

# Continuous Cayley Transform

A. Jadczyk

In complex analysis, the Cayley transform is a mapping  $w$  of the complex plane to itself, given by

$$w(z) = \frac{z - i}{z + i}.$$

This is a linear fractional transformation of the type

$$z \mapsto \frac{az + b}{cz + d}$$

given by the matrix

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

The determinant of this matrix is  $2i$ , and, noticing that  $(1 + i)^2 = 2i$ , we can as well replace  $C$  by  $\frac{1}{1+i}C$  in order to have  $C$  of determinant one - thus in the group  $SL(2, \mathbb{C})$ . The transformation itself does not change after such a replacement.

The Cayley transform is singular at the point  $z = -i$ . It maps the upper complex half-plane  $\{z : \text{Im}(z) > 0\}$  onto the interior of the unit disk  $D = \{w : |w| < 1\}$ , and the lower half-plane  $\{z : \text{Im}(z) < 0\}$ , except of the singular point, onto the exterior of the closure of  $D$ . The real line  $\{z : \text{Im}(z) = 0\}$  is mapped onto the unit circle  $\{w : |w| = 1\}$  - the boundary of  $D$ . The singular point  $z = -i$  can be thought of as being mapped to the point at infinity of the Riemann sphere - the one-point compactification of the complex plane.

By solving the equation  $w(z) = z$  we easily find that there are two fixed points  $(z_1, z_2)$  of the Cayley transform:

$$z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} + i \left( -\frac{1}{2} - \frac{\sqrt{3}}{2} \right),$$
$$z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2} + i \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} \right).$$

In order to interpolate between the identity mapping and the Cayley transform we will choose a one-parameter group of transformations in  $SL(2, \mathbb{C})$  sharing these two fixed points. A one-parameter group  $\alpha_t$  of  $SL(2, \mathbb{C})$  transformations is of the form

$$\alpha_t = \exp(tX),$$

where  $X$  is traceless matrix  $\text{tr}(X) = 0$ . In order to guarantee that  $\alpha_t(z_1) = z_1$  and  $\alpha_t(z_2) = z_2$ , we should have  $Xz_1 = Xz_2 = 0$ . Such a matrix  $X$  is determined up to a constant factor. The choice of this factor changes the overall scale of the parameter  $t$ . We choose  $X$  as having the most simple form:

$$X = \begin{pmatrix} 1 & 1-i \\ 1+i & 1 \end{pmatrix}.$$

We have then:

$$\exp(tX) = \begin{pmatrix} \frac{1}{3} (3 \cos(\sqrt{3}t) - i\sqrt{3} \sin(\sqrt{3}t)) & -\frac{(1+i) \sin(\sqrt{3}t)}{\sqrt{3}} \\ \frac{(1-i) \sin(\sqrt{3}t)}{\sqrt{3}} & \frac{1}{3} (3 \cos(\sqrt{3}t) + i\sqrt{3} \sin(\sqrt{3}t)) \end{pmatrix}$$

It can be checked that now  $\exp(tX) = C$  for  $t_{max} = \frac{\pi}{3\sqrt{3}}$ . Thus the following formula gives the interpolation between the identity transformation and the Cayley transform  $w$

$$w_t(z) = \frac{-\frac{(1+i) \sin(\sqrt{3}t)}{\sqrt{3}} + \frac{1}{3}z (3 \cos(\sqrt{3}t) - i\sqrt{3} \sin(\sqrt{3}t))}{\frac{(1-i)z \sin(\sqrt{3}t)}{\sqrt{3}} + \frac{1}{3} (3 \cos(\sqrt{3}t) + i\sqrt{3} \sin(\sqrt{3}t))}, \quad 0 \leq t \leq \frac{\pi}{3\sqrt{3}}.$$

