DIFFERENTIAL AND INTEGRAL GEOMETRY OF GRASSMANN ALGEGRAS

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We give a self-contained exposition of the differential geometry of Grassmann algebras. We also study elementary properties of these algebras from the point of view of Hochschild and cyclic cohomologies.

0. Introduction

Anticommuting Grassmann variables have found applications in physics for more than a decade, mainly in supersymmetric field theories, but also in superintegrable systems, etc. The present paper is a self-contained exposition of the differential geometry of Grassmann algebras, written in the spirit of the graded Lie-Cartan pairs framework [1], with the additional aim of studying the Berezin integral so as to prepare the discussion of its role in cyclic cohomology along the lines of [2]. Apart from this, most of the material that we discuss is either known to the experts or can be deduced from known facts; we refer the reader to the classical papers [3], [4].

The first part contains five paragraphs: general definitions, the canonical filtration, generators and frames, generalized parities, and automorphisms, the content of which can be summarized as follows:

General definitions: We define a Grassmann algebra $\mathscr{G}_n$ in terms of generators and relations. We establish the link between this definition and the definition via exterior algebra of a given vector space. The main point here is that the corresponding isomorphism is not canonical. Along with the $\mathbb{Z}/2$-graded algebra $\mathscr{G}_n$, we also introduce $\tilde{\mathscr{G}}_n$—the ungraded version of $\mathscr{G}_n$, obtained from the latter by forgetting the $\mathbb{Z}/2$-grading.

Canonical filtration: The powers of the ideal of nilpotents yield a canonical filtration. We define a canonical form τ on the Grassmann algebra, associating with each element of the algebra its "scalar" part (a complex number). We stress the fact that the dual $\mathscr{G}_n^*$ of $\mathscr{G}_n$ is a one-dimensional left $\mathscr{G}_n$-module (this property will be needed later for treatment of the Berezin integral).

Key-words: Grassmann algebras, supergeometry, cyclic cohomology.

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Generators and frames: We discuss generating systems in the Grassmann algebra. The important fact (stemming from the non-intrinsicality of the correspondence with the exterior algebra) is that, starting with a given system of generators, we can build others via a "change of basis", but that a matrix describing such a change of basis may have coefficients in the algebra $\mathscr{G}_n$ itself rather than in the complex numbers.

Generalized parities: An ungraded Grassmann algebra with an even number of generators has its even part determined by the algebra structure alone—it coincides with its center. The concept of a generalized parity is introduced to discuss the freedom of selecting an odd part. Ungraded Grassmann algebras have not found applications till now—in principle they would allow an extension of the concept of supersymmetry.

Automorphisms: We discuss mainly automorphisms of ungraded Grassmann algebras. The results of this paragraph are then used when discussing a general definition of the Berezin integral.

In the second part we discuss the differential geometry of a Grassmann algebra in the spirit of Lie-Cartan pairs. In particular we introduce the Lie superalgebra of derivations $\mathbb{L}_n$ of $\mathscr{G}_n$, and discuss its module properties. With the Lie-Cartan pair $(\mathbb{L}_n, \mathscr{G}_n)$ we associate the module of $\mathscr{G}_n$-valued graded alternate forms $\Lambda_n^* = \Lambda_{\mathscr{G}_n}^*(\mathbb{L}_n, \mathscr{G}_n)$. We then equip Λ_n^* with a **differential** d, **interior products** $\mathbf{i}(\xi)$, and **Lie derivatives** $L(\xi)$, $\xi \in \mathbb{L}_n$, and discuss the natural gradings and mutual relations of the latter. We also give an explicit proof of the Poincaré lemma.

The third part deals with the concept of Berezin integral. The conventional definition makes it explicitly dependent on the system of generators. After discussing properties coming from this definition, we describe an intrinsic definition as a module generator for the dual $\mathscr{G}_n^*$.

Elementary properties of Hochschild and cyclic cohomologies for Grassmann algebras are described: in the fourth part we list cyclic cocycles of low order, as the first step of a more complete study, to which we shall return at a later point.

