CONFORMAL STRUCTURES AND CONNECTIONS

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1. INTRODUCTION The primary aim of the present author was to understand "conformal invariant wave equation"

$$(\Box + \frac{R}{6})\varphi = \lambda \varphi^3 \tag{0}$$

This equation appears in [1] without given any explicite reason as to "why" it is invariant. A conformal structure is best described by a tensor density $\gamma_{\mu\nu}$ [2]. If φ is a scalar density of dimension -1, then, with $g_{\mu\nu} = \varphi^2 \gamma_{\mu\nu}$, the equation (0) is nothing but $R(g) \sim \lambda$. Therefore its conformal invariance is obvious and "improved energy - momentum tensor" of [3] is automatic [4]. This observation can not be taken as satisfactory enough. One would like to have a machine /like a "covariant derivative" in a metric case/ which generates conformal invariant field equations. Long ago Cartan [5] considered connections more general than principal ones. A theory of conformal connections has been then developed to a highly sofisticated degree [6,7], and more recently it was shown, to be equivalent to a theory of twistor connections [8,9]. We prefer to work with $O(4,2)$ vectors rather than spinors. The convenient mathematical apparatus is that of second order frames [7,10,11]. We give an interpretation of the bundle of $O(4,2)$-vectors in terms of jets of scalar densities and write down empty space field equations of gravitation in terms of the conformal connection. The theory is similar to one in [12]. More details can be found in [13].

2. CONFORMAL STRUCTURE OF SPACE - TIME Let M be a smooth, 4-dimensional manifold, thought of as being a set of space-time events. Let $B(M)$ be the bundle of linear frames over M. Then $B(M)$ is a principal bundle with the structure group $GL(4)$.

Let G be a Lie subgroup of $GL(4)$. A G-structure on M is a smooth subbundle of $B(M)$ with G as a structure group. In many interesting cases G can be identified as the stabilizer of some tensorial object on R^4. For example, to give M a pseudo - Riemannian structure is to give it an $O(1,3)$ -structure and $O(1,3)$ is the stabilizer of the standard metric tensor $\eta = (\eta_{ab}) = \mathrm{diag}(-1,1,1,1)$. Similarly, to give M a conformal structure is to give it CO $1,3$ -structure, and $CO_+(1,3)$ is the stabilizer of /pseudo-/ tensor

$$\overset{\circ}{\chi}{}^{ab}_{cd} = \frac{1}{2} \epsilon_{cdef} \eta^{ea} \eta^{fb} \tag{1}$$

Let P be a conformal structure on M. The frames in P are called conformal frames /of the first order/. Take any coordinate system x^μ around $p \in M$, and let (e^μ_a) be a conformal frame over p. Then the formula

$$\chi^{\alpha\beta}_{\gamma\delta} = e^\alpha_a\, e^\beta_b\, e^c_\gamma\, e^d_\delta\, \chi^{ab}_{cd} \tag{2}$$

defines a /pseudo-/ tensor χ at p, which is independent of (e^μ_a). According to a general theorem /see [7] /, P is integrable /flat/ iff each point $p \in M$ admits a coordinate neighbourhood, with local coordinates x^μ, with respect to which the components of χ coincide with the standard ones (1). The tensor (2) is nothing but a Hodge - $*$ - operator restricted to 2-forms. Therefore, modulo topological subtleties, to give M a conformal structure is to give it a smooth $*$-operator acting linearly on the bundle of 2-forms and satisfying

$$(i) \qquad *^2 = -I ,$$

(ii) $\ast F \wedge G = F \wedge \ast G$; $F, G \in \bigwedge^2(M)$.

/see [14] /

Although χ determines conformal structure completely, it is more convenient to deal with its "square root" i.e. with a tensor density $\gamma_{\mu\nu}$ uniquely defined by

$$\text{a)} \quad \chi^{\mu\nu}_{\sigma\rho} = \frac{1}{2}\epsilon_{\sigma\rho\alpha\beta}\,\gamma^{\alpha\mu}\gamma^{\beta\nu} \quad ,$$

$$\text{b)} \quad \det(\gamma_{\mu\nu}) = -1 \quad .$$

As it was above, a conformal structure is flat iff there are local coordinate systems in which $\gamma_{\mu\nu} \equiv \eta_{\mu\nu}$. Since this condition is known to be equivalent to vanishing of the Weyl conformal curvature tensor W , it should be possible to express W in terms of only.