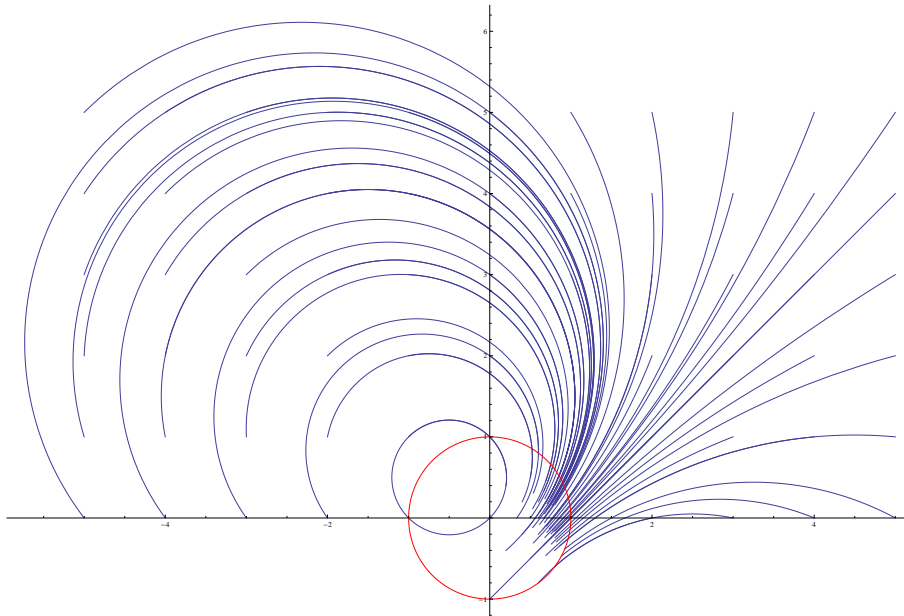


Fig. 1: Some examples of trajectories of  $w_t(z)$  starting with  $z$  being Gaussian integers (complex numbers with integer real and imaginary part) in the upper half-plane. The image may be, in some respect, misleading. For instance the trajectory starting at  $z = -2$  ends at  $z = \frac{3}{5} + \frac{4i}{5}$  on the unit circle. But this trajectory passes through  $z = 2i$ , and the trajectory starting at  $z = 2i$  ends at  $w = 1/3$  inside the unit circle, partially overlapping with the first trajectory.

In order to represent the continuous transformation graphically, let us start with the grid of lines along the coordinate axes as follows:

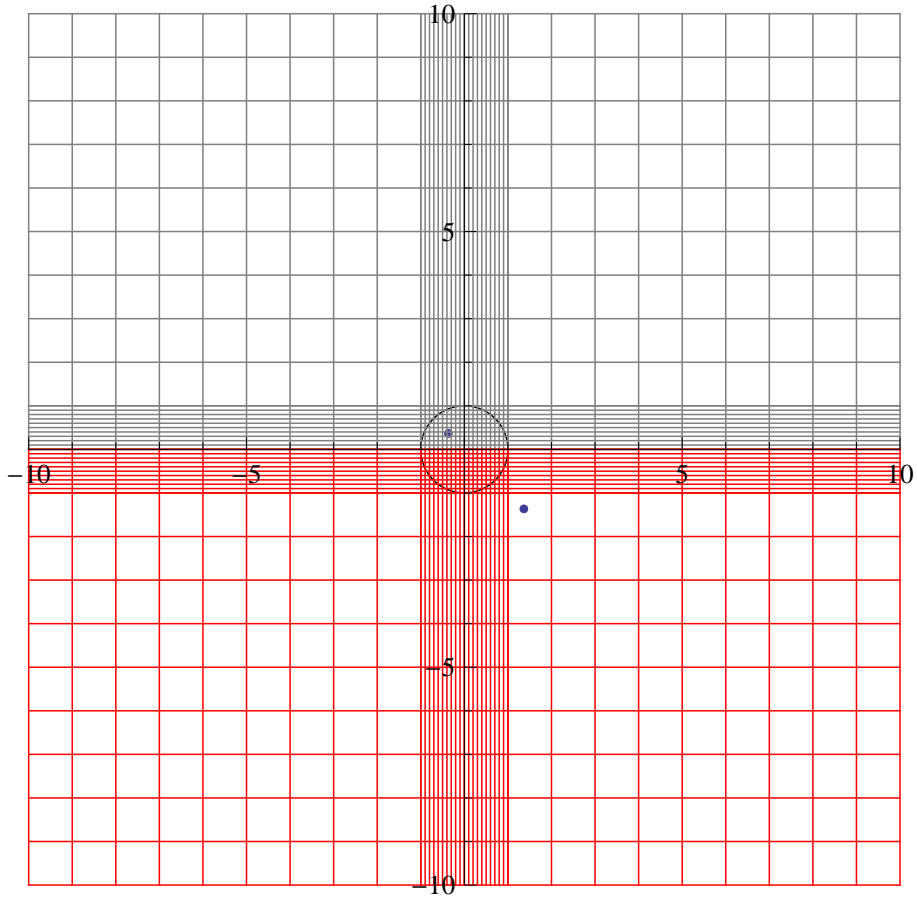


Fig. 2: The original grid lines, before the transformation. Added the unit circle and the two fixed points.

Then, for  $t = \frac{10}{75}t_{max}, t = \frac{20}{75}t_{max}$  we get the following deformations

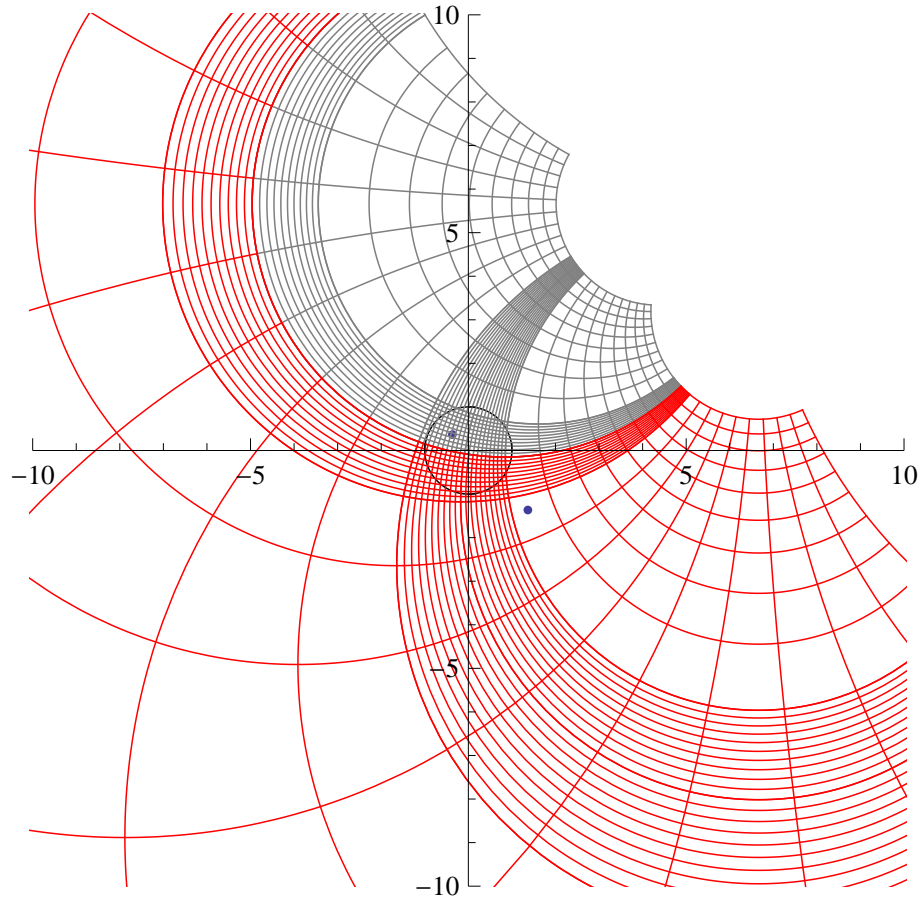


Fig. 3: The deformation for  $t = \frac{1}{75}t_{max}$ .