1. Grassmann Algebras

[1.0] Definition. (i) A $\mathbb{Z}/2$-**graded** vector space (real or complex) is a vector space E together with a direct sum decomposition $E = E^0 \oplus E^1$ (called the *grading of* E) into an **even subspace** E^0 and an **odd subspace** E^1. We shall denote $E^0 = E^0 \cup E^1$ the set of **homogeneous elements** and write ∂x for the **grade of** $x \in E^0$ defined by $\partial x = r \bmod 2$ if $x \in E^r$. The **grading involution** $\varepsilon : E \to E$ is defined[a] by $\varepsilon(x) = (1 - 2\partial x)x$, $x \in E^0$.

(ii) A $\mathbb{Z}/2$-**graded** (associative) **algebra**[b] over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is a vector space[c] $A = A^0 \oplus A^1$ with a bilinear product $A \times A \to A$ fulfilling A^0A^0, $A^1A^1 \subset A^0$, A^0A^1, $A^1A^0 \subset A^1$.

(iii) A **graded commutative** (or **supercommutative**) **algebra** is a $\mathbb{Z}/2$ graded algebra

[a] The grading of E can be defined by specifying ε, or the representation ρ of $\mathbb{Z}/2$ on E given by $\rho(1 \bmod 2) = \varepsilon$.

[b] In the sequel *algebra* always means associative algebra.

[c] Equivalently: the grading involution of A is an automorphism.

$A = A^0 \oplus A^1$ over $\mathbb{K}$ such that

$$vu = (-1)^{\partial u \partial v} uv, \qquad u, v \in A^0. \tag{1.1}$$

[1.1] Definition. Let $n \in \mathbb{N}$, $n > 0$.

The **Grassmann algebra** $\mathscr{G}_n$ is defined

(i) as the $\mathbb{Z}/2$-graded algebra $\mathscr{G}_n = \mathscr{G}_n^0 \oplus \mathscr{G}_n^1$ over $\mathbb{K}$ which admits odd generators $\theta_0^1, \theta_0^2, \ldots, \theta_0^n$, satisfying the relations

$$\theta_0^i \theta_0^j + \theta_0^j \theta_0^i = 0, \qquad i, j = 1, \ldots, n, \tag{1.2}$$

(ii) or alternatively as the graded commutative algebra of dimension 2^n with n odd generators.

The **parity-free Grassmann algebra** $\tilde{\mathscr{G}}_n$ is obtained by omitting the words: "$\mathbb{Z}/2$-graded" and "odd" in definition (i). Thus $\tilde{\mathscr{G}}_n$ is isomorphic to the algebra quotient of the free algebra[d] $\mathbb{F}\{\theta_0^i; i = 1, \ldots, n\}$ through the ideal generated by the left-hand side of the relations (1.2); alternatively, $\tilde{\mathscr{G}}_n$ is isomorphic (as an algebra) to the **exterior algebra** $\Lambda\mathbb{E}^*$ over $\mathbb{E}^*$, $\mathbb{E} \approx \mathbb{K}^n$ (with $\mathscr{G}_n$ obtained by equipping the latter with the $\mathbb{Z}/2$ grading stemming from the parity of tensors[e]). A handy definition of $\Lambda\mathbb{E}^*$ is obtained as follows: consider $\mathbb{E}^*$ as the dual of $\mathbb{E} \approx \mathbb{K}^n$, and take the direct sum

$$\Lambda\mathbb{E}^* = \bigoplus_{p \in \mathbb{N}} \Lambda_p(\mathbb{E}, \mathbb{K}), \tag{1.3}$$

($\Lambda_p(\mathbb{E}, \mathbb{K})$ the set of $\mathbb{K}$-valued ($\mathbb{K}$ is always $\mathbb{R}$ or $\mathbb{C}$) alternate p-linear forms over $\mathbb{E}$, $\Lambda_0(\mathbb{E}, \mathbb{K}) = \mathbb{K}$, $\Lambda_1(\mathbb{E}, \mathbb{K}) = \mathbb{E}^*$), equipped with the **wedge product**.

