and by $\mathcal{T}\boldsymbol{E}$ the Lie algebra of all tangents vector fields on $\boldsymbol{E}$. Moreover, we indicate by $\mathcal{F}\boldsymbol{E}$ the algebra of all (smooth) functions $\boldsymbol{E} \to \mathbb{R}$. Then from the previous results we easily obtain:

PROPOSITION 3.1. *The tangent lift*

$$\mathcal{A}^{\mathrm{P}} \to \mathcal{T}\boldsymbol{E}\colon\ f \mapsto X[f]$$

is an $\mathcal{F}\boldsymbol{E}$-linear epimorphism, with kernel $\mathcal{F}\boldsymbol{E} \subset \mathcal{A}^{\mathrm{P}}$, and an $\mathbb{R}$-Lie algebra morphism. Namely, we have

$$X[[f_1, f_2]] = [X[f_1], X[f_2]]. \qquad \diamond$$

Next, in view of quantum operators we start by looking for distinguished vector fields on $\boldsymbol{Q}^{\uparrow}$. Consider any vector field $Y^{\uparrow}\colon \boldsymbol{Q}^{\uparrow} \to T\boldsymbol{Q}^{\uparrow}$ which is projectable over some vector field $X^{\uparrow}\colon J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$, Hermitian linear and such that the vertical restriction of $L(Y^{\uparrow})Ч$ vanishes. Then it can be proved that $Y^{\uparrow}$ is of the type

$$Y^{\uparrow}_{\tau}[f] := f^{\#}_{\tau} \lrcorner Ч + i f и\colon \boldsymbol{Q}^{\uparrow} \to T\boldsymbol{Q}^{\uparrow},$$

where $и\colon \boldsymbol{Q}^{\uparrow} \to V\boldsymbol{Q}^{\uparrow}$ is the Liouville vector field (Subsection 2.1.5), $f\colon J_1\boldsymbol{E} \to \mathbb{R}$ is a function and τ a time scale. Moreover, $Y^{\uparrow}_{\tau}[f]$ turns out to be projectable over a vector field $Y[f]\colon \boldsymbol{Q} \to T\boldsymbol{Q}$ iff f is quantizable and $\tau = f''$. Then $Y[f]$ is called the *quantum phase vector field* corresponding to f, or the *quantum lift* of f. It has the coordinate expression

$$Y[f] = u^0 f'' \partial_0 - u_0 \frac{\hbar}{m} f^j \partial_j + i\left(u^{00} f'' \frac{m}{\hbar} a_0 - f^j a_j + f_{\circ}\right) z\partial z.$$

From this formula one sees that the space of all quantum phase vector fields on $\boldsymbol{Q}$ is just the Lie algebra $\mathcal{Q}$ of all Hermitian linear projectable vector fields $\boldsymbol{Q} \to T\boldsymbol{Q}$. A long calculation shows that the map $\mathcal{A}^{\mathrm{P}} \to \mathcal{Q}\colon f \mapsto Y[f]$ is an isomorphism of $\mathbb{R}$-Lie algebras, namely we have

$$Y[[f_1, f_2]] = [Y[f_1], Y[f_2]].$$

Recalling Subsection 2.1.5 we see that there is a natural way of defining $Y.\Psi\colon \boldsymbol{E} \to \boldsymbol{Q}$ for any linear vector field $Y\colon \boldsymbol{Q} \to T\boldsymbol{Q}$ projectable over $X\colon \boldsymbol{E} \to T\boldsymbol{E}$. If $Y = X^{\lambda}\partial_{\lambda} + iY^{z}\partial z$, we obtain the coordinate expression

$$Y.\Psi = (X^{\lambda}\partial_{\lambda}\psi - iY^{z}\psi)b.$$

The *almost-quantum operator* $\mathcal{Y}[f]$ corresponding to f, acting on quantum densities $\Psi^{\eta} := \Psi \otimes \sqrt{\eta}$,[7] is defined by

$$\mathcal{Y}[f](\Psi \otimes \sqrt{\eta}) := i(Y[f].(\Psi \otimes \sqrt{v})) \otimes \frac{1}{\sqrt{v}} \otimes \sqrt{\eta}.$$

Then, since $Y.\sqrt{v} = \frac{1}{2}(\operatorname{div} X)\sqrt{v}$, where divergence is taken with respect to the volume form v, we obtain

$$\mathcal{Y}[f](\Psi \otimes \sqrt{\eta}) = i(Y.\Psi + \tfrac{1}{2}(\operatorname{div} \underline{X})\Psi) \otimes \sqrt{\eta}.$$

We then obtain a natural $\mathbb{R}$-Lie algebra isomorphism between the Lie algebras of quantizable phase functions and almost-quantum operators, if the bracket of two almost-quantum operators $\mathcal{Y}[f_1]$ and $\mathcal{Y}[f_2]$ is defined by

$$[\mathcal{Y}[f_1], \mathcal{Y}[f_2]] := -i[\![\mathcal{Y}[f_1], \mathcal{Y}[f_2]]\!],$$

where[8]

$$[\![\mathcal{Y}[f_1], \mathcal{Y}[f_2]]\!] := \mathcal{Y}[f_1] \circ \mathcal{Y}[f_2] - \mathcal{Y}[f_2] \circ \mathcal{Y}[f_1].$$

[7]This extension to quantum densities is necessary in order to have symmetric operators (see Section 7.5).

[8]Throughout this paper we shall indicate commutators by this 'blackboard bold' bracket, as in general we shall have to distinguish them from Lie brackets.

The Euler–Lagrange operator [31, 9]

$$\mathcal{E}\colon J_2\boldsymbol{Q} \to \mathbb{L}^3 \otimes \wedge^4 T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}^*$$

derived from the quantum Lagrangian can be characterized, via the Hodge isomorphism and the real part of the Hermitian metric h, by a map

$$*\mathcal{E}^{\#}\colon J_2\boldsymbol{Q} \to \mathbb{T}^* \otimes \boldsymbol{Q}.$$

Then we define the *Schrödinger operator*, acting on quantum densities, by

$$\mathfrak{S}(\Psi^\eta) := -\tfrac{i}{2} *\mathcal{E}^{\#}[\Psi] \otimes \sqrt{\eta}.$$

It can be proved that $\mathfrak{S}$ is a symmetric operator with respect to the Hermitian product.

We shall sketch in the more general spin case (Section 7.5) the construction which yields the infinite-dimensional *pre-Hilbert bundle* $H'\boldsymbol{Q}^\eta \to \boldsymbol{T}$ over time (eventually, this will yield the *quantum Hilbert bundle* $H\boldsymbol{Q}^\eta \to \boldsymbol{T}$ by the completion procedure). Here we just observe that, if f is a quantizable phase function, then in general the operator $\mathcal{Y}[f]$ will not correspond to a fibred automorphism of $H'\boldsymbol{Q}^\eta$ over $\boldsymbol{T}$; in fact the expression of $\mathcal{Y}[f](\Psi^\eta)$, if $f'' \neq 0$, will contain the time derivative of Ψ. In order to construct from $\mathcal{Y}[f]$ such a fibred automorphism, which we shall indicate by $\widehat{f}$ and call a *pre-Hilbert quantum operator*, we have, in rough terms, to 'eliminate' the time derivative. There is a natural way of obtaining this result, namely by using the Schrödinger operator (whose kernel is constituted by the solutions of the generalized Schrödinger equation)[9] and setting

$$\widehat{f} := \mathcal{Y}[f] - i f'' \lrcorner\, \mathfrak{S}.$$

The operator $\widehat{f}$ is symmetric iff f'' is constant. This is true in all physically significant cases where f'' is either 0 or u_0. Thus, the above formula is our implementation of the principle of correspondence, achieved in a purely geometric way. In particular, in the flat spacetime case, these operators and their commutators correspond to the standard ones.

4. Classical spin

It is well known that quantum spin has no classical counterpart in a strict sense. However, we can give a mathematically self-consistent formulation of classical mechanics of a charged spinning particle, which under certain circumstances yields a good approximation of the real mechanics and, at the same time, will constitute the background for the quantum spin.

4.1. Classical spin particle

We first note that g can be seen as a (non-scaled) metric on the vector bundle $\mathbb{L}^* \otimes V\boldsymbol{E} \to \boldsymbol{E}$ (this will be the fundamental bundle for spin particles). The induced

[9] The Schrödinger operator can also be seen as a connection on the infinite-dimensional pre-Hilbert bundle.

'index-lowering' and 'index-raising' morphisms will be indicated, respectively, by

$$g^\flat\colon \mathbb{L}^* \otimes VE \to \mathbb{L} \otimes V^*E, \qquad g^\#\colon \mathbb{L} \otimes V^*E \to \mathbb{L}^* \otimes VE.$$

We shall denote by (e_r) a positively-oriented orthonormal frame of $\mathbb{L}^* \otimes VE$. The dual frame (ϵ^r) of $\mathbb{L} \otimes V^*E$ determines a linear fibred chart (x^λ, ϵ^r) on $\mathbb{L}^* \otimes VE$.

Consider any linear connection $C\colon VE \to T^*E \otimes_{VE} TVE$ on the bundle $VE \to E$. Clearly, C can be regarded also as a connection

$$C\colon \mathbb{L}^* \otimes VE \to T^*E \underset{\mathbb{L}^* \otimes VE}{\otimes} T(\mathbb{L}^* \otimes VE),$$

with the coordinate expression

$$C = dx^\lambda \otimes (\partial x_\lambda + C_{\lambda}{}^{p}{}_{r} \epsilon^r e_p),$$

where $C_{\lambda}{}^{p}{}_{r} := -\langle \epsilon^p, \nabla_\lambda[C] e_r \rangle$. Note that here λ is an index of the spacetime coordinates, while the Latin indices appearing in this formula are related to the linear coordinates ϵ^r, on the fibres of $\mathbb{L}^* \otimes VE$, that are not induced by the spacetime coordinates. Moreover, C is said to be *metrical* if $\nabla[C]g = 0$. Then, in particular, the vertical restriction K' of a metrical spacetime connection is a connection of this type.

We shall indicate by $UE \to E$ the subbundle of $\mathbb{L}^* \otimes VE$ whose fibres are unit 2-spheres. The history of a classical spinning particle will be described by a section $\upsilon\colon T \to UE$. Its projection $s\colon T \to E$ is the particle motion in the usual way, while the vertical vector field over it represents the particle's spin; more precisely, the classical intrinsic angular momentum of the particle is $\frac{1}{2}\hbar \upsilon$.

We can state the equation of motion for υ by means of a couple of connections: the spacetime connection $\Gamma := \Gamma_{q/m}$, where q is the charge and m is the mass of the considered particle, and a metrical linear connection $C := K'_{2\mu}$ on the bundle $VE \to E$ (which reduces to a connection on UE). Here,

$$\mu \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}$$

is a new coupling constant which we call the *spin-magnetic field coupling constant*. We shall also write μ as

$$\mu = G\frac{q}{2m}, \qquad G \in \mathbb{R}.$$

Latter on, by comparing the flat case with standard formulae [28], the section $\mu\upsilon\colon T \to \mathbb{T}^* \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{-1/2} \otimes VE$ will turn out to be the *magnetic moment* of the particle, and the real number G will turn out to be its *gyromagnetic ratio*. When $q = e$ is the positron's charge, then $\mu\hbar/G = e\hbar/2m$ is the so called *Bohr magneton*.

In an orthonormal frame (e_r) the components of $C := K'_{2\mu}$ are given by

$$C_{h}{}^{r}{}_{s} = \widetilde{\Gamma}^{\natural}{}_{h}{}^{r}{}_{s}, \qquad C_{0}{}^{r}{}_{s} = \widetilde{\Gamma}^{\natural}{}_{0}{}^{r}{}_{s} + u_0 \mu \widetilde{F}^{r}{}_{s} = \widetilde{\Gamma}^{\natural}{}_{0}{}^{r}{}_{s} + 2u_0 \mu \varepsilon^{r}{}_{sp} \widetilde{B}^{p},$$

where

$$B := \tfrac{1}{2} * \check{F}\colon E \to \mathbb{L}^{-5/2} \otimes \mathbb{M}^{1/2} \otimes VE$$

is the *magnetic field*.[10] The tilde over the components of $\Gamma^\natural$, $\check{F}$ and B indicates that

[10] In the Galilean context the magnetic field is observer-independent.

these are components in the frame (e_r). In particular we have

$$\widetilde{B}^p = \tfrac{1}{2}\varepsilon_r{}^{sp}\widetilde{F}^r{}_s = \tfrac{1}{2}\varepsilon^{rsp}\widetilde{F}_{rs}.$$

Furthermore, C: $VE \to T^*E \otimes_E TVE$ yields the map

$$\gamma' := \text{д} \lrcorner C\colon J_1E \underset{E}{\times} VE \to \mathbb{T}^* \otimes TVE,$$

with the coordinate expression

$$\gamma' = u^0 \otimes (\partial_0 + y_0^j \partial_j + \gamma'^r \dot{e}_r), \qquad \gamma'^r = (C_0{}^r{}_s + C_h{}^r{}_s y_0^h)\epsilon^s,$$

where $(\dot{e}_r)$ is the frame induced on $V_E VE$. The couple (Γ, C) is a linear connection on $J_1E \times_E UE \to E$. Thus the equation of motion for υ can be formulated as

$$\nabla[\gamma']\upsilon' := j_1\upsilon' - (\gamma, \gamma') \circ \upsilon' = 0,$$

where

$$\upsilon' := (j_1 s, \upsilon) : E \to J_1E \underset{E}{\times} UE.$$

Now the above equation splits into two equations: the equation of motion for s, which is the standard one (Section 3.2), and that for υ, which reads $\nabla[C]_{j_1 s}\upsilon = 0$ (thus a first-order equation: the covariant derivative of the spin vector along the particle motion vanishes). In coordinates it reads

$$\nabla[C]_{j_1 s}\upsilon = u^0(\partial_0 \upsilon^r - C_0{}^r{}_p \upsilon^p - C_h{}^r{}_p(\partial_0 s^h)\upsilon^p)e_r.$$

Moreover, the same equation can also be written as

$$\nabla^{\natural}_{j_1 s}\upsilon - \mu\upsilon \times B = 0;$$

in the flat spacetime case the above covariant derivative reduces to the ordinary derivative, so that we obtain the standard equation [15].