Assume now that a conformal structure $\gamma_{\mu\nu}$ is given, and let Φ be a scalar density of dimension l , i.e.

$$\Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{1/4} \phi(x).$$

If $l = -1$, then $g_{\mu\nu} \doteq \Phi^2 \gamma_{\mu\nu}$ is a metric tensor on M . In this way one gets a correspondence between conformal structures and classes of conformally equivalent pseudo-Riemannian metrics on M . It follows in particular, that each scalar density of dimension -1 determines a symmetric affine connection which preserves the conformal structure. However, no such an affine connection is distinguished.

3. <u>CONFORMAL FIELD EQUATIONS</u> Usually field equations are considered on a flat Minkowskian background. What is a deeper meaning of conformal invariance in such a case? The group of all automorphisms of flat causal structure is the Weyl group - semidirect product of Poincaré transformations and dilatations. Special conformal transformations are singular, and conformal inversion $x^\mu \mapsto x^\mu / x^2$ does

not preserve causal relations. If so, then why should one require full conformal invariance and not only invariance with respect to the Weyl group? To answer this question it is necessary to consider what will happen after local deformation of the flat light-cone structure. In a flat space the very difference between local and global aspects can easily be lost. The total space can be naturally identified with its tangent space at a given point. A point can be identified with intersection of two infinite lines etc. On the contrary, in a generic Riemannian space no such identifications are possible, and no automorphisms of its causal structure exist. Here conformal structure exhibits its true meaning. Rimannian metric separates into a volume element /or length scale/ Φ^4 and a causal structure $\gamma^{\mu\nu}$. Conformal invariance means that the field equations are governed only by the causal structure, and not by the length scale. In consequence energy-momentum tensor is automatically traceless, and field equations are automatically invariant with respect to all Killing vector fields X of the causal structure. In particular they remain conformally invariant when specified to the flat case.

It follows that covariance of field equations under conformal inversions $x^\mu \mapsto x^\mu / x^2$ should be considered only as a hint that the equations are stable under local deformations of the conformal /flat/ structure. And only such stable systems are of physical interest. Given a transformation law of fields under conformal inversions it is then usually possible to deduce what kind of a geometrical object one is dealing with, and to generalize field equations to a curved background. However, the situation here is not as simple as in a Riemannian case, since no straightforward recipe like "replace derivatives by the covariant ones" is possible. /In fact, even in Riemannian case one meets umbiguity of curvature terms/.

4. SECOND ORDER CONFORMAL FRAME

a) The bundle $P^2(M)$

A second order frame at $p \in M$ is characterized /with respect to a coordinate, system x^μ / by a set of numbers $(e^\mu = x^\mu(p)\ , e^\mu_a ,$ $e^\mu_{ab} = e^\mu_{ba}$) with transformation laws

$$e^{\mu'} = x^{\mu'}(p)$$

$$e^{\mu'}_a = \frac{\partial x^{\mu'}}{\partial x^\mu}(p)\ e^\mu_a$$

$$e^{\mu'}_{ab} = \frac{\partial x^{\mu'}}{\partial x^\mu}\ e^\mu_{ab} + \frac{\partial^2 x^{\mu'}}{\partial x^\alpha \partial x^\beta}(p)\ e^\alpha_a\ e^\beta_b\ .$$

The coordinates e^μ_a can be interpreted as determining linear /1-st order/ frame at p , and e^μ_{ab} can be interpreted as determining connection coefficients

$$e^\mu_{\rho\sigma} \equiv -e^r_\rho\ e^s_\sigma\ e^\mu_{rs}\ .$$

We denote by $P^2(M)$ the bundle of second-order frames.