$$u \wedge v = \frac{(p+q)!}{p!q!} A(u \otimes v), \qquad u \in \Lambda_p(\mathbb{E}, \mathbb{K}), \quad v \in \Lambda_q(\mathbb{E}, \mathbb{K}), \tag{1.4}$$

where $\otimes$ is the tensor product, and A the alternating idempotent, i.e., for $x_i \in \mathbb{E}$, $i = 1, \ldots, p+q$, we have

$$(u \otimes v)(x_1, \ldots, x_{p+q}) = u(x_1, \ldots, x_p) v(x_{p+1}, \ldots, x_{p+q}), \tag{1.5}$$

and

$$(Au)(x_1, \ldots, x_p) = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} (-1)_u^\sigma (x_{\sigma 1}, \ldots, x_{\sigma p}) \tag{1.6}$$

[d] $\mathbb{F}\{\theta_0^i; i = 1, \ldots, n\}$ consists of formal linear combinations of all "words" with "letters" θ_0^i, with words multiplying through concatenation.

[e] Observe that the division ideal is generated by even tensors.

(Σ_p the group of permutations of the first p integers). We recall that, under the product (1.4), $\Lambda\mathbb{E}^*$ is a unital algebra over $\mathbb{K}$, $\mathbb{N}$-**graded** in the sense

$$\Lambda_p(\mathbb{E},\mathbb{K}) \wedge \Lambda_q(\mathbb{E},\mathbb{K}) \subset \Lambda_{p+q}(\mathbb{E},\mathbb{K}),\ p, q \in \mathbb{N}, \tag{1.7}$$

thus $\mathbb{Z}/2$-graded for the even-odd grading

$$\left\{ \Lambda\mathbb{E}^* = (\Lambda\mathbb{E}^*)^0 \oplus (\Lambda\mathbb{E}^*)^1, \qquad \text{with} \begin{cases} (\Lambda\mathbb{E}^*)^0 = \bigoplus_{p\,\text{even}} \Lambda_p(\mathbb{E},\mathbb{K}) \\ (\Lambda\mathbb{E}^*)^1 = \bigoplus_{p\,\text{odd}} \Lambda_p(\mathbb{E},\mathbb{K}) \end{cases} \right. . \tag{1.8}$$

By what precedes, $\tilde{\mathcal{G}}_n$ (resp. $\mathcal{G}_n$) is isomorphic to $\Lambda\mathbb{E}^*$ as an algebra over $\mathbb{K}$ (resp. as a $\mathbb{Z}/2$-graded algebra over $\mathbb{K}$), *although all such isomorphisms are on equal footing.* Calling $\theta_0 : \Lambda\mathbb{E}^* \to \mathcal{G}_n$ one of these isomorphisms, we set

$$\left\{ \mathcal{G}_n^0 = \theta_0\{(\Lambda\mathbb{E}^*)^0\} \quad \text{with } \pi(a^0 + a^1) = a^0 - a^1,\ \mathcal{G}_n^1 = \theta_0\{(\Lambda\mathbb{E}^*)^1\} \begin{cases} a^0 \in \mathcal{G}_n^0 \\ a^1 \in \mathcal{G}_n^1 \end{cases} \right. \tag{19}$$

(π is called the **parity involution of** $\mathcal{G}_n$). The product of $\mathcal{G}_n$ is denoted by the usual product notation:

$$ab = \theta_0(a \wedge b), \qquad a, b \in \mathcal{G}_n. \tag{1.10}$$

[1.2] Remark. With $n \geq 1$, and a p-multiindex a sequence $I = i_1 i_2 \ldots i_p$ of positive integers $\leq n$, we shall, given p elements $\eta^1, \eta^2, \ldots, \eta^p$ of either $\Lambda\mathbb{K}^{n^*}$, $\mathcal{G}_n$, or $\tilde{\mathcal{G}}_n$ indexed by indices $i_k \leq n$, $k = 1, \ldots, p$, use for their product the following compact notation:

$$\eta^{i_1}\eta^{i_2}\ldots\eta^{i_p} = \eta^I \qquad \text{with } I = i_1 \ldots i_p. \tag{1.11}$$

The p-multiindex $I = i_1 \ldots i_p$ is called **lexicographic** whenever $1 \leq i_1 < i_2 < \ldots < i_p \leq n$. We will denote by $\mathbb{I}_n$, (resp, $\mathbb{L}_n$) the set of all p-multiindices (resp. of all lexicographic p-multiindices I, where the **length of** I, $|I| = p$ ranges from 1 to n).