For a classical charged particle in the flat case it is known [15] that the interaction between spin and magnetic field yields an energy

$$-\mu\hbar\, g(\upsilon, B) = -\mu\hbar *(\upsilon^\flat \wedge \check{F}) = -\tfrac{1}{2}\mu\hbar\,\varepsilon_p{}^{rs}\upsilon^p \widetilde{F}_{rs}.$$

This function is well-defined also in the general curved case. In order to see that it has the same meaning, we should postulate the effect of spin on the electromagnetic field, through a suitable current to be coupled to the field via the Maxwell equations, and study the energy balance in the present context. We omit such analysis and simply assume that the classical *spin Hamiltonian* H^{S}: $J_1E \times_E VE \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$ is given by[11]

$$H^{\mathrm{S}}[\upsilon] := H[s] - \mu\hbar\, g(\upsilon, B),$$

that is

$$H^{\mathrm{S}} = \tfrac{1}{2} m g_{jk} y_0^j y_0^k - m a_0 - \tfrac{1}{2}\mu\hbar\varepsilon_p{}^{rs}\epsilon^p \widetilde{F}_{rs}.$$

[11] Note that the first term on the right-hand side is observer-dependent, while the second is observer-independent.

We would be tempted to extend these arguments in order to include a spin-gravitation energy. For example, formal similarity might suggest a term of the form

$$\frac{1}{2}\varepsilon_p{}^{rs}\epsilon^p(\widetilde{\Gamma}^{\natural}{}_{0rs} + \widetilde{\Gamma}_{hrs}(\partial_0 s^h)).$$

Any interpretation of this kind, however, would need a more general approach to the classical theory of angular momentum, which should include orbital angular momentum in a general relativistic context. We shall address this question in a future work.

4.2. Quantizable functions

In view of quantum operators for spin particles we wish to extend the Lie algebra of quantizable functions, by considering functions $f\colon J_1\boldsymbol{E} \times_{\boldsymbol{E}} (\mathbb{L}^* \otimes V\boldsymbol{E}) \to \mathbb{R}$.

In Section 3.3 we showed how the Lie algebra $\mathcal{A}^{\mathrm{P}}$ of quantizable phase functions on $J_1\boldsymbol{E}$ arises naturally from the geometric arguments. Up to now, we are not able to extend those arguments to the spin case. Hence we present a more restricted approach which, however, encompasses the most physically interesting examples.

The space of *quantizable spin functions* is defined to be the space $\mathcal{A}^{\mathrm{S}} := \mathcal{A}^{\mathrm{SQ}} \oplus \mathcal{A}^{\mathrm{SL}}$ of all functions $\phi\colon \mathbb{L}^* \otimes V\boldsymbol{E} \to \mathbb{R}$ of the type $\phi = \phi^{\mathrm{Q}} + \phi^{\mathrm{L}}$, where $\phi^{\mathrm{L}} \in \mathcal{A}^{\mathrm{SL}}$ is linear, $\phi^{\mathrm{Q}} \in \mathcal{A}^{\mathrm{SQ}}$ is quadratic and proportional to g. Namely, the expression of $\phi \in \mathcal{A}^{\mathrm{S}}$ in an orthonormal frame is of the type

$$\phi = \phi'' \delta_{rs} \epsilon^r \epsilon^s + \phi_r \epsilon^r,$$

with $\phi'', \phi_r\colon \boldsymbol{E} \to \mathbb{R}$.

By means of the vertical isomorphism $g^{\#}$ any $\phi \in \mathcal{A}^{\mathrm{S}}$ yields[12] the section

$$X[\phi] =: \phi^{\mathrm{Q}\#} + \phi^{\mathrm{L}\#}\colon \boldsymbol{E} \to \otimes^2(\mathbb{L}^* \otimes V\boldsymbol{E}) \underset{\boldsymbol{E}}{\oplus} \mathbb{L}^* \otimes V\boldsymbol{E}.$$

Its orthonormal frame expression is

$$X[\phi] = \delta^{rs}(\phi'' e_r \otimes e_s + \phi_s e_r) := \delta^{rs}\phi'' e_r \otimes e_s + \phi^r e_r.$$

By analogy with the phase functions we call $X[\phi]$ the *tangent lift* of ϕ.

We indicate by $\mathcal{V}\boldsymbol{E}$ the space of all vertical-valued vector fields on $\boldsymbol{E}$. Then $\mathbb{L}^* \otimes \mathcal{V}\boldsymbol{E}$ is naturally equipped with the $\mathcal{F}\boldsymbol{E}$-Lie algebra structure given by the cross--product. Since the map $\mathcal{A}^{\mathrm{SL}} \to \mathbb{L}^* \otimes \mathcal{V}\boldsymbol{E}$: $\phi^{\mathrm{L}} \mapsto X[\phi]$ is an $\mathcal{F}\boldsymbol{E}$-linear isomorphism, it induces an $\mathcal{F}\boldsymbol{E}$-Lie algebra structure on $\mathcal{A}^{\mathrm{SL}}$. Moreover, we define an $\mathcal{F}\boldsymbol{E}$-Lie algebra structure on $\mathcal{A}^{\mathrm{S}}$ by assuming $\mathcal{A}^{\mathrm{SQ}}$ to be an Abelian ideal. Then we have

$$[\phi, \theta] := (\phi^{\mathrm{L}\#} \times \theta^{\mathrm{L}\#})^{\flat}, \qquad \text{or} \qquad [\phi, \theta] = \varepsilon_p{}^{rs}\phi_r\theta_s\epsilon^p.$$

Namely, only the linear parts of ϕ and θ contribute to $[\phi, \theta]$.

Now we note that $\mathcal{A}^{\mathrm{P}} \cap \mathcal{A}^{\mathrm{S}} = \{0\}$, and set

$$\mathcal{A} := \mathcal{A}^{\mathrm{P}} \oplus \mathcal{A}^{\mathrm{S}}.$$

[12] An equivalent construction may be given by using the natural symplectic structure [11] of any Riemannian manifold (here, all spacetime fibres). This fact might be useful for future generalizations of this approach.

We are going to define a bracket on $\mathcal{A}$. Since we have brackets on $\mathcal{A}^{\mathrm{P}}$ and $\mathcal{A}^{\mathrm{S}}$, it suffices to define the bracket between any $f \in \mathcal{A}^{\mathrm{P}}$ and any $\phi \in \mathcal{A}^{\mathrm{S}}$. Then we set

$$[f, \phi] := \nabla[C]_{X[f]}\phi^{\mathrm{L}} \in \mathcal{A}^{\mathrm{SL}},$$

and $[\phi, f] := -[f, \phi]$. Then $\mathcal{A}^{\mathrm{S}}$ and $\mathcal{A}^{\mathrm{SL}}$ are ideals of $\mathcal{A}$. We have the coordinate expression

$$[f, \phi]_s = \left(u^0 f'' \partial_0 - u_0 \frac{\hbar}{m} f^j \partial_j\right)\phi_s + \left(u^0 f'' C_{0}{}^{r}{}_{s} - u_0 \frac{\hbar}{m} f^h C_{h}{}^{r}{}_{s}\right)\phi_r.$$

The new bracket fulfils the Jacobi identity in all cases except when one and only one of the three factors belongs to $\mathcal{A}^{\mathrm{SL}}$. In fact, by a straightforward calculation we prove

PROPOSITION 4.1. *Let* $f_1, f_2 \in \mathcal{A}^{\mathrm{P}}$, $\phi, \theta \in \mathcal{A}^{\mathrm{S}}$. *Then*

$$[f_1, [\phi, \theta]] + [\phi, [\theta, f_1]] + [\theta, [f_1, \phi]] = 0;$$

$$[f_1, [f_2, \phi]] + [f_2, [\phi, f_1]] + [\phi, [f_1, f_2]] = \underline{R}[C](X[f_1], X[f_2], \phi^{\mathrm{L}\#}). \qquad \diamond$$

Then $\mathcal{A} := \mathcal{A}^{\mathrm{P}} \oplus \mathcal{A}^{\mathrm{S}}$ will be called the $\mathbb{R}$-*algebra of quantizable functions*. The tangent lift of $f + \phi \in \mathcal{A}$ is defined to be $X[f + \phi] := X[f] + X[\phi]$. Then we obtain a map

$$\mathcal{A} \to T\boldsymbol{E} \oplus \vee^2(\mathbb{L}^* \otimes V\boldsymbol{E}) \oplus (\mathbb{L}^* \otimes V\boldsymbol{E}),$$

where $\vee$ denotes the symmetrized tensor product. This is an $\mathcal{F}\boldsymbol{E}$-linear epimorphism, and it turns out to be an $\mathbb{R}$-algebra morphism if we take, on the right-hand space, the bracket

$$(u, v) \mapsto \begin{cases} [u, v], & u, v \in T\boldsymbol{E}; \\ u \times v, & u, v \in \mathbb{L}^* \otimes V\boldsymbol{E}; \\ \nabla[C]_u v, & u \in T\boldsymbol{E},\ v \in \mathbb{L}^* \otimes V\boldsymbol{E}; \\ 0, & u \in \vee^2(\mathbb{L}^* \otimes V\boldsymbol{E}). \end{cases}$$

The most important quantizable function is the classical spin Hamiltonian (Section 4.1), which can be written as

$$H := u_0 \frac{H^{\mathrm{S}}}{\hbar} := u_0\left(\frac{H}{\hbar} - \mu B^\flat\right).$$

5. Spin bundle and connection

In this chapter we shall introduce two basic mathematical objects: the spin bundle and the Pauli map (a kind of 'soldering form'); the latter, together with a spacetime connection, yields in a natural way a connection on the spin bundle. In the next chapter, this will allow us to formulate quantum mechanics of a particle with spin along the lines of the scalar theory.

5.1. Spin bundle

Consider a complex vector bundle $\pi_S\colon S \to E$ with fibres of (complex) dimension 2, endowed with a Hermitian metric

$$h_S\colon E \to \overline{S^\star} \underset{E}{\otimes} S^\star,$$

where $S^\star$ and $\overline{S^\star}$ are the complex dual and antidual bundles, respectively (namely the bundles of linear and antilinear morphisms $S \to \mathbb{C}$ over E). We shall also be involved with the 'conjugate' bundle $S^\bullet := (\overline{S^\star})^\star \equiv \overline{(S^\star)^\star}$ (whose transition maps are conjugate to those of S).

Consider an h_S-orthonormal frame (ζ_A) of S, $A = 1, 2$, and its dual frame (z^A). Then we have the linear fibred coordinate chart (x^λ, z^A) on S. The conjugate chart on $S^\bullet$ will be denoted by $(x^\lambda, \bar{z}^{A^\bullet})$. The induced frame of VS will be denoted by $(\partial_A := \partial z_A)$; its dual and antidual frames by $(d^A := dz^A)$ and $(\bar{d}^{A^\bullet} := d\bar{z}^{A^\bullet})$. Since S admits a bundle atlas constituted by h_S-orthonormal charts, it can be regarded as a bundle associated with the principal bundle of all h_S-orthonormal frames, with the structure group $U(2)$.

We shall also consider the case when S is endowed with an h_S-normalized non-singular 2-form

$$\varepsilon_S\colon E \to \wedge^2 S^\star,$$

and define a *normal spin frame* to be an ordered h_S-orthonormal frame such that $\varepsilon_S = z^1 \wedge z^2$. Then S can be regarded as a bundle associated with the principal bundle of normal spin frames, with the structure group $SU(2)$.

Now we focus our attention on the vector bundle $\mathrm{End}(S) \equiv S \otimes_E S^\star$ of complex linear endomorphisms, whose fibres are equipped with the standard structure of associative algebra, given by $\phi\theta := \phi \circ \theta$, and with the induced structure of Lie algebra, given by $[\phi, \theta] := [\![\phi, \theta]\!] := \phi\theta - \theta\phi$. This bundle splits naturally into the direct sum of the real subbundles of all Hermitian and anti-Hermitian endomorphisms:

$$\mathrm{End}(S) = \boldsymbol{H} \underset{E}{\oplus} i\boldsymbol{H}.$$

Moreover, $\boldsymbol{H}$ splits into the direct sum of the vector subbundle $\langle \mathbf{1} \rangle$ generated by the identity and the vector subbundle $\boldsymbol{H}_0$ of all traceless endomorphisms, according to the formula

$$\phi = \tfrac{1}{2}(\mathrm{Tr}\,\phi)\,\mathbf{1} + (\phi - \tfrac{1}{2}(\mathrm{Tr}\,\phi)\,\mathbf{1}).$$

Then we obtain

$$\mathrm{End}(S) = \langle \mathbf{1} \rangle \underset{E}{\oplus} \boldsymbol{H}_0 \underset{E}{\oplus} \langle i\mathbf{1} \rangle \underset{E}{\oplus} i\boldsymbol{H}_0.$$

The bundle $\boldsymbol{H}_0 \to E$ will play an essential role in the Galilean quantum theory of spin. For this reason we are going to make a fairly detailed study of its rich algebraic structure. Note that $\boldsymbol{H}_0$ is constituted by all endomorphisms ϕ whose matrix, in any

h_S-orthonormal frame of S, is of the type $(\phi^A{}_B) = \begin{pmatrix} r & c \\ \bar{c} & -r \end{pmatrix}$, with $r \in \mathbb{R}$, $c \in \mathbb{C}$; actually, the fibres of $\boldsymbol{H}_0$ have (real) dimension 3.

We first observe that the fibred map over $\boldsymbol{E}$

$$k\colon \boldsymbol{H}_0 \underset{\boldsymbol{E}}{\times} \boldsymbol{H}_0 \to \mathbb{R}\colon (\phi, \theta) \mapsto \tfrac{1}{2}\operatorname{Tr}(\phi \circ \theta),$$

turns out to be an Euclidean metric on the fibres of $\boldsymbol{H}_0$. Hence, we can regard $\boldsymbol{H}_0$ as a bundle associated with the principal bundle of all k-orthonormal frames, with the structure group $O(3)$.

LEMMA 5.1. *Let (ζ_A) be an orthonormal frame of S, and (σ_r) an orthonormal frame of $\boldsymbol{H}_0$. Then, for each $P \in U(2)$, the endomorphisms*

$$\sigma'_r := \sigma_r{}^A{}_B P(\zeta_A) \otimes (P^*)^{-1}(z^B), \qquad r = 1, 2, 3,$$

constitute a k-orthonormal frame with the same orientation as (σ_r). Hence, there is a unique $\widetilde{P} \in SO(3)$ such that $\sigma'_s = \widetilde{P}^r{}_s \sigma_r$. The map $U(2) \to SO(3)$: $P \mapsto \widetilde{P}$ is a group epimorphism (which depends on the choice of (ζ_A) and (σ_r)). In particular, the map $SU(2) \to SO(3)$ is two-to-one.[13] ◇

The following lemma is the key for studying those structures of $\boldsymbol{H}_0$ which arise from the algebra $\operatorname{End}(S)$.

LEMMA 5.2. *For each $\phi, \theta \in \boldsymbol{H}_0$ we have*

$$\phi\theta = k(\phi, \theta)\,\mathbf{1} + i\,\xi,$$

where $\xi \in \boldsymbol{H}_0$ and

$$k(\xi, \xi) = k(\phi, \phi)\, k(\theta, \theta) - (k(\phi, \theta))^2,$$
$$k(\phi, \xi) = k(\theta, \xi) = 0.$$

Moreover, we have $\theta\phi = k(\phi, \theta)\,\mathbf{1} - i\,\xi$. ◇

Thus $\boldsymbol{H}_0$ is closed neither under the associative multiplication $(\phi, \theta) \mapsto \phi\theta$ nor under the commutator $(\phi, \theta) \mapsto [\![\phi, \theta]\!] := \phi\theta - \theta\phi$. However, we shall see that these operations are related to further structures on $\boldsymbol{H}_0$.