The $G^2(4)$ set of all second-order frames at $0 \in R^4$ is a group with the multiplication law

$$(h^a{}_b, h^a{}_{bc})\ (k^a{}_b, k^a{}_{bc}) = (h^a{}_r\ k^r{}_b, h^a{}_{rs}\ k^r{}_b\ k^s{}_c + h^a{}_r k^r{}_{bc})$$

If $h^a{}_b, h^a{}_{bc}$ are replaced by e^μ_a , e^μ_{ab} , then the last formula gives natural action of $G^2(4)$ on $P^2(M)$ which makes $P^2(M)$ a principal bundle.

b) reduction of $P^2(M)$ induced by conformal structure.

Let a conformal structure C be given on M in terms of $\gamma_{\mu\nu}$. A general symmetric affine connection which preserves C is of the form

$$\Gamma^\alpha_{\mu\nu} = \hat{\Gamma}^\alpha_{\mu\nu} + (\delta^\alpha_\mu\, \rho_\nu + \delta^\alpha_\nu\, \rho_\mu - \gamma_{\mu\nu}\gamma^{\alpha\beta}\rho_\beta) \tag{3}$$

where
$$\hat{\Gamma}^\alpha_{\mu\nu} = \frac{1}{2}\gamma^{\alpha\beta}(\partial_\mu\gamma_{\nu\beta} + \partial_\nu\gamma_{\mu\beta} - \partial_\beta\gamma_{\mu\nu}) .$$

Therefore $P^2(M)$ can be reduced to $P^2_C(M)$ defined as consisting of second order frames e such that e^μ_a are "orthonormal" conformal frames, and $e^\mu_{\sigma\rho}$ are coefficients of connections (3) .

The structure group of $P^2_C(M)$ is $G = CO(1,3) \times R^{4*}$ with the multiplication law

$$(h^a{}_b, v_a)(k^a{}_b, w_a) = (h^a{}_r k^r{}_b, v_r k^r{}_a + w_a)$$

identified with a subgroup of $G^2(4)$ by

$$(h^a{}_b, v_a) \mapsto (h^a{}_b, h^a{}_{bc})$$

where

$$h^a{}_{bc} = h^a{}_r \left(\delta^r_b v_c + \delta^r_c v_b - \eta_{bc} \eta^{rs} v_s \right).$$

It is evident that, conversely, each reduction of $P^2(M)$ to G determines a conformal structure C. There is, however, a distinguished class of reductions, namely those of the form $P^2_c(M)$.

c) bundle of jets of scalar densities.

Let ϕ be a scalar density of dimension 1 .
The first jet of ϕ at p is parametrized by 1+4 parameters:

$$\phi = \phi(p)$$
$$z_\mu = \partial_\mu \phi(p)$$

If e is a conformal frame in P^2_c , then one defines coordinates of (ϕ, z) with respect to e by

$$\varphi(e) \doteq e^{1/4}\phi \quad , \quad e = |e^\mu_a|$$
$$z^a(e) \doteq \eta^{ar} e^\mu_r e^{1/4}\left(z_\mu + \tfrac{1}{4}\phi \Gamma_\mu\right)$$

where $\Gamma_\mu = e^\tau_{\mu\tau}$. From the transformation character of $(\varphi(e), z(e))$ one then finds that the bundle of jets of scalar densities of dimension 1 can be considered as an associated bundle $P^2_{c,l}$ of $P^2_c(M)$ corresponding to the following representation D_l of G on R^{1+4} :

$$D_1(\Lambda,\theta) = \theta^{(1-1)}\begin{pmatrix} \theta & , & 0 \\ 1\theta & , & \Lambda \end{pmatrix}$$

where $h^a{}_b = \theta \Lambda^a{}_b$, and $\Lambda \eta \Lambda^T = \eta$.