With $\{e_i = (\delta_i^k)\}_{i=1,\ldots,n}$ the canonical basis of $\mathbb{K}^n = \mathbb{E}$ and $\{\varepsilon^i\}_{i=1,\ldots,n}$ the dual basis of $\mathbb{E}^* = \Lambda_1(\mathbb{E},\mathbb{K})$, we then have that the ε^I, $I \in \mathbb{L}_n$, together with $\mathbb{1} = \varepsilon^{\varnothing}$, build a linear basis of ΛE^*, which thus has the dimension[f] 2^n. Accordingly, with

$$\theta_0^i = \theta_0(\varepsilon^i), \qquad i = 1, \ldots, n, \tag{1.12}$$

the θ_0^I, $I \in \mathbb{L}_n$, yield, together with $\mathbb{1} = \theta_0(\mathbb{1})$, a linear basis of $\mathcal{G}_n$ (or, for that matter, of $\tilde{\mathcal{G}}_n$).

[f] Note that $\dim(\Lambda E^*)^0 = \dim(\Lambda E^*)^1 = 2^{n-1}$.

Note that one has

$$\varepsilon^J \varepsilon^{\bar{J}'} = \pm\delta_{J'J}\varepsilon^{1,2\ldots n}, \qquad J, J' \in \mathbb{L}_n \cup \{\varnothing\}, \qquad |J| + |J'| = n, \tag{1.13}$$

where J' is the lexicographic $(n - |J|)$-index containing all indices notifying in J (and analogous facts for the θ_0^J).

The canonical filtration

The next definitions and proposition, which we phrase for $\mathscr{G}_n$, in fact essentially pertain to $\tilde{\mathscr{G}}_n$.

[1.3] Definitions and notation. (i) We denote by $\mathscr{N}$ the set of nilpotent elements of $\mathscr{G}_n$:[g]

$$\mathscr{N} = \{a \in \mathscr{G}_n;\, a^k = 0 \text{ for some } k \in \mathbb{N}\} \tag{1.14}$$

and set, for $k \geq 1$,

$$\mathscr{N}^k = \text{linear closure of } \{a_1 \ldots a_k;\, a_1, a_2, \ldots, a_k \in \mathscr{N}\}. \tag{1.15}$$

(ii) $\mathscr{G}_n^*$ denotes the dual (set of $\mathbb{K}$-valued linear forms) of $\mathscr{G}_n$.

The **polar** of $\mathscr{S} \subset \mathscr{G}_n$, resp. $\Phi \subset \mathscr{G}_n^*$, is by definition

$$\mathscr{S}^\perp = \{\varphi \in \mathscr{G}_n^*;\, \varphi|_S = 0\}, \tag{1.16}$$

resp.

$$\Phi^\perp = \{a \in \mathscr{G}_n;\, \varphi(a) = 0 \text{ for all } \varphi \in \Phi\}. \tag{1.17}$$

(iii) For $\varphi \in \mathscr{G}_n^*$ and $a \in \mathscr{G}_n$ we define φa, $a\varphi \in \mathscr{G}_n^*$ as

$$\begin{cases} (\varphi a)(x) = \varphi(ax) \\ (a\varphi)(x) = \varphi(xa) \end{cases} \qquad x \in \mathscr{G}_n, \tag{1.18}$$

and define the parity of $\mathscr{G}_n^*$ by transposition:

$$\mathscr{G}_n^* \ni \varphi \to \varphi \circ \pi \in \mathscr{G}_n^*. \tag{1.19}$$

(iv) We define $\tau \in \mathscr{G}_n^*$ by setting[h]

$$\begin{cases} \tau|_{\mathscr{N}} = 0 \\ \tau(\mathbb{1}) = 1 \end{cases}. \tag{1.20}$$

[g] We write $\mathscr{N} = \mathscr{N}_n$ if we want to stress the reference to $\mathscr{G}_n$.

[h] As will follow from (1.23).

(v) We denote by $\mathscr{Z}_n$ the center of $\mathscr{G}_n$.