In particular, if $\phi \in \boldsymbol{H}_0$ and $\|\phi\| = 1$, then

$$\phi\phi = \mathbf{1},$$

if $\phi, \theta \in \boldsymbol{H}_0$, $\|\phi\| = \|\theta\| = 1$ and $k(\phi, \theta) = 0$, then

$$\phi\theta = i\xi,$$

with $\xi \in \boldsymbol{H}_0$, $\|\xi\| = 1$, $k(\phi, \xi) = k(\theta, \xi) = 0$.

The above result yields a distinguished global orientation on the bundle $H_0 \to \boldsymbol{E}$. In fact, for each k-orthonormal frame (σ_r), the condition $\sigma_1\sigma_2 = i\sigma_3$ determines an orientation which does not depend on the frame choice.

[13] This last statement is a geometric reformulation, in our context, of a well-known algebraic result.

The metric k and the above orientation yield a global volume form $\tilde{\eta}$: $\boldsymbol{E} \to \wedge^3 \boldsymbol{H}_0$. Accordingly, the bundle $\boldsymbol{H}_0 \to \boldsymbol{E}$ can be seen as associated with the principal bundle of all positively oriented k-orthonormal frames, with the structure group $SO(3)$.

A positively oriented orthonormal frame is called a set of *Pauli endomorphisms*. Moreover, we set $\sigma_0 := \mathbf{1}_S$, so that (σ_α), $\alpha = 0,1,2,3$, is a frame of $\boldsymbol{H}$.

For any h_S-orthonormal frame (ζ_A) we may consider, in particular, these elements (σ_r) in $\boldsymbol{H}_0$ whose matrix expressions $\sigma_r = \sigma_r{}^A{}_B \zeta_A \otimes z^B$ are given by the *Pauli matrices*:

$$(\sigma_r{}^A{}_B) := \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \qquad r = 1,2,3.$$

Then (σ_r) is a set of Pauli endomorphisms. Conversely, in virtue of the double covering $SU(2) \to SO(3)$, for any given set (σ_r) of Pauli endomorphisms, there exists an orthonormal frame (ζ_A) such that $(\sigma_r{}^A{}_B)$ are the Pauli matrices. However, this particular matrix representation will play no essential role in our treatment.

In terms of a set of Pauli endomorphisms the volume form $\tilde{\eta}$ reads

$$\tilde{\eta} = \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = \tfrac{1}{3!}\varepsilon^{prs}\sigma_p \wedge \sigma_r \wedge \sigma_s,$$

and the statement of Lemma 5.2 reads

$$\sigma_r \sigma_s = \delta_{rs}\sigma_0 + i\,\varepsilon^p{}_{rs}\sigma_p.$$

The metric k and the volume form $\tilde{\eta}$ yield the *cross-product* Lie algebra structure on $\boldsymbol{H}_0$ given by

$$(\phi, \theta) \mapsto \phi \times \theta := \tilde{\eta}(k^\flat(\phi) \wedge k^\flat(\theta)).$$

In terms of any set of Pauli endomorphisms this reads

$$\sigma_r \times \sigma_s = \varepsilon^p{}_{rs}\,\sigma_p.$$

The type fibre of this Lie algebra is $\mathfrak{su}(2)$, namely the Lie algebra of the Lie group $SU(2)$, which is usually called the *angular momentum algebra*.

The cross-product Lie algebra is related to the Lie algebra End($\boldsymbol{S}$) by the formula

$$\phi \times \theta = -\tfrac{i}{2}[\![\phi, \theta]\!]$$

which, in a set of Pauli endomorphisms, reads

$$[\![\sigma_r, \sigma_s]\!] = 2i\,\varepsilon^p{}_{rs}\,\sigma_p, \qquad \text{or} \qquad [\![-\tfrac{i}{2}\sigma_r, -\tfrac{i}{2}\sigma_s]\!] = \varepsilon^p{}_{rs} \cdot (-\tfrac{i}{2}\sigma_p).$$

Then we see that $i\boldsymbol{H}_0$ is closed under the Lie bracket of End($\boldsymbol{S}$), and the map $\boldsymbol{H}_0 \to i\boldsymbol{H}_0 : \phi \mapsto -\tfrac{i}{2}\phi$ is a Lie algebra isomorphism.

Remark 5.1: For all $\phi, \theta \in \boldsymbol{H}_0$ we have

$$\phi\theta + \theta\phi = 2k(\phi, \theta)\mathbf{1}.$$

In terms of a set of Pauli endomorphisms this formula reads

$$\sigma_r\sigma_s + \sigma_s\sigma_r = 2\,\delta_{rs}\,\mathbf{1}.$$

Then one sees easily that the Clifford algebra bundle of $\boldsymbol{H}_0$ (see [12]) coincides with the real vector bundle underlying End($\boldsymbol{S}$) $\equiv \boldsymbol{S} \otimes \boldsymbol{S}^\star$, with the product given by ordinary composition. This result agrees with $\dim_{\mathbb{R}} \mathrm{End}(S) = 8 = 2^{\dim \boldsymbol{H}_0}$. A set of Pauli endomorphisms yields the following set of generators of the Clifford algebra:

$$\sigma_0, \quad \sigma_1, \quad \sigma_2, \quad \sigma_3,$$

$$\sigma_1\sigma_2 = i\sigma_3, \quad \sigma_2\sigma_3 = i\sigma_1, \quad \sigma_3\sigma_1 = i\sigma_2,$$

$$\sigma_1\sigma_2\sigma_3 = i\sigma_0.$$

This Clifford algebra will not enter our treatment in the Galilean context. However, it is important for a comparison with the Einstein case. •

Remark 5.2: The Hermitian metric h_S yields an isomorphism $\boldsymbol{S}\otimes\boldsymbol{S}^\star \to \boldsymbol{S}\otimes\boldsymbol{S}^\bullet$. The latter is the space of world spinors [35] that carries a natural Lorentz structure defined via ε. An analogous Lorentz metric can be defined on $\boldsymbol{H}$, and the above isomorphism is an isometry. Once h_S has been assigned, the two constructions are equivalent. Then k is just the restriction of the Lorentz metric to the canonical spacelike subbundle $\boldsymbol{H}_0$, while $\langle \mathbf{1}\rangle$ is its orthogonal timelike subbundle. Moreover, (σ_α) is an orthonormal frame of $\boldsymbol{H}$. •

5.2. Spin connections

Henceforth we assume that $\boldsymbol{S}$ is endowed with a Hermitian metric h_S and a non-singular h_S-normalized 2-form ε_S: $\boldsymbol{E} \to \wedge^2\boldsymbol{S}^\star$.

The coordinate expression of a linear connection Б: $\boldsymbol{S} \to T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} T\boldsymbol{S}$ on the bundle $\boldsymbol{S} \to \boldsymbol{E}$ is of the type[14]

$$\text{Б} = d^\lambda \otimes (\partial_\lambda + i\text{Б}_{\lambda}{}^{A}{}_{B} z^B\, \partial_A),$$

with $\text{Б}_{\lambda}{}^{A}{}_{B}$: $E \to \mathbb{C}$. (The choice of writing the coefficients of the connection with the factor i is merely a convention.) Moreover, we have the *conjugate* linear connection $\text{Б}^\bullet$: $\boldsymbol{S}^\bullet \to T^*\boldsymbol{E}\otimes T\boldsymbol{S}^\bullet$, with the coordinate expression

$$\text{Б}^\bullet = dx^\lambda \otimes (\partial x_\lambda - i\text{Б}_{\lambda}{}^{A^\bullet}{}_{B^\bullet} \bar{z}^{B^\bullet}\, \partial_{A^\bullet}),$$

where $\text{Б}_{\lambda}{}^{A^\bullet}{}_{B^\bullet} = \overline{\text{Б}_{\lambda}{}^{A}{}_{B}}$. We also have the induced linear connections on $\boldsymbol{S}^\star$ and $\overline{\boldsymbol{S}^\star}$, with coefficients $\text{Б}_{\lambda B}{}^{A} = -\text{Б}_{\lambda}{}^{A}{}_{B}$ and $\text{Б}_{\lambda B^\bullet}{}^{A^\bullet} = -\text{Б}_{\lambda}{}^{A^\bullet}{}_{B^\bullet} = \overline{\text{Б}_{\lambda B}{}^{A}}$.

A linear connection Б on $\boldsymbol{S}$ will be called *Hermitian* if it fulfils $\nabla[\text{Б}]h_S = 0$.

LEMMA 5.3. *A linear connection* Б *on* $\boldsymbol{S}$ *is Hermitian iff the coefficients of* Б *in a normal spin frame are given by*

$$\text{Б}_{\lambda}{}^{A}{}_{B} = \text{Б}_{\lambda}^{\mu}\, \sigma_{\mu}{}^{A}{}_{B},$$

where Б_{λ}^{μ}: $\boldsymbol{E} \to \mathbb{R}$, *and* (σ_j) *is any set of Pauli endomorphisms.*

Proof: In any linear coordinate chart the condition $\nabla[\text{Б}]h_S = 0$ reads

$$\partial_\lambda h_{A^\bullet B} - ih_{C^\bullet B}\, \text{Б}_{\lambda}{}^{C^\bullet}{}_{A^\bullet} + ih_{A^\bullet C}\, \text{Б}_{\lambda}{}^{C}{}_{B} = 0.$$

[14] Б is the Cyrillic character corresponding to Latin B.

According to this formula, in an orthonormal chart the components $Б_\lambda{}^A{}_B$, for each fixed λ, constitute Hermitian 2×2 matrices, and thus, for any set of Pauli endomorphisms, are linear combinations of the matrices $(\sigma_\mu{}^A{}_B)$. ∎

LEMMA 5.4. *A Hermitian connection* Б *on* $\boldsymbol{S}$ *fulfils* $\nabla[Б]\varepsilon_S = 0$ *iff* $Б^0_\lambda = 0$ *in a normal spin frame, that is iff we have*

$$Б_\lambda{}^A{}_B = Б^r_\lambda \, \sigma_r{}^A{}_B, \qquad r = 1, 2, 3,$$

for any set of Pauli endomorphisms.

Proof: In any linear coordinate chart the condition $\nabla[Б]\varepsilon_S = 0$ reads

$$\partial_\lambda \varepsilon_{AB} + i\varepsilon_{CB} Б_\lambda{}^C{}_A + i\varepsilon_{AC} Б_\lambda{}^C{}_B = 0.$$

In a normal spin chart we have $\partial_\lambda \varepsilon_{AB} = 0$, hence the matrices $(Б_\lambda{}^A{}_B)$, for each fixed λ, are traceless. ∎

DEFINITION 5.1. A *spin connection* is a linear connection Б on $\boldsymbol{S}$ such that $\nabla[Б]h_S = 0$ and $\nabla[Б]\varepsilon_S = 0$. ♣

In the particular case when the matrices of the considered Pauli endomorphisms are the usual Pauli matrices, the components of a spin connection are given by

$$(Б_\lambda{}^A{}_B) = \begin{pmatrix} Б^3_\lambda & Б^1_\lambda - iБ^2_\lambda \\ Б^1_\lambda + iБ^2_\lambda & -Б^3_\lambda \end{pmatrix}.$$

Henceforth, by Б we shall always indicate a spin connection.

Remark 5.3. A spin connection preserves also the Euclidean metric k, as one sees from its definition via ε (or also by a direct calculation), namely $\nabla[Б]k = 0$. •

LEMMA 5.5. *We have*:

$$\begin{aligned} \nabla_\lambda[Б]\sigma_0 &= 0, \\ \nabla_\lambda[Б]\sigma_s &= -Б^p_\lambda [\![\sigma_p, \sigma_s]\!]^A{}_B \, \zeta_A \otimes z^B \\ &= -2Б^p_\lambda \varepsilon^r{}_{sp} \sigma_r{}^A{}_B \zeta_A \otimes z^B \;=\; -2Б^p_\lambda \varepsilon^r{}_{sp} \sigma_r. \end{aligned}$$

Proof:

$$\begin{aligned} \nabla_\lambda[Б]\sigma_\alpha &= \nabla_\lambda[Б](\sigma_\alpha{}^A{}_B \zeta_A \otimes z^B) \\ &= \sigma_\alpha{}^A{}_B(-iБ_\lambda{}^C{}_A \zeta_C \otimes z^B + iБ_\lambda{}^B{}_C \zeta_A \otimes z^C) \\ &= -iБ^p_\lambda(\sigma_p{}^A{}_B \sigma_\alpha{}^C{}_B - \sigma_\alpha{}^A{}_C \sigma_p{}^C{}_B)\zeta_A \otimes z^B \\ &= -iБ^p_\lambda [\![\sigma_p, \sigma_\alpha]\!]^A{}_B \, \zeta_A \otimes z^B. \end{aligned}$$ ∎

PROPOSITION 5.1. *The natural extension of* Б *to* $\boldsymbol{S} \otimes \boldsymbol{S}^\star$ *gives rise, through restriction, to a real linear connection* $\widetilde{Б}$: $\boldsymbol{H}_0 \to T^*\boldsymbol{E} \otimes T\boldsymbol{H}_0$ *on the Hermitian traceless subbundle* $\boldsymbol{H}_0 \to \boldsymbol{E}$. *In a frame of Pauli endomorphisms the coefficients of* $\widetilde{Б}$ *are given by* $\widetilde{Б}_\lambda{}^r{}_s = 2Б^p_\lambda \varepsilon^r{}_{sp}$. ◇

Conversely, we have

PROPOSITION 5.2. *Suppose that* $\mathrm{B}\colon \boldsymbol{H}_0 \to T^*\boldsymbol{E}\otimes T\boldsymbol{H}_0$ *is a linear connection such that* $\nabla[\mathrm{B}]k = 0$. *Then there exists a unique spin connection* $\mathrm{Б}$ *such that* $\widetilde{\mathrm{Б}} = \mathrm{B}$. *Its coefficients are given by*

$$\mathrm{Б}^p_\lambda := \tfrac{1}{4}\varepsilon_r{}^{sp}\mathrm{B}^{\,r}_{\lambda\,s},$$

that is

$$\mathrm{Б}^{\;A}_{\lambda\;B} = \tfrac{1}{4}\varepsilon_r{}^{sp}\mathrm{B}^{\,r}_{\lambda\,s}\sigma_p{}^A{}_B.$$

Proof: Uniqueness: If $\mathrm{Б}$ exists, then $\nabla[\mathrm{Б}]\sigma_s = \nabla[\mathrm{B}]\sigma_s \Rightarrow \widetilde{\mathrm{Б}}^{\,r}_{\lambda\,s} = \mathrm{B}^{\,r}_{\lambda\,s}$, that is

$$\mathrm{B}^{\,r}_{\lambda\,s} = 2\mathrm{Б}^p_\lambda\varepsilon^r{}_{sp}.$$

This equality determines the coefficients $\mathrm{Б}^p_\lambda$ (and then also the coefficients $\mathrm{Б}^{\;A}_{\lambda\;B}$), since it can be reversed as:

$$\mathrm{Б}^p_\lambda := \tfrac{1}{4}\varepsilon_r{}^{sp}\mathrm{B}^{\,r}_{\lambda\,s}.$$

Existence: The spin connection whose real coefficients $\mathrm{Б}^p_\lambda$ are given by the previous formula satisfies $\widetilde{\mathrm{Б}} = \mathrm{B}$. ∎

From the above results we see how one is naturally involved with $\boldsymbol{H}_0$ when considering Hermitian connections.