This representation admits no invariant bilinear form. If, however, a new parameter ψ is introduced, then the following quadratic form:

$$Z^2 = \begin{pmatrix} \varphi \\ z \\ \psi \end{pmatrix}^2 = z^2 - 2\varphi\psi$$

is scale invariant, provided D_1 is prolonged to $\tilde{D}_1$ given by

$$\tilde{D}_1 = \begin{pmatrix} & D_l & , & 0 \\ & & , & 0 \\ \frac{1}{2}^2\theta\, v^2 , & 1\, v\eta\Lambda & , & \theta^{-1} \end{pmatrix}$$

The representation $\tilde{D}_1$ preserves the scalar product given by the matrix

$$S = \begin{pmatrix} 0 & , & 0 & , & -1 \\ 0 & , & \eta & , & 0 \\ -1 & , & 0 & , & 0 \end{pmatrix}$$

up to a factor:

$$D_1^T S D_1 = \theta^{2(1-1)} S .$$

In particular, for $1=1$, the quadratic form $Z^T S Z$ is invariant.

5. <u>CONFORMAL CONNECTIONS</u> The above prolongation $\tilde{P}^2_{c,1}$ of $P^2_{c,1}$ carries natural invariant bilinear form (Z,Z') with $O(2,4)$ as an invariance group (precisely speaking the stability group of S is <u>isomorphic</u> to $O(4,2)$). It carries also a natural one-dimensional isotropic line subbundle I defined by $\varphi = 0$, $z = 0$.
A <u>natural</u> linear connection in $\tilde{P}^2_{c,1}$ should preserve both. No such connection exists. There exists connection which preserve scalar product /but no I / . It is given by

$$\nabla_\mu Z = \partial_\mu Z + \Gamma_\mu Z$$

where

$$\Gamma_\mu = t_\mu D + \frac{1}{2}\omega_\mu{}^r{}_s M^s{}_r + v_{\mu r} K^r + w_\mu^r + P_r =$$

$$= \begin{pmatrix} t_\mu , & w_{\mu a} , & 0 \\ v_\mu{}^a , & \omega_\mu{}^a{}_b , & w_\mu{}^a \\ 0 , & v_{\mu a} , & -t_\mu \end{pmatrix}$$

and

$$w_\mu^a = - e_\mu^a$$
$$\omega_\mu{}^a{}_b = e^a{}_\sigma \nabla_\mu e_b^\sigma$$
$$v_{\mu a} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{6} R g_{\mu\nu}\right) e^\nu{}_a$$
$$t_\mu \text{ - arbitrary one-form.}$$

Here ∇_μ and $R_{\mu\nu}$ are covariant derivative and Ricci tensor of $g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b$. Since t_μ is a one-form, one can put $t_\mu = 0$ to get what is called "canonical normal Cartan connection".

It should be observed that this connection preserve the cone $Z^2 = 0$, and so defines a /nonlinear/ connection in a "compactified tangent bundle".

6. FIELD EQUATIONS FOR GRAVITATION. The equations $\nabla_\mu Z = 0$ are equivalent to the field equations of gravitation in empty space. In fact, $\nabla_\mu Z = 0$ implies Z^2 = const, and then $\nabla_\mu Z = 0$ is equivalent to $R(\tilde{g})$ = const, where $\tilde{g}_{\mu\nu} = \varphi^{-2} e_\mu^a \eta_{ab} e_\nu^b$ /observe that Z is always refered to section (e_a^μ) of P_c^1, which determines section of P_c^2 /. In particular Z^2 is proportional to a cosmological constant, and $R(\tilde{g})$ = const is nothing but the familar "wave equation"(6).

Equations of the above type can not be obtained from a Lagrangean. It is, however, interesting to notice that $D = K^\mu \nabla_\mu$ is an invariant operator /like the Dirac operator/, and Z satisfies $\nabla_\mu Z = 0$ iff it satisfies

i) $D Z = 0$

ii) Z^2 = const.

Therefore $Z^t\, D\, Z$ can be taken as proportional to Lagrangean density which, together with the constraint ii) gives us standard Einstein vacuum equations.

The connection Γ can be defined in every associated $O(2,4)$ bundle, in particular, it defines covariant derivative of twistors. However, in the twistorial /four-valued/ representation of $O(2,4)$ $K_\mu K_\nu \equiv 0$. Therefore the invariant operator D has no curvature terms in this case. Therefore $DZ = 0$ has a different meaning for a twistorial section Z.

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