In the next proposition we introduce the intrinsic filtration of $\mathscr{G}_n$. We also study the properties of the unique non-trivial "character" τ of $\mathscr{G}_n$. τ assigns to each element of the Grassmann algebra its "scalar" part; in the context of supermanifolds it is often called the "body map".

[1.4] Proposition. (i) *The $\mathscr{N}^k$ are a decreasing sequence of ideals of $\mathscr{G}_n$:*

$$\mathscr{G}_n \supset \mathscr{N} \supset \mathscr{N}^2 \supset \ldots \supset \mathscr{N}^n \supset \mathscr{N}^{n+1} = \{0\}, \tag{1.21}$$

yielding a filtration (called the **canonical filtration**) *in the sense that*

$$\mathscr{N}^p \mathscr{N}^q \subset \mathscr{N}^{p+q}, \qquad p, q \geq 1; \tag{1.22}$$

in fact we have[i]

$$\mathscr{N}^k = \bigoplus_{p=k}^{n} \Lambda_p(\mathbb{E}, \mathbb{K}) = \theta_0(\mathscr{N}^k). \tag{1.23}$$

Moreover each $\mathscr{N}^k$ is a graded (i.e., $\pi(\mathscr{N}^k) = \mathscr{N}^k$, with π defined in (1.9)) ideal.

(ii) *τ is determined by* (B.20) *as the unique character (non-vanishing multiplicative linear form) of $\mathscr{G}_n$. We have $\mathscr{N} = \tau^\perp$.*

(iii) *One has the dual direct decompositions*

$$\mathscr{G}_n = \mathbb{K}\mathbb{1} \oplus \mathscr{N}, \tag{1.24}$$

$$\mathscr{G}_n^* = \mathbb{K}\tau \oplus \{\mathbb{1}\}^\perp \tag{1.25}$$

both as vector spaces and as $\mathscr{G}_n$-bimodules.[j] *The corresponding expansions for $a \in \mathscr{G}_n$ and $\varphi \in \mathscr{G}_n^*$ are*

$$a = \tau(a)\mathbb{1} + \underline{a}, \qquad \underline{a} \in \mathscr{N}, \tag{1.24a}$$

$$\varphi = \varphi(1)\tau + \underline{\varphi}, \qquad \underline{\varphi} \in \{\mathbb{1}\}^\perp. \tag{1.25a}$$

(iv) *An element $u \in \mathscr{G}_n$ is invertible iff $\tau(u) \neq 0$. Its inverse is then given by the formula (where $(1 + x) = u/\tau(u)$)*

$$(\mathbb{1} + x)^{-1} = \mathbb{1} + \sum_{k=1}^{n} (-1)^k x^k \qquad (= 1 - x \textit{ if } x \textit{ is odd}). \tag{1.26}$$

[i] Showing that $\mathscr{G}_n^1 \subset \mathscr{N}$ and that $\mathscr{N}^n$ is one-dimensional.

[j] $\mathscr{G}_n$ with the obvious $\mathscr{G}_n$-bimodule structure, $\mathscr{G}_n^*$ with that defined by (1.18). Observe that $\mathscr{N}$ is a sub-bimodule (an ideal) and that all polars of subsets stable under parity are sub-bimodules.

(v) *$\mathscr{G}_n^*$ is simply generated (hence $\approx \mathscr{G}_n$) as a right (or left) $\mathscr{G}_n$-module, in fact $\varphi \in \mathscr{G}_n^*$ generates $\mathscr{G}_n^*$ iff $\varphi|_{\mathscr{N}^n} \neq 0$.*

(vi) *One has $\mathscr{Z}_n = \mathscr{G}_n^0$ for n even and $\mathscr{Z}_n = \mathscr{G}_n^0 \oplus \mathscr{N}^n$ for n odd.*

Proof. (i) One has $\mathscr{N} = \theta_0(\bar{\mathscr{N}})$, $\bar{\mathscr{N}}$ being the set of nilpotent elements of $\Lambda\mathbb{E}^*$. Now $\bar{\mathscr{N}} = \bigoplus_{p\geq 1} \Lambda_p(\mathbb{E}, \mathbb{K})$, since the $(n+1)^{\text{th}}$ power of each element of this set vanishes, whilst all powers of $\mathbb{1} + x$, $x \in \bar{\mathscr{N}}$, are non-vanishing. Since $\bar{\mathscr{N}}^k = \bigoplus_{p\geq k} \Lambda_k(\mathbb{E}, \mathbb{K})$, we established (1.23), from which the rest of (i) follows.