5.3. Pauli map

An orientation-preserving linear fibred isometry over $\boldsymbol{E}$:

$$\Sigma\colon \mathbb{L}^*\otimes V\boldsymbol{E} \to \boldsymbol{H}_0,$$

will be called a *Pauli map*. If (e_r) is a positively-oriented orthonormal frame of $\mathbb{L}^*\otimes V\boldsymbol{E}$, then $(\sigma_r) := (\Sigma(e_r))$ is a set of Pauli endomorphisms. Henceforth, when dealing with Σ we shall use the linear fibred charts on $\mathbb{L}^*\otimes V\boldsymbol{E}$ and $\boldsymbol{H}_0$ induced by a given frame (e_r) and the corresponding frame (σ_r). So, the information relative to Σ is encoded in the choice of such an adapted chart.

A Pauli map is, obviously, an isomorphism of cross-product Lie algebras (see Section 5.1). Moreover, we have the Lie algebra isomorphism $-\frac{i}{2}\Sigma\colon \mathbb{L}^*\otimes V\boldsymbol{E} \to i\boldsymbol{H}_0$.

A Pauli map can be naturally extended to tensor products by setting

$$\Sigma^2\colon \otimes^2(\mathbb{L}^*\otimes V\boldsymbol{E}) \to \boldsymbol{S}\underset{\boldsymbol{E}}{\otimes}\boldsymbol{S}^\star\colon u\otimes v \mapsto \Sigma(u)\circ\Sigma(v) \in \boldsymbol{H}_0\circ\boldsymbol{H}_0 \subset \boldsymbol{S}\underset{\boldsymbol{E}}{\otimes}\boldsymbol{S}^\star.$$

PROPOSITION 5.3. *Let C be a metrical linear connection on* $V\boldsymbol{E}\to\boldsymbol{E}$ (*Section* 4.1). *Then there exists a unique spin connection* $\mathrm{Б}$ *on* $\boldsymbol{S}$ *such that for any section* $v\colon \boldsymbol{E} \to \mathbb{L}^*\otimes V\boldsymbol{E}$ *one has*

$$\Sigma(\nabla[C]v) = \nabla[\mathrm{Б}](\Sigma(v)).$$

Namely, we have

$$\mathrm{Б}^{\;A}_{\lambda\;B} = \tfrac{1}{4}\varepsilon_r{}^{sp}C^{\,r}_{\lambda\,s}\sigma_p{}^A{}_B.$$

Proof: Since Σ is an isomorphism, the connection C induces a connection B on $\boldsymbol{H}_0$ according to the above requirement. We have $\Sigma(\nabla[C]e_s) = \nabla[\mathrm{B}]\sigma_s$, that is $\mathrm{B}^{\,r}_{\lambda\,s} = C^{\,r}_{\lambda\,s}$. Since $\nabla[C]g{=}0$, we also have $\nabla[\mathrm{B}]k{=}0$. Thus we only need to apply Proposition 5.2. ∎

We shall be concerned with the curvature tensor $R[Б]$ of Б. We have the coordinate expression $R[Б] = R_{\lambda\mu}{}^{A}{}_{B} z^{B} d^{\lambda} \wedge d^{\mu} \otimes \partial_{A}$, where

$$R_{\lambda\mu}{}^{A}{}_{B} = i\partial_{[\lambda} Б_{\mu]}{}^{A}{}_{B} + Б_{[\lambda}{}^{A}{}_{C} Б_{\mu]}{}^{C}{}_{B}.$$

If we replace the coefficients $Б_{\lambda}{}^{A}{}_{B}$ in the previous formula with their expression given in Proposition 5.3, we obtain, after some calculations, the following result.

PROPOSITION 5.4. *We have*

$$R[Б] = -\tfrac{i}{4}\Sigma(*\underline{R}[C]),$$

where

$$\underline{R}[C]\colon\ \boldsymbol{E} \to \wedge^{2}T^{*}\boldsymbol{E} \otimes \wedge^{2}(\mathbb{L} \otimes V^{*}\boldsymbol{E})$$

is the completely covariant curvature tensor of C. *The coordinate expression of* $R[Б]$ *is*

$$R_{\lambda\mu}{}^{A}{}_{B} = \tfrac{i}{4}\varepsilon_{r}{}^{sp} R[C]_{\lambda\mu}{}^{r}{}_{s} \sigma_{p}{}^{A}{}_{B},$$

where

$$R[C]_{\lambda\mu}{}^{r}{}_{s} = \partial_{\lambda} C_{\mu}{}^{r}{}_{s} + C_{\lambda}{}^{q}{}_{s} C_{\mu}{}^{r}{}_{q}. \qquad \diamond$$

In particular we shall be involved with the connection $Б_{2\mu}$ induced by $C := K'_{2\mu}$ (Section 4.1). In that case, Proposition 5.4 is the analogous, for the spin connection, of the formula $R[Ч] = i(m/\hbar)\Omega \otimes \mathbf{1}_{Q}$ for the quantum connection.

6. Quantum spin

6.1. Quantum spin connection

In addition to the postulates of the classical theory (Section 3.1) and of the scalar quantum theory (Section 3.2), we have the two following basic geometric postulates of the quantum spin theory.

POSTULATE QS1. The *spin bundle* is a complex vector bundle $\boldsymbol{S} \to \boldsymbol{E}$ with fibres of (complex) dimension 2, endowed with a Hermitian metric h_{S} and a non-singular h_{S}-normalized 2-form ε_{S}. ♣

POSTULATE QS2. The *Pauli map* is an orientation-preserving linear fibred isometry over $\boldsymbol{E}$:

$$\Sigma\colon\ \mathbb{L}^{*} \otimes V\boldsymbol{E} \to \boldsymbol{H}_{0}. \qquad ♣$$

Then we define the *quantum spin bundle* to be the tensor product

$$\pi_{W}\colon\ \boldsymbol{W} := \boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{S} \to \boldsymbol{E}.$$

The Hermitian metrics h_{Q} and h_{S}, defined respectively on $\boldsymbol{Q}$ and $\boldsymbol{S}$, yield a Hermitian metric $h := h_{Q} \otimes h_{S}$ on $\boldsymbol{W}$. We shall indicate by $b_{A} := b \otimes \zeta_{A}$ the orthonormal frame of $\boldsymbol{W}$ induced by a normal frame b of $\boldsymbol{Q}$ and by a normal spin frame (ζ_{A}) of $\boldsymbol{S}$. The corresponding linear coordinates induced on $\boldsymbol{W}$ are denoted by $w^A := z \otimes z^A$, and the frame induced on $V\boldsymbol{W} \to \boldsymbol{W}$ by (∂w_A).

Quantum histories will be described as sections Ψ: $\boldsymbol{E} \to \boldsymbol{W}$. Locally

$$\Psi = \Psi^A \otimes \zeta_A = \psi^A b \otimes \zeta := \psi^A b_A,$$

where $\Psi^A := \psi^A b$: $\boldsymbol{E} \to \boldsymbol{Q}$ is a scalar quantum history $(A = 1, 2)$, ψ^A: $\boldsymbol{E} \to \mathbb{C}$.

We consider a particle with given values q of the charge, m of the mass and μ of the spin-magnetic field coupling constant. Then we have (Section 4.1) the two spacetime connections $K_{q/m}$ and $K_{2\mu}$. The first yields a quantum connection $Ч_{q/m}$ on $\boldsymbol{Q}^\uparrow$ (henceforth denoted simply as Ч). The second yields a connection $C :\mp K'_{2\mu}$ on $\mathbb{L}^* \otimes V\boldsymbol{E} \to \boldsymbol{E}$ (Section 4.1); this, in turn, yields via Σ a spin connection $Б_{2\mu}$, henceforth denoted simply by Б, whose components in a normal spin frame are given by

$$\begin{aligned}
Б_j{}^A{}_B &= \tfrac{1}{4}\varepsilon_r{}^{sp}\widetilde{\Gamma}^{\natural}{}_j{}^r{}_s\sigma_p{}^A{}_B = Б^{\natural}{}_j{}^A{}_B, \\
Б_0{}^A{}_B &= \tfrac{1}{4}\varepsilon_r{}^{sp}(\widetilde{\Gamma}^{\natural}{}_0{}^r{}_s + u_0\mu\widetilde{F}^r{}_s)\sigma_p{}^A{}_B \\
&= \tfrac{1}{4}\varepsilon_r{}^{sp}\widetilde{\Gamma}^{\natural}{}_0{}^r{}_s\sigma_p{}^A{}_B + \tfrac{1}{2}u_0\mu\widetilde{B}^p\sigma_p{}^A{}_B \\
&= Б^{\natural}{}_0{}^A{}_B + \tfrac{1}{2}u_0\mu\widetilde{B}^p\sigma_p{}^A{}_B,
\end{aligned}$$

where $Б^\natural$ is the spin connection arising from $\Gamma^\natural$ (vanishing coupling constant).

The quantum connection and the spin connection yield a Hermitian linear connection $Ч^{\boldsymbol{W}} := Ч \otimes Б$, called the *quantum spin connection*, on the vector bundle

$$\boldsymbol{W}^\uparrow := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{W} \to J_1\boldsymbol{E}.$$

The components of $Ч^{\boldsymbol{W}}$ can be synthetically written as

$$Ч_\lambda{}^A{}_B = ч_\lambda^\alpha\sigma_\alpha{}^A{}_B = ч_\lambda^0\delta^A{}_B + ч_\lambda^p\sigma_p{}^A{}_B,$$

where we have set

$$ч_\lambda^0 := Ч_\lambda, \qquad ч_\lambda^h := Б_\lambda^h,$$

that is (Section 3.2):

$$\begin{aligned}
Ч_0{}^A{}_B &= -u_0\frac{H}{\hbar}, \qquad Ч_j{}^A{}_B = \frac{p_j}{\hbar} \quad \text{if } A = B, \\
Ч_\lambda{}^A{}_B &= \tfrac{1}{4}\varepsilon_r{}^{sp}C_\lambda{}^r{}_s\sigma_p{}^A{}_B \quad \text{if } A \neq B.
\end{aligned}$$

The corresponding covariant derivative of a section Ψ turns out to be the section $\nabla\Psi$: $J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} \boldsymbol{W}$ given by

$$\begin{aligned}
\nabla_\lambda\Psi &:= (\nabla_\lambda\Psi^A) \otimes \zeta_A + \Psi^A \otimes (\nabla_\lambda\zeta_A) \\
&= (\nabla_\lambda b) \otimes (\Psi^A\zeta_A) + b \otimes \nabla_\lambda(\Psi^A\zeta_A).
\end{aligned}$$

The coordinate expression of $\nabla\Psi$ is

$$\nabla\Psi = (\partial_\lambda\psi^A - iЧ_\lambda\psi^A - iБ_\lambda{}^A{}_B\psi^B)d^\lambda \otimes b_A.$$

We also have the derivatives:

$$\overset{\circ}{\nabla}\Psi := \text{д} \lrcorner \nabla\Psi\colon\ J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{W},$$
$$\check{\nabla}\Psi := \nabla\Psi_{|V\boldsymbol{E}}\colon\ J_1\boldsymbol{E} \to V^*\boldsymbol{E} \otimes \boldsymbol{W},$$

where д: $J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ is the natural map introduced in Section 3.1. Their coordinate expressions are:

$$\overset{\circ}{\nabla}\Psi = u^0\left((\partial_0 + y_0^j\partial_j)\psi^A - i(\text{Ч}_0 + y_0^j\text{Ч}_j)\psi^A - i(\text{Б}_{0}{}^{A}{}_{B} + y_0^j\text{Б}_{j}{}^{A}{}_{B})\psi^B\right) b_A,$$
$$\check{\nabla}\Psi = (\partial_j\psi^A - i\text{Ч}_j\psi^A - i\text{Б}_{j}{}^{A}{}_{B}\psi^B)d^j \otimes b_A.$$

We shall also be concerned with the curvature tensor of the quantum spin connection. By a simple calculation one sees that this is essentially the sum of the curvature tensors of Ч and Б (see Proposition 5.4):

$$\begin{aligned} R[\text{Ч}\otimes\text{Б}] &= R[\text{Ч}]\otimes \mathbf{1}_S + \mathbf{1}_Q \otimes R[\text{Б}] \\ &= i\frac{m}{\hbar}\Omega_{q/m}\otimes \mathbf{1}_W - \tfrac{i}{4}\mathbf{1}_Q \otimes \Sigma(*\underline{R}[C]) \\ &= (R[\text{Ч}]_{\lambda\mu}\delta^A{}_B + R[\text{Б}]_{\lambda\mu}{}^{A}{}_{B})w^B d^\lambda \wedge d^\mu \otimes b_A. \end{aligned}$$

6.2. Quantum spin Lagrangian and momentum

We have the following distinguished observer-dependent 4-forms over $\boldsymbol{E}$:

$$\overset{\circ}{\mathcal{L}}[\Psi] := \tfrac{1}{2}(h(\Psi, i\overset{\circ}{\nabla}\Psi) + h(i\overset{\circ}{\nabla}\Psi, \Psi))\upsilon\colon\ \boldsymbol{E} \to \mathbb{L}^3 \otimes \wedge^4 T^*\boldsymbol{E},$$
$$\check{\mathcal{L}}[\Psi] := \frac{\hbar}{2m}(g^\# \otimes h)(\check{\nabla}\Psi, \check{\nabla}\Psi)\upsilon\colon\ \boldsymbol{E} \to \mathbb{L}^3 \otimes \wedge^4 T^*\boldsymbol{E},$$

where υ is the spacetime volume form (Section 3.1). As in the theory without spin we obtain a Lagrangian independent of any observer by the projectability principle. Namely:

PROPOSITION 6.1. *The form*

$$\mathcal{L}[\Psi] := \overset{\circ}{\mathcal{L}}[\Psi] - \check{\mathcal{L}}[\Psi]$$

is the unique linear combination (up to an overall factor) of $\overset{\circ}{\mathcal{L}}$ *and* $\check{\mathcal{L}}$ *which turns out to be independent of the observer.*

Proof: A rather long computation shows that this is the unique linear combination of $\overset{\circ}{\mathcal{L}}$ and $\check{\mathcal{L}}$ such that the coordinates y_0^j disappear in its coordinate expression. ∎

Then we have the main dynamical postulate of the quantum spin theory:

POSTULATE QS3. The form $\mathcal{L}$ of Proposition 6.1 is assumed to be the *quantum spin Lagrangian*. ♣

In the scalar case it is known [16] that the analogous procedure yields what is essentially the unique natural and physically meaningful Lagrangian. Morever, note that adding to our Lagrangian a term proportional to the natural function

$$\frac{q}{m} h(\Psi, \Sigma(B)\Psi)\colon \boldsymbol{E} \to \mathbb{R}$$

would simply amount to modifying the gyromagnetic ratio.