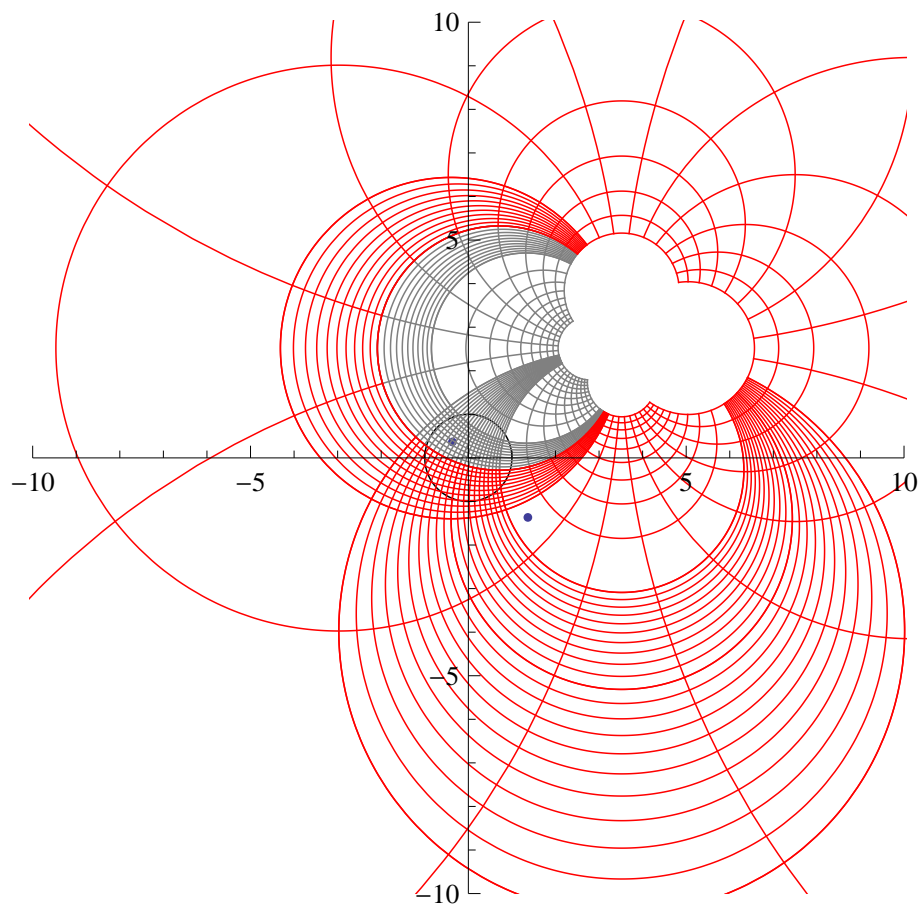


Fig. 4: The deformation for  $t = \frac{20}{73}t_{max}$ .

The final deformation, for  $t = t_{max}$  looks as follows:

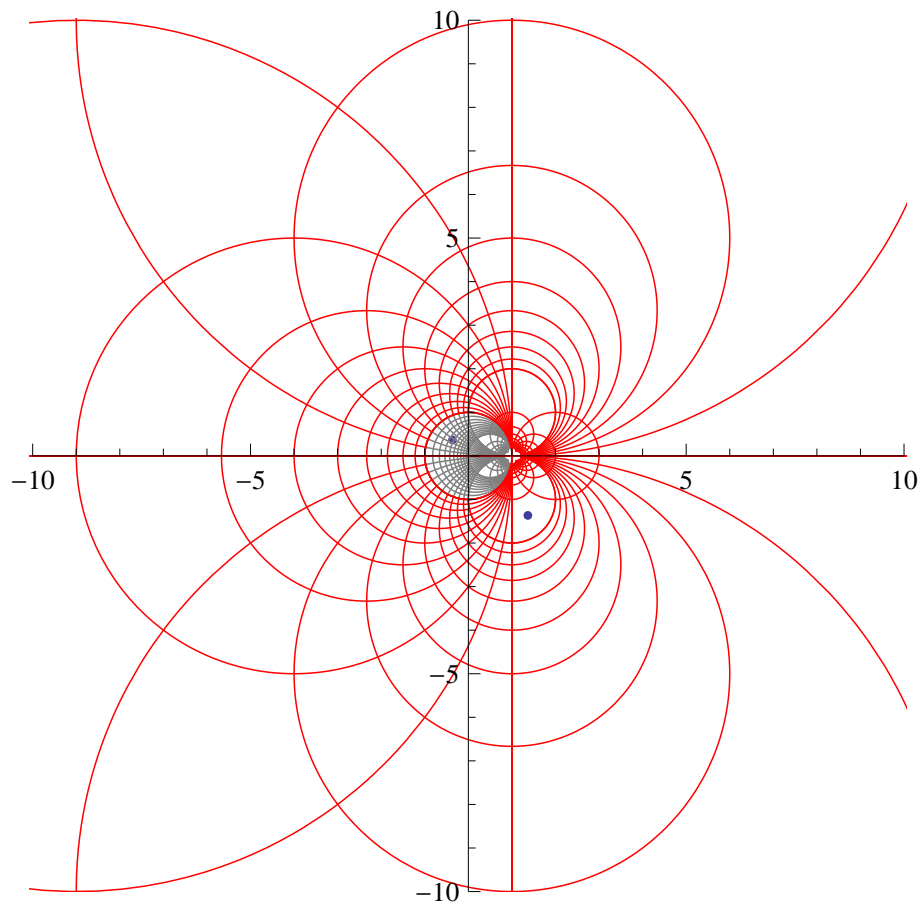


Fig. 5: The final deformation for  $t = t_{max}$ . - The Cayley transform.

The image below represents a zoom of the above - so that the inside of the unit circle is clearly represented.

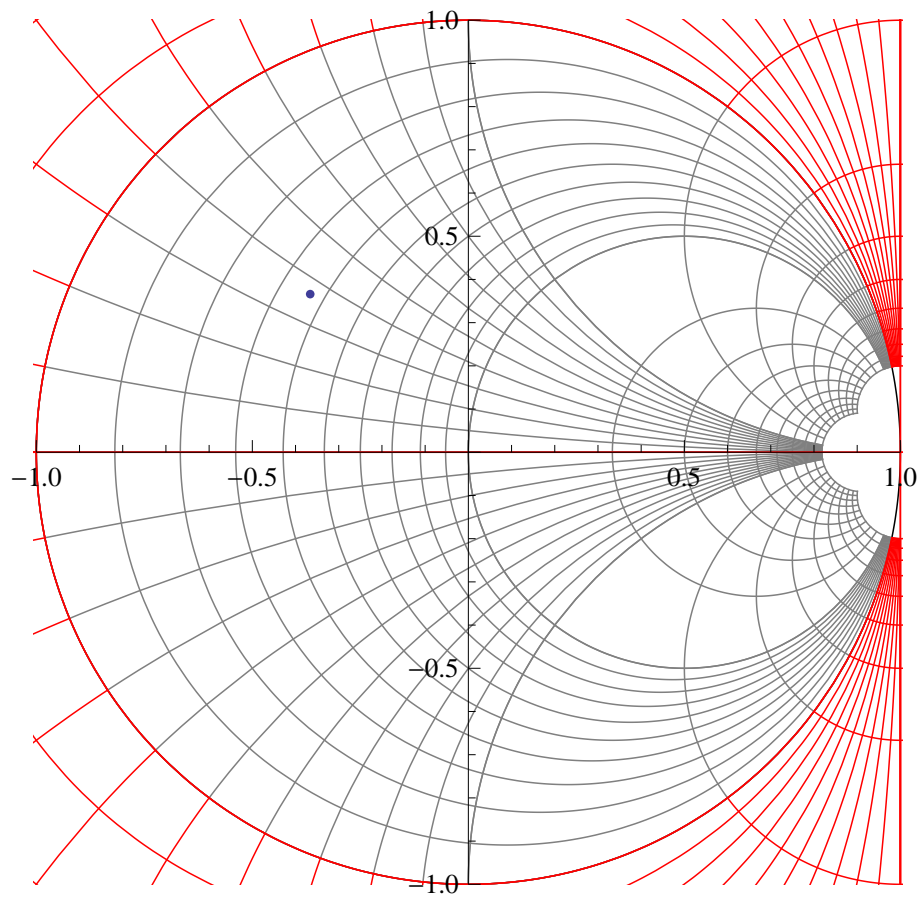


Fig. 6: The Cayley transform. Zoom into the fine structure of the unit disk.

The whole animation, consisting of the sequence of 75 snapshots can be found at the URL [http://www.icca7.ups-tlse.fr/Cayley\\_transform.gif](http://www.icca7.ups-tlse.fr/Cayley_transform.gif)

Arkadiusz Jadczyk  
 Center CAIROS  
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<http://www.icca7.ups-tlse.fr/cairos.htm>