(ii) and (iii) (1.24) follows from (1.23) and justifies the definition (1.20) of τ, which in turn implies (1.24a). For $\varphi \in \mathscr{G}_n^*$ and $\lambda \in \mathbb{K}$, $\varphi - \lambda\tau$ then vanishes on $\mathbb{1}$ iff $\lambda = \varphi(\mathbb{1})$, whence the existence and uniqueness of the decomposition (1.25a), and hence (1.25). We now show that τ is the unique character of $\mathscr{G}_n$: for $a = \tau(a)\mathbb{1} + \underline{a}$, $b = \tau(b)\mathbb{1} + \underline{b} \in \mathscr{G}_n$, $\underline{a}$, $\underline{b} \in \mathscr{N}$ we have

$$ab = \tau(a)\tau(b)\mathbb{1} + \tau(a)\underline{b} + \tau(b)\underline{a} + \underline{a}\,\underline{b}, \tag{1.27}$$

whence $\tau(ab) = \tau(a)\tau(b)$ by the uniqueness of (1.24a). Now let $\tau' \in \mathscr{G}_n^*$, $\tau' \neq 0$ be multiplicative, and let $w \in \mathscr{N}$, so that $w^N = 0$ for some $N \geq 1 : \tau'(w)^N$, and hence $\tau'(w)$, vanishes, hence τ' belongs to $\mathscr{N}^\perp$, hence equals $\lambda\tau$ for some $\lambda \in \mathbb{K}$: but $\tau'(\mathbb{1}^2) = \tau'(\mathbb{1}) = \lambda^2 = \lambda$, whence $\lambda = 1$ owing to $\tau' \neq 0$.

(iv) If $\tau(u) = 0$, u is nilpotent by (1.24a), hence not invertible. And if $\tau(u) \neq 0$, u is invertible, since this holds for $u/\tau(u)$ owing to (1.26), which readily follows from the fact that *uP* vanishes for $p > n$ (and for $p > 1$ if x is odd).

(v) Let $\varphi \in \mathscr{G}_n^*$ be non-vanishing on $\mathscr{N}^n$. To prove the equality $\varphi\mathscr{G}_n = \mathscr{G}_n^*$ it suffices to show that $(\varphi\mathscr{G}_n)^\perp = \{0\}$, i.e., to prove the implication

$$x \in \mathscr{G}_n : \varphi(x\theta_0^I) = 0 \qquad \text{for all } I \in \mathbb{L}_n \Rightarrow x = 0. \tag{1.28}$$

Let $x \in \mathscr{G}_n$: expressing $\varphi(x\theta_0^I) = 0$ for multiindices I of successive lengths n, $n-1, \ldots, 1$, it follows successively from (1.13) and $\varphi|_{\mathscr{N}^n} \neq 0$ that $x \in \mathscr{N}^k$ for $k = 1, 2, \ldots, n$, thus $x = 0$ from $\varphi(x\mathbb{1}) = 0$. Conversely, if $\varphi|_{\mathscr{N}^n} = 0$, we have $\varphi(ax) = 0$ for all $a \in \mathscr{G}_n$, and hence $\varphi\mathscr{G}_n \neq \mathscr{G}_n^*$.

(vi) Let $z = z^0 + z^1 \in \mathscr{Z}_n$, $z^0 \in \mathscr{G}_n^0$, $z^1 \in \mathscr{G}_n^1$: since $z^0 \in \mathscr{Z}_n$ we must have $z^1 \in \mathscr{Z}_n$, hence, for each $a \in \mathscr{G}_n^1$,

$$z^1 a = a z^1 = -z^1 a \Rightarrow z^1 a = 0. \tag{1.29}$$

Our result then follows from

[1.5] Lemma. *Let $z \in \Lambda\mathbb{E}^*$, $\mathbb{E} = \mathbb{K}^n$, be such that zx vanishes for each $x \in \mathbb{E}^*$: it follows that $z \in \Lambda_n(\mathbb{E}, \mathbb{K})$.*

Proof. Straightforward from (1.13).