We have the coordinate expression

$$\begin{aligned}\mathcal{L}[\Psi] = \tfrac{1}{2} h_{C^\bullet A}\Big(& i(\overline{\psi}^{C^\bullet}\partial_0\psi^A - \psi^A\partial_0\overline{\psi}^{C^\bullet}) - u_0\frac{\hbar}{m} g^{jk}\partial_j\psi^A\partial_k\overline{\psi}^{C^\bullet} + \\ & + i g^{jk} a_k(\psi^A\partial_j\overline{\psi}^{C^\bullet} - \overline{\psi}^{C^\bullet}\partial_j\psi^A) + u^0\frac{m}{\hbar}\psi^A\overline{\psi}^{C^\bullet}(2a_0 - g^{jk}a_j a_k) + \\ & + 2(\mathrm{B}_0{}^A{}_B - g^{jk}a_k \mathrm{B}_j{}^A{}_B)\psi^B\overline{\psi}^{C^\bullet} + u_0\frac{i\hbar}{m} g^{jk}\mathrm{B}_k{}^A{}_B(\psi^B\partial_j\overline{\psi}^{C^\bullet} - \overline{\psi}^{C^\bullet}\partial_j\psi^B) + \\ & + u_0\frac{\hbar}{m} g^{jk}\mathrm{B}_j{}^E{}_B \mathrm{B}_k{}^A{}_E\psi^B\overline{\psi}^{C^\bullet}\Big)\sqrt{|g|}\,\omega.\end{aligned}$$

Note that to simplify this expression for $\mathcal{L}$ we used the property that the coefficients $\mathrm{B}_j{}^A{}_B$ are Hermitian: $h_{C^\bullet A}\mathrm{B}_\lambda{}^{C^\bullet}{}_{B^\bullet} = h_{B^\bullet C}\mathrm{B}_\lambda{}^C{}_A$.

In an h-orthonormal frame (b_A) we have $h_{C^\bullet A} = \delta_{C^\bullet A}$, and then the Lagrangian splits as

$$\mathcal{L}[\Psi] = \mathcal{L}[\Psi^1] + \mathcal{L}[\Psi^2] + \mathcal{L}[\Psi]_{\text{spin}},$$

where $\mathcal{L}[\Psi^1]$ and $\mathcal{L}[\Psi^2]$ (first two lines) are exactly the Lagrangians of the scalar wave functions Ψ^1 and Ψ^2. The spin Lagrangian $\mathcal{L}[\Psi]_{\text{spin}}$ is the new part (with respect to the scalar case) and contains interaction terms. By using Proposition 5.3, after some calculations we can express it in terms of the vertical spacetime connection C.

PROPOSITION 6.2. *We have*

$$\begin{aligned}\mathcal{L}[\Psi]_{\text{spin}} = \tfrac{1}{2} h_{C^\bullet A}\Big(& \tfrac{1}{2}(C_0{}^r{}_s - g^{jk}a_k C_j{}^r{}_s)\varepsilon_r{}^{sp}\sigma_p{}^A{}_B\psi^B\overline{\psi}^{C^\bullet} + \\ & + u_0\frac{i\hbar}{4m} g^{jk} C_k{}^r{}_s\varepsilon_r{}^{sp}\sigma_p{}^A{}_B(\psi^B\partial_j\overline{\psi}^{C^\bullet} - \overline{\psi}^{C^\bullet}\partial_j\psi^B) + \\ & - u_0\frac{\hbar}{8m} g^{jk} C_j{}^r{}_s C_k{}^r{}_s\psi^A\overline{\psi}^{C^\bullet}\Big)\sqrt{|g|}\,\omega.\end{aligned}$$ ◇

It is interesting to look at the spin part of the Lagrangian in the flat case. Setting $C_j{}^r{}_s = 0$, $C_0{}^r{}_s = u_0\mu\widetilde{F}^r{}_s$ we obtain

$$\begin{aligned}\mathcal{L}[\Psi]_{\text{spin}} &= \tfrac{1}{4}u_0\mu\, h_{C^\bullet A}(\widetilde{F}^r{}_s\varepsilon_r{}^{sp}\sigma_p{}^A{}_B\psi^B\overline{\psi}^{C^\bullet})\sqrt{|g|}\,\omega \\ &= \tfrac{1}{2}u_0\mu\, h_{C^\bullet A}\widetilde{B}^p\sigma_p{}^A{}_B\psi^B\overline{\psi}^{C^\bullet}\sqrt{|g|}\,\omega = \tfrac{1}{2}u_0\mu\, h(\Psi, \Sigma(B)\Psi)\sqrt{|g|}\,\omega.\end{aligned}$$

This is just the *Pauli term* which appears in the standard Lagrangian of a particle with spin.

We shall denote by

$$\mathcal{L}\colon J_1\boldsymbol{W} \to \mathbb{L}^3 \otimes \wedge^4 T^*\boldsymbol{E}$$

the fibred morphism over $\boldsymbol{E}$ characterized by $\mathcal{L} \circ j_1\Psi = \mathcal{L}[\Psi]\ \forall\Psi$. Here (and everywhere) $J_1\boldsymbol{W} \to \boldsymbol{W}$ is the jet bundle of $\boldsymbol{W}$ with respect to the base space $\boldsymbol{E}$. The coordinate expression of $\mathcal{L}$ is obtained from that of $\mathcal{L}[\Psi]$ by replacing ψ^A with w^A and $\partial_\lambda\psi^A$ with w^A_λ. In order to write down the field equation for a section Ψ, it is convenient to express the Lagrangian as $\mathcal{L} := \ell\omega$, with $\omega := d^0 \wedge d^1 \wedge d^2 \wedge d^3$. We have

$$\begin{aligned}\ell = \tfrac{1}{2}\delta_{C^\bullet A}\sqrt{|g|}\Big(& i(\overline{w}^{C^\bullet} w^A_0 - w^A\overline{w}^{C^\bullet}_0) - u_0\frac{\hbar}{m}g^{jk}w^A_j\overline{w}^{C^\bullet}_k + \\ & + ig^{jk}a_k(w^A\overline{w}^{C^\bullet}_j - \overline{w}^{C^\bullet}w^A_j) + \chi w^A\overline{w}^{C^\bullet} + \\ & + \chi^p\sigma_p{}^A{}_B w^B\overline{w}^{C^\bullet} + i\chi^{pj}\sigma_p{}^A{}_B(w^B\overline{w}^{C^\bullet}_j - \overline{w}^{C^\bullet}w^B_j)\Big),\end{aligned}$$

where $\chi, \chi^p, \chi^{pj}\colon \boldsymbol{E} \to \mathbb{R}$ are defined as the following shorthands:

$$\begin{aligned}\chi &:= u^0\frac{m}{\hbar}(2a_0 - g^{jk}a_j a_k) - u_0\frac{\hbar}{8m}g^{jk}C_j{}^r{}_s C_k{}^s{}_r,\\ \chi^p &:= \tfrac{1}{2}\varepsilon_r{}^{sp}(C_0{}^r{}_s - g^{jk}a_k C_j{}^r{}_s),\\ \chi^{pj} &:= u_0\frac{\hbar}{4m}\varepsilon_r{}^{sp}g^{jk}C_k{}^r{}_s.\end{aligned}$$

Recalling that a jet bundle is affine, and since $\boldsymbol{W} \to \boldsymbol{E}$ is a vector bundle, we have the following identification:

$$V_{\boldsymbol{W}}J_1\boldsymbol{W} \equiv J_1\boldsymbol{W} \underset{\boldsymbol{W}}{\times} (T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{W}).$$

Then applying the vertical functor to the morphism $\mathcal{L}$, after a contraction with the spacetime volume form we obtain a map

$$*V_{\boldsymbol{W}}\mathcal{L}\colon J_1\boldsymbol{W} \to \mathbb{T}^* \otimes T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{W}^*,$$

where $\boldsymbol{W}^*$ is the *real dual* bundle of $\boldsymbol{W}$. The real part of the Hermitian metric h is a positive-defined metric on the fibres of $\boldsymbol{W}$, and allows us to trasform the above morphism into the *quantum momentum*

$$\mathfrak{p}\colon J_1\boldsymbol{W} \to \mathbb{T}^* \otimes T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{W},$$

which has the coordinate expression

$$\mathfrak{p}[\Psi] = u^0\Big(\psi^A\partial_0 - i\frac{\hbar}{m}g^{jk}\Big(u_0\partial_k - i\frac{m}{\hbar}a_k\Big)\psi^A\partial_j - \chi^{rj}\sigma_r{}^A{}_B\psi^B\partial_j\Big) \otimes b_A.$$

6.3. Generalized Pauli equation

The *generalized Pauli equation* for a section $\Psi\colon \boldsymbol{E} \to \boldsymbol{W}$ is defined to be the Euler–Lagrange equation

$$\mathcal{E}[\Psi] := \mathcal{E} \circ j_2\Psi = 0,$$

where

$$\mathcal{E}\colon J_2\boldsymbol{W} \to \mathbb{L}^3 \otimes \wedge^4 T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{W}^*$$

is the Euler–Lagrange operator [31, 9], which can be characterized, via contraction with the spacetime volume form, by a morphism

$$*\mathcal{E}\colon J_2\boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{W}^*,$$

whose coordinate expression is of the type

$$*\mathcal{E} = \mathcal{E}_A dw^A + \mathcal{E}_{A^\bullet} d\overline{w}^{A^\bullet}.$$

The components $\mathcal{E}_A$ and $\mathcal{E}_{A^\bullet}$, which are conjugate to each other, can be calculated from the standard Euler–Lagrange formula by treating formally w^A and $\overline{w}^{A^\bullet}$ as independent real coordinates. Moreover, through the real part of the Hermitian metric h we can transform $*\mathcal{E}$ into a morphism

$$*\mathcal{E}^{\#}\colon J_2\boldsymbol{W} \to \mathbb{T}^* \otimes \boldsymbol{W}.$$

This has the coordinate expression $*\mathcal{E}^{\#} = \mathcal{E}^C b_C$, where $\mathcal{E}^C := 2h^{A^\bullet C}\mathcal{E}_{A^\bullet}$. We obtain

LEMMA 6.1. *The components of the Euler–Lagrange operator of the quantum spin Lagrangian are given by*

$$\begin{aligned}\mathcal{E}^C &= 2iu^0 w_0^C - 2iu^0 g^{jk} a_k w_j^C + \frac{\hbar}{m}\cdot\frac{1}{\sqrt{|g|}}\partial_k(\sqrt{|g|}g^{jk})w_j^C + \frac{\hbar}{m}g^{jk}w_{jk}^C + \\ &+ u^0\left(\chi + \frac{i}{\sqrt{|g|}}\partial_0\sqrt{|g|} - \frac{i}{\sqrt{|g|}}\partial_k(\sqrt{|g|}g^{jk}a_j)\right)w^C + \\ &+ u^0\left(\chi^r - \frac{i}{\sqrt{|g|}}\partial_k(\sqrt{|g|}\chi^{rk})\right)\sigma_r{}^C{}_B w^B - 2iu^0\chi^{rj}\sigma_r{}^C{}_B w_j^B,\end{aligned}$$

where $\chi, \chi^r, \chi^{rj}\colon \boldsymbol{E} \to \mathbb{R}$ *are the functions defined in Section* 6.2.

Note how the above expression for $\mathcal{E}^C$ splits into the sum of a non-interaction part and an interaction part. The interaction part consists of all those terms which contain the sigma's (last line). The non-interaction part is identical to the Euler–Lagrange operator without spin for each component of Ψ, plus the new term $-(\hbar/8m)g^{jk}C_j{}^r{}_s C_k{}^s{}_r \psi^C$ (contained in χ).

Next we would like to write the generalized Pauli equation in a more compact way. We shall accomplish this by defining two observer-dependent differential operators D^o and $\check{\Delta}^o$, which are immediate generalizations of the analogous operators defined in [18].

Recall that the connection $Ч^W := Ч \otimes Б$ is a map

$$Ч^W\colon J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{W} \to T^*\boldsymbol{E} \underset{\boldsymbol{W}}{\otimes} T\boldsymbol{W}.$$

Given an observer $o\colon \boldsymbol{E} \to J_1\boldsymbol{E}$, consider its natural jet prolongation $jo\colon J_1\boldsymbol{E} \to J_1J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$, given by $jo = u^0 \otimes \partial_0$ in adapted coordinates. Consider the map

$$\widetilde{o} := (jo \lrcorner Ч^W) \circ o\colon \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{W},$$

or, in adapted coordinates:

$$\widetilde{o} = u^0 \otimes [\partial_0 + i(\text{Ч}_0 w^A + \text{Б}_0{}^A{}_B w^B) b_A].$$

Recalling Subsection 2.1.5 we set

$$D^o\Psi := \left\langle \frac{1}{\sqrt{v}}, L_{\tilde{o}}(\Psi \otimes \sqrt{v}) \right\rangle: \boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{W}$$

with the coordinate expression

$$D^o\Psi = u^0\left(\partial_0\psi^A + \frac{\partial_0\sqrt{|g|}}{2\sqrt{|g|}}\psi^A - iu^0\frac{m}{\hbar}a_0\psi^A - i\text{Б}_0{}^A{}_B\psi^B\right)b_A.$$

The observer-dependent vertical covariant derivative of Ψ is defined to be

$$\check{\nabla}^o\Psi := \check{\nabla}\Psi \circ o: \boldsymbol{E} \to V^*\boldsymbol{E} \otimes \boldsymbol{W}.$$

In a coordinate chart adapted to the observer ($y_0^j \circ o = 0$), this derivative has the expression

$$\check{\nabla}^o\Psi = \left(\delta^A_B\partial_j - iu^0\frac{m}{\hbar}\delta^A_B a_j - i\text{Б}_j{}^A{}_B\right)\psi^B d^j \otimes b_A.$$

Then one defines the observer-dependent vertical Laplacian as

$$\check{\Delta}^o\Psi := \langle g^{\#}, \check{\nabla}^o\check{\nabla}^o\Psi\rangle: \boldsymbol{E} \to \boldsymbol{W},$$

with the coordinate expression

$$(\check{\Delta}^o\Psi)^A = g^{jk}\left(\delta^A_B\left(\partial_j - iu^0\frac{m}{\hbar}a_j\right) - i\text{Б}_j{}^A{}_B\right)\left(\delta^B_C\left(\partial_k - iu^0\frac{m}{\hbar}a_k\right) - i\text{Б}_k{}^B{}_C\right)\psi^C.$$

Then, taking into account the identity $g^{jk}\Gamma_k{}^h{}_j = \frac{1}{\sqrt{|g|}}\partial_j(g^{hj}\sqrt{|g|})$, after some calculations one proves

PROPOSITION 6.3. *The Euler–Lagrange operator can be written as*

$$*\mathcal{E}^{\#}[\Psi] = 2\left(iD^o\Psi + \frac{\hbar}{2m}\check{\Delta}^o\Psi\right). \qquad \diamond$$

Another formulation of the generalized Pauli equation can be obtained by introducing the differential $d[\text{Ч}^{\mathrm{S}}]$ associated with the connection Ч^{S} via the Frölicher–Nijenhuis bracket [30], and the related divergence-type operator $\operatorname{div}[\text{Ч}^{\mathrm{S}}]$ defined through the spacetime volume form v.

PROPOSITION 6.4. *If* Ψ: $\boldsymbol{E} \to \boldsymbol{W}$ *is any quantum spin history, then* $*\mathcal{E}^{\#}[\Psi]$ *is the unique linear combination (up to a scalar factor) of* $\mathring{\nabla}[\Psi]$ *and* $\operatorname{div}[\text{Ч}^{\mathrm{S}}]\mathfrak{p}[\Psi]$ *which projects over E. Namely*

$$*\mathcal{E}^{\#}[\Psi] = i(\mathring{\nabla}[\Psi] + \operatorname{div}[\text{Ч}^{\mathrm{S}}]\mathfrak{p}[\Psi]): \boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{W}. \qquad \diamond$$

In particular, let us write down the field equation in the flat case. By setting $\widetilde{\Gamma}\natural^{r}_{\lambda s} = 0$ and $|g| := \det(g) = 1$ (i.e. by using orthonormal Cartesian coordinates), since ma_λ reduces to the electromagnetic potential A_λ, we obtain the familiar Pauli equation

$$i\hbar\partial_0\psi^C = u_0\frac{1}{2m}g^{jk}(-i\hbar\partial_j - u^0A_j)(-i\hbar\partial_k - u^0A_k)\psi^C - u^0A_0\psi^C + \tfrac{1}{2}u_0\mu\hbar\widetilde{B}^r\sigma_r{}^C{}_B\psi^B.$$

For an electron $\mu = -e/m$ ($G = 2$), thus the last term equals $-(e\hbar/2m)\Sigma(B)\Psi$.

Next we focus our attention on quantum densities $\Psi^\eta := \Psi\otimes\sqrt{\eta}$, whose coordinate expression will be written as

$$\Psi^\eta = \psi^{\eta A}b_A\otimes\sqrt{\omega}, \qquad \psi^{\eta A} := \sqrt[4]{|g|}\psi^A.$$

The Euler–Lagrange operator yields the *Pauli operator*

$$\mathfrak{P}(\Psi^\eta) := -\tfrac{i}{2}*\mathcal{E}^{\#}[\Psi]\otimes\sqrt{\eta},$$

which is the analogous, for the spin case, of the *Schrödinger operator* introduced in Section 3.2. We obtain

$$\mathfrak{P}(\Psi^\eta) = u^0\left(\partial_0\psi^{\eta A} - iu_0\frac{\hbar}{2m}\check{\Delta}^o\Psi^{\eta A} - iu^0\frac{m}{\hbar}a_0\psi^{\eta A} - i\text{Б}_0{}^A{}_B\psi^{\eta B}\right)b_A\otimes\sqrt{\check{\omega}_0}.$$

One then sees that Ψ satisfies the generalized Pauli equation $\mathcal{E}[\Psi] = 0$ iff Ψ^η satisfies the equation $\mathfrak{P}(\Psi^\eta) = 0$, that is:

$$i\partial_0\psi^{\eta A} = -u_0\frac{\hbar}{2m}\check{\Delta}^o\Psi^{\eta A} - u^0\frac{m}{\hbar}a_0\psi^{\eta A} - \text{Б}_0{}^A{}_B\psi^{\eta B}.$$

6.4. Symmetries

We recall [9, 31] that the Nöther theorem can be expressed in geometric form through the Poincaré–Cartan form Θ. The Poincaré–Cartan form of the Lagrangian $\mathcal{L}$ can be calculated, similarly to the Euler–Lagrange form, by treating (w^A) and $(\overline{w}^{A^\bullet})$ as formally independent coordinates. We obtain

$$\begin{aligned}\Theta = \frac{\sqrt{|g|}}{2}h_{C^\bullet A}\Big[&i(\overline{w}^{C^\bullet}dw^A - w^Ad\overline{w}^{C^\bullet})\wedge\omega_0 + \Big(u_0\frac{\hbar}{m}g^{jk}(w^A_kd\overline{w}^{C^\bullet} - \overline{w}^{C^\bullet}_kdw^A) + \\ &+ig^{jk}a_k(w^Ad\overline{w}^{C^\bullet} - \overline{w}^{C^\bullet}dw^A) + i\chi^{rj}\sigma_r{}^A{}_B(w^Bd\overline{w}^{C^\bullet} - \overline{w}^{C^\bullet}dw^B)\Big)\wedge\omega_j + \\ &+\Big(u_0\frac{\hbar}{m}g^{jk}w^A_j\overline{w}^{C^\bullet}_k + \chi w^A\overline{w}^{C^\bullet} + \chi^r\sigma_r{}^A{}_Bw^B\overline{w}^{C^\bullet}\Big)\omega\Big],\end{aligned}$$

where $\omega_\lambda := \partial_\lambda \lrcorner\, \omega$.

Consider the natural action of the group $U(1)$ on $\boldsymbol{W}$ given by

$$\mathbb{R}\times\boldsymbol{W}\to\boldsymbol{W}\colon (\phi,\zeta)\mapsto e^{-i\phi}\zeta.$$

This action can be naturally prolonged to actions on $T\boldsymbol{W}$ and $J_1\boldsymbol{W}$. We then have two one-parameter groups generated, respectively, by the vector fields $\underline{v}\colon \boldsymbol{W}\to T\boldsymbol{W}$ and

$v\colon J_1\boldsymbol{W} \to TJ_1\boldsymbol{W}$, whose coordinate expressions are:

$$\underline{v} = -iw^A\partial w_A, \qquad v = -i(w^A\partial w_A + w^A_\lambda \partial w^\lambda_A).$$

Moreover, v is the natural prolongation of $\underline{v}$ [30]. It is immediate to check that the Lagrangian $\mathcal{L}$ is invariant with respect to the natural action of the group $U(1)$. We have

$$L_v\mathcal{L} = L_v\Theta = 0,$$

thus for each critical section Ψ we have the conserved probability current

$$(j\Psi)^*(\underline{v} \lrcorner \Theta) = \sqrt{|g|}h_{C^\bullet A}\Big[\overline{\psi}^{C^\bullet}\psi^A\omega_0 + \Big(iu_0\frac{\hbar}{2m}g^{jk}(\overline{\psi}^{C^\bullet}\psi^A_k - \overline{\psi}^{C^\bullet}_k\psi^A) - -g^{jk}a_k\overline{\psi}^{C^\bullet}\psi^A - \chi^{rj}\overline{\psi}^{C^\bullet}\sigma_r{}^A{}_B\psi^B\Big)\omega_j\Big].$$

The corresponding conserved quantity is the ω_0 component, i.e. the probability density $h(\Psi,\Psi)\eta$.

We have a larger simmetry in the case of flat spacetime and vanishing electro-magnetic field (set $C^{\;r}_{\lambda\;s} = 0$ and $\widetilde{F}^r{}_s = 0$). In this case the Lagrangian is invariant with respect to the action of the group $SU(2)$ given by

$$SU(2)\times \boldsymbol{W} \to \boldsymbol{W}\colon\ (P,u) \mapsto P^A{}_B u^B \partial w_A,$$

and its jet prolongation. In particular we have, for $r = 1,2,3$ and $\phi \in \mathbb{R}$, the actions of $\exp(\frac{i\phi}{2}\sigma_r)$ which yields the vector fields $\underline{v}_r$ (on $\boldsymbol{W}$) and their jet prolongation v_r whose coordinate expressions are:[15]

$$\underline{v}_r = \tfrac{i}{2}\sigma_r{}^A{}_B w^B\partial w_A, \qquad v_r = \tfrac{i}{2}\sigma_r{}^A{}_B(w^B\partial w_A + w^B_\lambda\partial w^\lambda_A).$$

The related conserved current is

$$(j\Psi)^*(\underline{v}_r \lrcorner \Theta) = \sqrt{|g|}h_{C^\bullet A}\,\sigma_r{}^A{}_B\Big[-\overline{\psi}^{C^\bullet}\psi^B\omega_0 + +g^{jk}\Big(iu_0\frac{\hbar}{2m}(\overline{\psi}^{C^\bullet}\psi^B_k - \overline{\psi}^{C^\bullet}_k\psi^B) + a_k\overline{\psi}^{C^\bullet}\psi^B\Big)\omega_j\Big].$$

The conserved quantity is, up to integration, the expectation value of spin, that is $h(\Psi,\Sigma(\Psi))\eta$.

7. Quantum operators

We shall construct the algebra of quantum operators by a procedure which generalizes that used in the scalar case, and is divided into analogous steps. Starting from the algebra $\mathcal{A}$ of all quantizable functions, we first construct the algebra $\mathcal{W}$ of quantum vector fields $\boldsymbol{W} \to T\boldsymbol{W}$, then the algebra $\mathcal{O}$ of almost-quantum operators acting on quantum densities and, finally, the algebra $\widehat{\mathcal{O}}$ of quantum operators on

[15]This action depends on the considered basis. However, the Lie algebra generated by the fields $\underline{v}_r$ is independent of the basis.

the Hilbert bundle. At each step we put together 'phase' objects, coming from the Lie algebra $\mathcal{A}^{\mathrm{P}}$ of quantizable functions on the phase space $J_1\boldsymbol{E}$, and 'spin' objects, coming from the Lie algebra $\mathcal{A}^{\mathrm{S}}$ of quantizable spin functions on $\mathbb{L}^* \otimes V\boldsymbol{E}$.

7.1. Quantum phase vector fields

In this section we examine the natural prolongation of quantum phase vector fields on $\boldsymbol{Q}$ to vector fields on $\boldsymbol{W}$.

LEMMA 7.1. *There is a natural construction which, for each Hermitian linear vector field $Y\colon \boldsymbol{Q} \to T\boldsymbol{Q}$ projectable over a vector field $X\colon \boldsymbol{E} \to T\boldsymbol{E}$, yields a Hermitian linear vector field*

$$Y^{\boldsymbol{W}}\colon \boldsymbol{W} \to T\boldsymbol{W}$$

projectable over X. Let $Y = X^\lambda \partial_\lambda + iY^z z\partial z$, with $X^\lambda, Y^z\colon \boldsymbol{E} \to \mathbb{R}$, be the coordinate expression of Y. Then the coordinate expression of $Y^{\boldsymbol{W}}$ is

$$Y^{\boldsymbol{W}} = X^\lambda \partial_\lambda + i(X^\lambda \text{Б}_{\lambda}{}^{A}{}_{B} w^B + Y^z w^A)\partial w_A.$$

Proof: Consider the horizontal lift of X by Б, i.e. the vector field

$$X \lrcorner \text{Б} = X^\lambda \partial_\lambda + iX^\lambda \text{Б}_{\lambda}{}^{A}{}_{B} z^B \partial w_A\colon \boldsymbol{E} \to T\boldsymbol{S},$$

which is also Hermitian and projectable over X. Then we have the tensor product

$$Y \otimes (X \lrcorner \text{Б})\colon \boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{S} \to T\boldsymbol{Q} \underset{T\boldsymbol{E}}{\otimes} T\boldsymbol{S}.$$

Now the universal property of the fibred tensor product over $T\boldsymbol{E}$ yields a linear fibred morphism

$$\theta\colon T\boldsymbol{Q} \underset{T\boldsymbol{E}}{\otimes} T\boldsymbol{S} \to T(\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{S}) := T\boldsymbol{W}$$

over $T\boldsymbol{E}$, with the coordinate expression

$$(w^A, \dot{w}^A) \circ \theta = (z \cdot z^A, z \cdot \dot{z}^A + \dot{z} \cdot z^A).$$

Thus by setting

$$Y^{\boldsymbol{W}} := \theta \circ (Y \otimes (X \lrcorner \text{Б}))$$

we obtain the claimed result. ∎

We shall denote by $\mathcal{W}$ the space of all Hermitian linear projectable vector fields on $\boldsymbol{W}$. Clearly $\mathcal{W}$ is an $\mathcal{F}\boldsymbol{E}$-modulus, and an $\mathbb{R}$-Lie algebra with respect to the standard bracket. From the above lemma we see that the map $\mathcal{Q} \to \mathcal{W}\colon Y \mapsto Y^{\boldsymbol{W}}$ is an $\mathcal{F}\boldsymbol{E}$-linear isomorphism. In general, this is not an isomorphism of Lie algebras; namely, by direct calculation one shows the following

LEMMA 7.2. *If $Y_1, Y_2\colon \boldsymbol{Q} \to T\boldsymbol{Q}$ are both projectable, linear and Hermitian, then also their Lie bracket is such, and we have*

$$[Y_1^{\boldsymbol{W}}, Y_2^{\boldsymbol{W}}] = [Y_1, Y_2]^{\boldsymbol{W}} + R^{\boldsymbol{W}}[\text{Б}](X_1, X_2),$$

where

$$R^{W}[\text{Б}]\colon \boldsymbol{E} \to \wedge^2 T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{W} \underset{\boldsymbol{W}}{\otimes} V^*\boldsymbol{W}$$

is obtained from

$$R[\text{Б}]\colon \boldsymbol{E} \to \wedge^2 T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{S} \underset{\boldsymbol{S}}{\otimes} V^*\boldsymbol{S}$$

by tensor product with the identity form $\boldsymbol{E} \to V\boldsymbol{Q} \otimes_{\boldsymbol{Q}} V^*\boldsymbol{Q}$ *(in these formulae all vertical spaces are taken with respect to the base space* $\boldsymbol{E}$*). In coordinates we have*

$$\begin{aligned}[][Y_1^W, Y_2^W] = {} & (X_1^\mu \partial_\mu X_2^\lambda - X_2^\mu \partial_\mu X_1^\lambda)\partial_\lambda + i(X_1^\mu \partial_\mu X_2^\lambda - X_2^\mu \partial_\mu X_1^\lambda)\text{Б}_\lambda{}^A{}_B w^B \partial w_A + \\ & + i(X_1^\lambda \partial_\lambda Y_2^z - X_2^\lambda \partial_\lambda Y_1^z) w^A \partial w_A + R[\text{Б}]_{\lambda\mu}{}^A{}_B X_1^\lambda X_2^\mu w^B \partial w_A. \end{aligned} \qquad \diamond$$