We next investigate the generating systems of $\mathscr{G}_n$.

Generators and frames

[1.6] Definitions and notation. (i) A **generating system** of $\mathcal{N}$ is a subset $\eta \subset \mathcal{G}_n$ generating $\mathcal{N}$ as an algebra, i.e., such that the products of elements of η generate $\mathcal{N}$ linearly.

We denote by G_n the set of generating systems of $\mathcal{N}$.

(ii) A **frame of** $\tilde{\mathcal{G}}_n$ (resp. **frame** of $\mathcal{G}_n$) is an n-uple $\{\theta^i\}_{i=1,\dots,n}$ of elements of $\mathcal{N}$ (resp. of $\mathcal{N}^{\text{odd}} = \mathcal{N} \cap \mathcal{G}_n^1$) fulfilling the anticommuting relations

$$\theta^i\theta^j + \theta^j\theta^i = 0 \tag{1.30}$$

and such that

$$\theta^1\theta^2\dots\theta^n \neq 0. \tag{1.31}$$

We denote by $\tilde{\Theta}_n$, resp. Θ_n, the sets of frames of $\tilde{\mathcal{G}}_n$, resp. frames of $\mathcal{G}_n$.

(iii) We denote by Aut $\tilde{\mathcal{G}}_n$ the group of automorphisms (=multiplicative linear isomorphisms) of $\tilde{\mathcal{G}}_n$, and by Aut $\mathcal{G}_n$ the group of automorphisms[k] (parity preserving multiplicative linear isomorphisms) of $\mathcal{G}_n$.

[1.7] Remarks. (i) For odd θ^i's the left-hand side, cf. (1.30), equals the anticommutator $\{\theta^i, \theta^j\}$ which vanishes by graded commutativity: hence (1.30) can be omitted in the definition of the frames of $\mathcal{G}_n$. On the other hand the condition (1.31) expresses conveniently the desired algebraic independence of frame generators.

(ii) Obviously $\Theta_n \subset \tilde{\Theta}_n$. As it will be shown later, one has $\tilde{\Theta}_n \subset G_n$.

(iii) $\theta_0 = \{\theta_0^i\}_{i=1,\dots,n}$ as defined in (1.12) is a frame of $\mathcal{N}$ known to belong to G_n.

(iv) The generating systems of $\mathcal{G}_n$ (or $\tilde{\mathcal{G}}_n$) are obviously obtained by adjoining the unit to the elements of G_n.

The next two results describe a characterization of elements of G_n, resp. Θ_n and $\tilde{\Theta}_n$.

[1.8] Proposition. *The generating systems of $\mathcal{N}$ have at least n elements.*[l] *Choosing a particular $\eta_0 = \{\eta_0^i\}_{i=1,\dots,n} \in G_n$, the following are equivalent for $\eta = \{\eta^i\}_{i=1,-,n} \subset \mathcal{N}$:*

(i) $\eta \in G_n$;

(ii) $\{\eta^i \bmod \mathcal{N}^2\}_{i=1,\dots,n}$ *is a basis of* $\mathcal{N}/\mathcal{N}^2$;

(iii) *one has*[m]

$$\eta^i = \alpha_k^i\eta_0^k + \alpha^i \tag{1.32}$$

for some $\alpha_k^i \in \mathbb{K}$ *with* $\det(\alpha_k^i) \neq 0$, *and* $\alpha^i \in \mathcal{N}^2$, $i, k = 1, \dots, n$.

Proof. Let $\{\eta^i\}_{i=1,\dots,m} \in G_n$: the η^I, $I \in \mathbb{I}_n$, span $\mathcal{N}$ linearly, thus the $\eta^I \bmod \mathcal{N}^2$, $I \in \mathbb{I}_n$, span $\mathcal{N}/\mathcal{N}^2$ linearly; however, since $\eta^I \in \mathcal{N}^2$ for $|I| > 1$, the latter consisting of the $\eta^i \bmod \mathcal{N}^2$, $i = 1, \dots, m$: it follows that $m \geq \dim \mathcal{N}/\mathcal{N}^2 = n$ and also that (i) $\Rightarrow$ (ii).