If $f\colon J_1\boldsymbol{E} \to \mathbb{R}$ is a quantizable phase function (Section 3.3), then the quantum vector field $Y[f]\colon \boldsymbol{Q} \to T\boldsymbol{Q}$ yields a vector field $Z[f] := Y^{W}[f]\colon \boldsymbol{W} \to T\boldsymbol{W}$, which we still call the *quantum phase vector field* corresponding to f, or the *quantum lift* of f. Its coordinate expression is

$$\begin{aligned} Z[f] = {} & u^0 f'' \partial_0 - u_0 \frac{\hbar}{m} f^j \partial_j + \\ & + i\bigg(\Big(u^{00}\frac{m}{\hbar} f'' a_0 - f^j a_j + f_\circ\Big) w^A + \Big(u^0 f'' \text{Б}_0{}^A{}_B - u_0 \frac{\hbar}{m} f^j \text{Б}_j{}^A{}_B\Big) w^B\bigg)\partial w_A. \end{aligned}$$

Remark 7.4: The quantum phase vector field $Z[f]$ can be recovered also by a procedure similar to that used in the scalar case. In fact, the 'upper' vector field

$$f^\# \lrcorner \text{Ч}^W + i f\,\text{и}\colon \boldsymbol{W}^\uparrow \to T\boldsymbol{W}^\uparrow,$$

where $\text{и}\colon \boldsymbol{W}^\uparrow \to V\boldsymbol{W}^\uparrow$ is the Liouville vector field, turns out to be projectable exactly over $Z[f]$. •

The quantum lift $\mathcal{A}^{\mathrm{P}} \to \mathcal{W}\colon f \mapsto Z[f]$ is an $\mathcal{F}\boldsymbol{E}$-linear monomorphism. In general, however, it is not an $\mathbb{R}$-Lie algebra isomorphism. In fact from Lemma 7.2 we obtain

PROPOSITION 7.1. *Let* $f_1, f_2\colon J_1\boldsymbol{E} \to \mathbb{R}$ *be quantizable phase functions. Then we have*

$$[Z[f_1], Z[f_2]] = Z[[f_1, f_2]] + R^{W}[\text{Б}](X[f_1], X[f_2]). \qquad \diamond$$

7.2. Quantum spin vector fields

We can naturally associate a quantum vector field with each quantizable spin function. Namely, for any $\phi^{\mathrm{Q}} + \phi^{\mathrm{L}} \in \mathcal{A}^{\mathrm{S}} := \mathcal{A}^{\mathrm{SQ}} \oplus \mathcal{A}^{\mathrm{SL}}$ (Section 4.2) we consider the section

$$\widetilde{\phi} := \tfrac{1}{4}\Sigma^2(X[\phi^{\mathrm{Q}}]) + \tfrac{1}{2}\Sigma(X[\phi^{\mathrm{L}}])\colon \boldsymbol{E} \to \boldsymbol{H},$$

which has the coordinate expression

$$\widetilde{\phi} = \tfrac{3}{4}\phi''\sigma_0 + \tfrac{1}{2}\phi^r\sigma_r.$$

Now we observe that $\widetilde{\phi}$ can be regarded as a linear fibred morphism $\widetilde{\phi}\colon \boldsymbol{S} \to \boldsymbol{S}$ over $\boldsymbol{E}$. Hence, by tensorializing it with $\mathbf{1}_{\boldsymbol{Q}}\colon \boldsymbol{Q} \to \boldsymbol{Q}$, we obtain the linear fibred morphism

$\mathbf{1}_Q \otimes \tilde{\phi}$: $\boldsymbol{W} \to \boldsymbol{W}$ over $\boldsymbol{E}$. Finally we recall that $V\boldsymbol{W} \equiv \boldsymbol{W} \times_{\boldsymbol{E}} \boldsymbol{W}$ (as $\boldsymbol{W} \to \boldsymbol{E}$ is a vector bundle, see Subsection 2.1.5), and define the *quantum spin vector field* corresponding to ϕ, or the *quantum lift* of ϕ, to be the Hermitian vertical vector field

$$Z[\phi]\colon \boldsymbol{W} \to V\boldsymbol{W}\colon w \mapsto (w, i(\mathbf{1}_Q \otimes \tilde{\phi})(w)),$$

whose coordinate expression is

$$Z[\phi] = i(\tfrac{3}{4}\phi'' \sigma_0{}^A{}_B + \tfrac{1}{2}\phi^r \sigma_r{}^A{}_B) w^B \partial w_A.$$

The map $\phi \mapsto Z[\phi]$ is an $\mathcal{F}E$-linear isomorphism and an $\mathbb{R}$-Lie algebra isomorphism from $\mathcal{A}^{\mathrm{S}}$ to the space $\mathcal{V}\mathcal{W}$ of all *vertical* (Hermitian) vector fields of $\mathcal{W}$. Moreover, this isomorphism associates the subalgebra $\mathcal{A}^{\mathrm{SL}} \subset \mathcal{A}^{\mathrm{S}}$ with the subalgebra $\mathcal{V}_0\mathcal{W}$ of traceless vector fields, and the Abelian ideal $\mathcal{A}^{\mathrm{SQ}} \subset \mathcal{A}^{\mathrm{S}}$ with the Abelian ideal $\mathcal{V}_1\mathcal{W}$ generated by $\mathbf{1}_{\boldsymbol{W}}$. In fact, let $\phi, \theta \in \mathcal{A}^{\mathrm{S}}$; then by a straightforward calculation one finds

$$[Z[\phi], Z[\theta]] = Z[[\phi, \theta]] = Z[[\phi^{\mathrm{L}}, \theta^{\mathrm{L}}]],$$

or, in orthonormal coordinates,

$$[Z[\phi], Z[\theta]] = \tfrac{i}{2}\phi_r \theta_s \varepsilon^{prs} \sigma_p{}^A{}_B w^B \partial w_A.$$

7.3. Quantum vector fields

In previous sections we defined quantum lifts of phase and spin quantizable functions. Now, the direct sum of these lifts yields the *quantum lift of quantizable functions*:

$$Z\colon \mathcal{A} := \mathcal{A}^{\mathrm{P}} \oplus \mathcal{A}^{\mathrm{S}} \to \mathcal{W}\colon f + \phi \mapsto Z[f + \phi] := Z[f] + Z[\phi],$$

with the coordinate expression

$$Z[f + \phi] = u^0 f'' \partial_0 - u_0 \frac{\hbar}{m} f^j \partial_j + i\Bigg(\Big(u^{00}\frac{m}{\hbar} f'' a_0 - f^j a_j + f_\circ + \tfrac{3}{4}\phi''\Big) w^A +$$
$$+ \Big(u^0 f'' \mathrm{B}_0{}^A{}_B - u_0 \frac{\hbar}{m} f^j \mathrm{B}_j{}^A{}_B + \tfrac{1}{2}\phi^r \sigma_r{}^A{}_B\Big) w^B\Bigg)\partial w_A.$$

From the above formula we see that the map $Z\colon \mathcal{A}^{\mathrm{P}} \oplus \mathcal{A}^{\mathrm{SL}} \to \mathcal{W}$ is an $\mathcal{F}E$-linear isomorphism; the map $Z\colon \mathcal{A} \to \mathcal{W}$ is an $\mathcal{F}E$-linear epimorphism whose kernel is constituted by quantizable functions $f + \phi \in \mathcal{F}E \oplus \mathcal{A}^{\mathrm{SQ}}$ such that $f'' = -\frac{3}{4}\phi''$.

By a straightforward calculation we get

LEMMA 7.3. *Let $Y\colon \boldsymbol{Q} \to T\boldsymbol{Q}$ be any linear vector field projectable over $X\colon \boldsymbol{E} \to T\boldsymbol{E}$. Let $\phi = \phi^{\mathrm{Q}} + \phi^{\mathrm{L}} \in \mathcal{A}^{\mathrm{S}} := \mathcal{A}^{\mathrm{SQ}} \oplus \mathcal{A}^{\mathrm{SL}}$. Then*

$$[Y^{\boldsymbol{W}}, Z[\phi]] = Z[\nabla[C]_X \phi],$$

or, in coordinates,

$$[Y^{\boldsymbol{W}}, Z[\phi]] = \tfrac{i}{2} X^\lambda (\partial_\lambda \phi^r - \phi^s C_\lambda{}^r{}_s) \sigma_r{}^A{}_B w^B \partial w_A.$$

Hence, the behaviour of the quantum lift Z with respect to the algebra structures of $\mathcal{A}$ and $\mathcal{W}$ can be summarized as follows.

THEOREM 7.1. *Let* $f_1, f_2 \in \mathcal{A}^{\mathrm{P}}$, $\phi_1, \phi_2 \in \mathcal{A}^{\mathrm{S}}$. *Then*

$$[Z[f_1], Z[f_2]] = Z[[f_1, f_2]] + R^{W}[\text{Б}](X[f_1], X[f_2]),$$
$$[Z[\phi_1], Z[\phi_2]] = Z[[\phi_1, \phi_2]],$$
$$[Z[f_1], Z[\phi_1]] = Z[[f_1, \phi_1]]. \qquad \diamond$$

So we see that, if the curvature of C vanishes, then $\mathcal{A}$ is an $\mathbb{R}$-Lie algebra and the quantum lift is a morphism of Lie algebras.

7.4. Almost-quantum operators

Next we pass from quantum vector fields to operators. Like in the scalar case, there is a natural way of applying the quantum vector field Z to a quantum section with spin Ψ (see also Subsection 2.1.5); we obtain

$$Z.\Psi = (X^{\lambda}\partial_{\lambda}\psi^{A} - i(X^{\lambda}\text{Б}_{\lambda}{}^{A}{}_{B}\psi^{B} + Y^{z}\psi^{A}))b_{A}.$$

The corresponding operator which acts on quantum densities

$$\Psi^{\eta} := \Psi \otimes \sqrt{\eta}\colon \boldsymbol{E} \to \boldsymbol{W}^{\eta} := \mathbb{L}^{3/2} \otimes \boldsymbol{W} \otimes \sqrt{\wedge^{3} V^{*}\boldsymbol{E}},$$

is defined by[16]

$$\mathcal{Z}(\Psi \otimes \sqrt{\eta}) := i(Z.(\Psi \otimes \sqrt{\upsilon})) \otimes \frac{1}{\sqrt{\upsilon}} \otimes \sqrt{\eta},$$

and called an *almost-quantum operator*. Then we have

$$\mathcal{Z}(\Psi \otimes \sqrt{\eta}) = i((Z.\Psi) + \tfrac{1}{2}(\operatorname{div} X)\Psi) \otimes \sqrt{\eta}.$$

We shall denote by $\mathcal{O}$ the space of almost-quantum operators, and define the *almost-quantum operator lift* to be the composition

$$\mathcal{A} \to \mathcal{W} \to \mathcal{O}\colon\ f + \phi \mapsto Z[f] \mapsto \mathcal{Z}[f + \phi],$$

which is an $\mathcal{F}\boldsymbol{E}$-linear morphism.

We define the bracket of any two (local) Hermitian operators $\mathcal{Z}_1$ and $\mathcal{Z}_2$ to be the (local) Hermitian operator

$$[\mathcal{Z}_1, \mathcal{Z}_2] := -i\,[\![\mathcal{Z}_1, \mathcal{Z}_2]\!],$$

where $[\![\mathcal{Z}_1, \mathcal{Z}_2]\!] := (\mathcal{Z}_1 \circ \mathcal{Z}_2 - \mathcal{Z}_2 \circ \mathcal{Z}_1)$ is the commutator of $\mathcal{Z}_1$ and $\mathcal{Z}_2$. Then, by a straighforward calculation, recalling Proposition 7.1, we obtain the following result:

THEOREM 7.2. *The brackets of the almost-quantum operators corresponding to the quantizable phase functions* $f_1, f_2 \in \mathcal{A}^{\mathrm{P}}$ *and to the quantizable spin functions* ϕ_1, ϕ_2

[16]The reason for multiplying by i is that we want Hermitian operators, while Hermitian vector fields give rise to *anti*-Hermitian operators. The reason for the 'odd' multiplication and division by υ is that Z does not act naturally on the spacelike object η, but acts naturally on the spacetime object υ. We guess that this point might be formulated in a more satisfactory way within a fully Einsteinian approach.

$\in \mathcal{A}^{\mathrm{S}}$ *are given by*

$$[\mathcal{Z}[f_1], \mathcal{Z}[f_2]](\Psi^\eta) = \mathcal{Z}[[f_1, f_2]](\Psi^\eta) + iR^W[\mathrm{B}](X[f_1], X[f_2]).\Psi^\eta,$$
$$[\mathcal{Z}[\phi_1], \mathcal{Z}[\phi_2]](\Psi^\eta) = \mathcal{Z}[[\phi_1, \phi_2]](\Psi^\eta),$$
$$[\mathcal{Z}[f_1], \mathcal{Z}[\phi_1]](\Psi^\eta) = \mathcal{Z}[[f_1, \phi_1]](\Psi^\eta). \qquad \diamond$$

Note that the bracket of the almost-quantum operators corresponding to the quantizable phase functions f_1 and f_2 (first formula in the above theorem) has a term of the spin type, corresponding to the linear quantizable spin function $\phi \in \mathcal{A}^{\mathrm{SL}}$ whose components are given by

$$\phi^p = \tfrac{1}{2}\varepsilon_r{}^{sp} R[C]_{\lambda\mu}{}^r{}_s X[f_1]^\lambda X[f_2]^\mu.$$

7.5. Quantum operators on the Hilbert bundle

So far, quantum theory has been developed on a finite-dimensional bundle $\boldsymbol{W}^\eta \to \boldsymbol{E}$ over the spacetime. Now, we sketch how to introduce in a natural way an infinite-dimensional Hilbert bundle $H\boldsymbol{W}^\eta \to \boldsymbol{T}$ over time and obtain Hilbert operators from almost-quantum operators. Essentially, the construction is the same in the scalar and spin cases (we just replace $\boldsymbol{W}^\eta$ for $\boldsymbol{Q}^\eta$).

We focus our attention on the double fibred manifold $\boldsymbol{W}^\eta \to \boldsymbol{E} \to \boldsymbol{T}$. Each (smooth) local *tube* section $\Psi^\eta\colon \boldsymbol{E} \to \boldsymbol{W}^\eta$ (i.e. each section which is defined on a 'tubelike' open set of $\boldsymbol{E}$) yields, for any given $\tau \in \boldsymbol{T}$, a (smooth) section $\Psi^\eta_\tau\colon \boldsymbol{E}_\tau \to \boldsymbol{W}^\eta_\tau$. Next we consider the fibred set $\mathcal{S}\boldsymbol{W}^\eta \to \boldsymbol{T}$, where the fibre $\mathcal{S}\boldsymbol{W}^\eta_\tau$, $\tau \in \boldsymbol{T}$, is defined to be the set of all (smooth) sections $\widehat{\Psi}^\eta_\tau\colon \boldsymbol{E}_\tau \to \boldsymbol{W}^\eta_\tau$. Then clearly we have a natural injection $\Psi^\eta \mapsto \widehat{\Psi}^\eta$ from all (smooth) tube sections $\Psi^\eta\colon \boldsymbol{E} \to \boldsymbol{W}^\eta$ to all sections $\widehat{\Psi}^\eta\colon \boldsymbol{T} \to \mathcal{S}\boldsymbol{W}^\eta$.

In order to study geometrically the fibred set $\mathcal{S}\boldsymbol{W}^\eta \to \boldsymbol{T}$, one could use the standard methods of infinite-dimensional manifolds. But we can skip this unnecessary hard machinery and achieve our goal in a much simpler way by using the concept of *smoothness* due to Frölicher (see [7, 18]). Accordingly, a section $\widehat{\Psi}^\eta\colon \boldsymbol{T} \to \mathcal{S}\boldsymbol{W}^\eta$ is smooth iff it corresponds to a smooth section $\Psi^\eta\colon \boldsymbol{E} \to \boldsymbol{W}^\eta$.

We can repeat the above construction for any subsheaf of tube sections of the double fibred manifold $\boldsymbol{W}^\eta \to \boldsymbol{E} \to \boldsymbol{T}$, and obtain a fibred subset of $\mathcal{S}\boldsymbol{W}^\eta \to \boldsymbol{T}$; it is remarkable that this inclusion preserves smoothness automatically. In particular, we consider the fibred space $H'\boldsymbol{W}^\eta \to \boldsymbol{T}$ associated with (smooth) tube sections $\Psi^\eta\colon \boldsymbol{E} \to \boldsymbol{W}^\eta$ with compact support. The fibres of $H'\boldsymbol{W}^\eta$ are naturally endowed with a smooth pre-Hilbert structure. Namely we define, $\forall \tau \in \boldsymbol{T}$, a (non-complete) scalar product on $H'\boldsymbol{W}^\eta_\tau$ by

$$\langle \widehat{\Psi}^\eta_\tau \mid \widehat{\Psi}'^\eta_\tau \rangle = \int h_\tau(\Psi_\tau, \Psi'_\tau)\eta_\tau.$$

Our next goal is to obtain a pre-Hilbert bundle operator from each almost-quantum operator $\mathcal{Z}$. Let us consider a quantizable function $f + \phi \in \mathcal{A}$ and the associated almost-quantum operator $\mathcal{Z}[f + \phi]$. If $f'' = 0$, then $\mathcal{Z}[f + \phi]$, which acts on smooth sections $\widehat{\Psi}^\eta\colon \boldsymbol{T} \to \mathcal{S}\boldsymbol{W}^\eta$ only through vertical derivatives and multiplication by scalar functions, can be regarded as a linear fibred automorphism of the pre-Hilbert bundle over $\boldsymbol{T}$. In other words, $\mathcal{Z}[f + \phi]$ can be regarded as a pre-Hilbert operator. On the contrary, if $f'' \neq 0$, then the expression of $\mathcal{Z}[f + \phi](\Psi^\eta)$ contains the time derivative of Ψ^η. This means that $\mathcal{Z}[f + \phi]$ cannot be regarded as a pre-Hilbert operator. However, we can solve this problem by 'eliminating' the time derivative in the following natural and general way.

Consider the Pauli operator $\mathfrak{P}$ (Section 6.3) acting on quantum densities (its kernel is constituted by the solutions of the generalized Pauli equation).[17] Then for any $f + \phi \in \mathcal{A}$ we consider a linear fibred automorphism of the pre-Hilbert bundle over $\boldsymbol{T}$:

$$\widehat{f + \phi} = \mathcal{Z}[f + \phi] - if'' \lrcorner \mathfrak{P},$$

and call it the *pre-Hilbert quantum operator* associated with $f + \phi$. In particular, if $f'' = 0$ (this is equivalent to $Z[f + \phi]$ being a vertical field), then $\widehat{f + \phi} = \mathcal{Z}[f + \phi]$.

Let $\widehat{\mathcal{O}}$ be the set of all Hermitian linear fibred automorphisms of the pre-Hilbert bundle over $\boldsymbol{T}$. Then the map

$$\mathcal{A} \to \widehat{\mathcal{O}}\colon\ f + \phi \mapsto \widehat{f + \phi}$$

is our *correspondence principle.*

THEOREM 7.3. *Let $f + \phi \in \mathcal{A}$ be a quantizable function such that $f'' =$ constant. Then the corresponding quantum pre-Hilbert operator $\widehat{f + \phi}$ is symmetric, i.e.*

$$\langle \widehat{\Psi}^\eta_\tau \mid \widehat{f + \phi}(\widehat{\Psi}'^\eta_\tau) \rangle = \langle \widehat{f + \phi}(\widehat{\Psi}^\eta_\tau) \mid \widehat{\Psi}'^\eta_\tau \rangle .$$

Proof: It follows from the symmetry of the observer-dependent spacelike Laplacian, from Gauss' theorem and from the fact that the coefficients of the quantum spin connection are Hermitian. ∎

Next we give the explicit expressions of the pre-Hilbert quantum operators corresponding to the physically most important quantizable functions. Consider first the coordinates x^λ and the classical momenta $p_j/\hbar$; these are quantizable phase functions $J_1\boldsymbol{E} \to \mathbb{R}$, whose quantum lifts are vertical-valued (for simplicity we assume that spacetime fibres admit global spacelike coordinates, and refer to such charts). We obtain

$$\widehat{x^\lambda}(\Psi^\eta) \equiv \mathcal{Z}[x^\lambda](\Psi^\eta) = x^\lambda \Psi^\eta,$$

$$\widehat{p_j/\hbar}(\Psi^\eta) \equiv \mathcal{Z}[p_j/\hbar](\Psi^\eta) = -(i\partial_j \psi^A + \text{Б}_j{}^A{}_B \psi^B) b_A \otimes \sqrt{\eta} = -i(\nabla_j[\text{Б}]\Psi) \otimes \sqrt{\eta}.$$

These formulae enable us to write the observer-dependent vertical Laplacian as the following generalization of a well-known formula:

$$\check{\Delta}^o \Psi^\eta = -g^{jk}\left(\widehat{p_j/\hbar} - u^0 \frac{m}{\hbar} a_j\right)\left(\widehat{p_k/\hbar} - u^0 \frac{m}{\hbar} a_k\right)\Psi^\eta .$$

[17] Incidentally, we observe that this operator can be nicely interpreted as a linear covariant differential on the infinite-dimensional pre-Hilbert bundle [18].

Let $\phi \in \mathcal{A}^{\mathrm{S}}$ be a quantizable spin function. Then

$$\widehat{\phi} \equiv \mathcal{Z}[\phi](\Psi^\eta) = (\tfrac{3}{4}\phi''\delta^A{}_B + \tfrac{1}{2}\phi^r\sigma_r{}^A{}_B)\psi^B b_A \otimes \sqrt{\eta}.$$

Remark 7.5: Through the metric g, any vector field v: $\boldsymbol{E} \to \mathbb{L}^* \otimes V\boldsymbol{E}$ can be identified with the quantizable spin function $v^\flat$. Then we can define the quantum spin vector field associated with v as $S[v] := Z[v^\flat] = \frac{i}{2}\Sigma(v) \otimes \mathbf{1}_Q$, and the corresponding quantum spin operator as $\widehat{S}[v](\Psi^\eta) := iS[v].\Psi^\eta$. On the other hand, the quadratic spin function associated with g ($\phi'' = 1$) yields the operator $\widehat{S}^2$, called the *square of spin*, given by

$$\widehat{S}^2 = \delta^{rs}\widehat{S}[e_r] \circ \widehat{S}[e_s] = \widehat{S}[e_1] \circ \widehat{S}[e_1] + \widehat{S}[e_2] \circ \widehat{S}[e_2] + \widehat{S}[e_2] \circ \widehat{S}[e_2] = \tfrac{3}{4}\mathbf{1}.$$

Here one recovers the well-known facts about the spin operators. The operator $\widehat{S}^2$ is the Casimir invariant [14, 10] of this representation of $\mathfrak{su}(2)$. For any unit vector field u, $\widehat{S}^2$ and $\widehat{S}[u]$ constitute a maximal set of commuting operators, with the eigenvalues $\frac{1}{2}(\frac{1}{2} + 1) = \frac{3}{4}$ and $\pm\frac{1}{2}$, respectively. •

From Sections 4.1 and 4.2 we recall that for a classical spinning particle we have the Hamiltonian $H^{\mathrm{S}} := H - \mu\hbar B^\flat$. Consider the *Hamiltonian function*

$$H := u_0 H^{\mathrm{S}}/\hbar\colon J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\mathbb{L}^* \otimes V\boldsymbol{E}) \to \mathbb{R}.$$

This is the main example of a quantizable function which has both phase and spin components. We have the quantum vector lift

$$Z[H] := Z[u_0 H/\hbar] - S[u_0\mu B],$$

with the coordinate expression

$$\begin{aligned} Z[H] &= \partial_0 + \tfrac{i}{4}(\varepsilon_r{}^{sp}C_0{}^r{}_s - 2u_0\mu\widetilde{B}^p)\sigma_p{}^A{}_B w^B b_A \\ &= \partial_0 + \tfrac{i}{4}\varepsilon_r{}^{sp}\widetilde{\Gamma}^\natural{}_0{}^r{}_s\sigma_p{}^A{}_B w^B b_A = \partial_0 + i\mathrm{Б}^\natural{}_0{}^A{}_B w^B b_A. \end{aligned}$$

The corresponding almost-quantum operator is then given by

$$\mathcal{Z}[H](\Psi^\eta) = (i\partial_0\psi^{\eta A} + \tfrac{1}{4}\varepsilon_r{}^{sp}\widetilde{\Gamma}^\natural{}_0{}^r{}_s\sigma_p{}^A{}_B\psi^{\eta B})b_A \otimes \sqrt{\omega}.$$

We obtain the following commutators:

$$\begin{aligned} &[\mathcal{Z}[x^\lambda], \mathcal{Z}[x^\mu]](\Psi^\eta) = 0, \\ &[\mathcal{Z}[x^0], \mathcal{Z}[p_j/\hbar]](\Psi^\eta) = 0, \\ &[\mathcal{Z}[y^j], \mathcal{Z}[p_k/\hbar]](\Psi^\eta) = i\delta^j_k\Psi^\eta, \\ &[\mathcal{Z}[p_j/\hbar], \mathcal{Z}[p_k/\hbar]](\Psi^\eta) = R[\mathrm{Б}]_{jk}{}^A{}_B\psi^B b_A \otimes \sqrt{\eta}, \\ &[\mathcal{Z}[x^\lambda], \mathcal{Z}[\phi]](\Psi^\eta) = 0, \end{aligned}$$

$$
\begin{aligned}
&[\mathcal{Z}[p_j/\hbar], \mathcal{Z}[\phi]](\Psi^\eta) = -i\mathcal{Z}[\nabla_j[C]\phi](\Psi^\eta),\\
&[\mathcal{Z}[y^j], \mathcal{Z}[H]](\Psi^\eta) = 0,\\
&[\mathcal{Z}[x^0], \mathcal{Z}[H]](\Psi^\eta) = -i\Psi^\eta,\\
&[\mathcal{Z}[p^j/\hbar], \mathcal{Z}[H]](\Psi^\eta) = -\tfrac{i}{4}\varepsilon_r{}^{sp} R[\Gamma^\natural]_{j0}{}^r{}_s \sigma_p{}^A{}_B \psi^B b_A \otimes \sqrt{\eta},\\
&[\mathcal{Z}[H], \mathcal{Z}[\phi]](\Psi^\eta) = i\mathcal{Z}[\nabla_0[\Gamma^\natural]\phi](\Psi^\eta).
\end{aligned}
$$

The Hamiltonian function is also the main example of a quantizable function whose associated sheaf and pre-Hilbert operators do not coincide. We have

$$\widehat{H} = \mathcal{Z}[H] - iu_0 \mathfrak{P},$$

that is

$$
\begin{aligned}
\widehat{H}(\Psi^\eta) &= \mathcal{Z}[H](\Psi^\eta) - \tfrac{1}{2}u_0 *\mathcal{E}^\#[\Psi] \otimes \sqrt{\eta}\\
&= -u_0 \frac{\hbar}{2m} \check{\Delta}^o \Psi^\eta - u^0 \frac{m}{\hbar} a_0 \Psi^\eta - u_0 \frac{\mu}{2} \Sigma(B)\Psi^\eta\\
&= u_0 \frac{\hbar}{2m} g^{jk} \left(\widehat{p_j/\hbar} - u^0 \frac{m}{\hbar} a_j\right)\left(\widehat{p_k/\hbar} - u^0 \frac{m}{\hbar} a_k\right)\Psi^\eta -\\
&\quad - u^0 \frac{m}{\hbar} a_0 \Psi^\eta - u_0 \frac{\mu}{2} \Sigma(B)\Psi^\eta.
\end{aligned}
$$

The generalized Pauli equation can now be written as

$$(i\partial_0 \Psi^{\eta A} + \text{Б}^\natural{}_0{}^A{}_B \Psi^{\eta B}) b_A \otimes \sqrt{\omega} = \widehat{H}(\Psi^\eta).$$

Then it would be nice if we were able to interpret the second term on the left-hand side as arising from the quantization of the energy of interaction between spin and gravitational field, to be included in the total spin energy operator composed of a spin-gravitation term and a spin-magnetic field term. An interpretation of this kind would need a deeper understanding of classical and quantum energy in the general relativistic Galilean context. We shall address this question in a future work.

Finally, the pre-Hilbert bundle yields the Hilbert bundle $\boldsymbol{HW}^\eta \to \boldsymbol{T}$ by the standard completion procedure. This bundle carries the standard probabilistic interpretation of quantum mechanics. We stress that we do not have a unique Hilbert space, but a Hilbert bundle over time. Indeed, a unique Hilbert space would be in conflict with the Galilean principle of relativity. On the other hand, a global observer yields an isometry between the fibres of the quantum Hilbert bundle.

Moreover, our symmetric pre-Hilbert operators will yield selfadjoint Hilbert operators under suitable functional hypotheses concerning the quantizable functions involved and the potentials of the concrete background spacetime.

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