[k] Identifying $\mathcal{G}_n$ and $\tilde{\mathcal{G}}_n$ as algebras, Aut $\mathcal{G}_n$ consists of the elements of Aut $\tilde{\mathcal{G}}_n$ which commute with π.

[l] And possibly exactly n, as does θ_0.

[m] We sum over repeated indices.

Proof of (ii) ⇔ (iii). Since $\{\eta_0^i \bmod \mathcal{N}^2\}_{i=1,\dots,n}$ is, by what precedes, another basis of $\mathcal{N}/\mathcal{N}^2$, (iii) means that for some $\alpha_k^i \in \mathbb{K}$ with $\det(\alpha_k^i) \neq 0$, we have

$$\eta^i \bmod \mathcal{N}^2 = \alpha_k^i(\eta_0^k \bmod \mathcal{N}^2), \tag{1.33}$$

synonymous with (1.32).

Proof of (iii) ⇒ (i). Since, as shown by the last step, statement (iii) does not depend upon the choice of $\eta_0 \in G_n$; it suffices to show that statement (iii) with the choice $\eta_0 = \theta_0$ (cf. (1.12)) implies (i). Now, let $a_k^i \in \mathbb{K}$, $i, k = 1, \dots, n$, be such that $\det(a_k^i) \neq 0$: obviously one has $\{\eta^i\} \in G_n$ iff $\{a_k^i \eta^k\} \in G_n$. Taking for (a_k^i) the matrix inverse of (α_k^i), we reduce the proof to the case $\alpha_k^i = \delta_k^i$. Let us thus show that $\{\theta_0^i + \alpha^i\}_{i=1,\dots,n} \in G_n$, i.e., that

$$\xi^i = \theta_0^i + \sum_{\substack{I \in \mathbb{L}_n \\ |I| > 1}} \lambda_I \varepsilon^I, \qquad \lambda_I \in \mathbb{K}, \tag{1.34}$$

generate $\mathcal{G}_n$ as an algebra. Utilizing (1.13), we can successively find sums of products of ξ^i yielding elements of the type

$$\theta_0^i + \sum_{\substack{I \in \mathbb{L}_n \\ |I| > p}} \mu_I \varepsilon^I \tag{1.35}$$

for $p = 2, 3, \dots$, whence our result after n steps.

[1.9] Proposition. *Let* $\theta = \{\theta_0^i\}_{i=1,\dots,n} \subset \mathcal{N}$.
The following are equivalent:
(i) $\theta \in \Theta_n$.
(ii) $\theta^i = \alpha(\theta_0^i)_{i=1,\dots,n}$, *for some* $\alpha \in \operatorname{Aut} \mathcal{G}_n$.
(iii) $\theta \subset \mathcal{G}_n^1$ *and* $\theta \in G_n$.
(iv) *One has*

$$\theta^i = \alpha_k^i \theta_0^k + \alpha^i \tag{1.36}$$

for some $\alpha_k^i \in \mathbb{K}$ *with* $\det(\alpha_k^i) \neq 0$, *and* $\alpha^i \in \mathcal{N}^3 \cap \mathcal{G}_n^1$, $i, k = 1, \dots, n$.
One the other hand, one has the inclusion $\tilde{\Theta}_n \subset G_n$. *And the following are equivalent*:
(ia) $\theta \in \tilde{\Theta}_n$.
(iia) $\theta^i = \alpha(\theta_0^i)_{i=1,\dots,n}$, *for some automorphism* α *of* $\tilde{\mathcal{G}}_n$.
(iiia) *One has* $\theta \in G_n$ *and* θ^i *fulfill* (1.30).

Proof. We prove (ia)⇔(iia) from which (i)⇔(ii) readily follows. Let $\theta \in \tilde{\Theta}_n$, we claim that the θ^I, $I \in \mathbb{L}_n$, are linearly independent. Indeed assume the existence of a relation

$$\sum_{I \in \mathbb{L}_n} \lambda_I \theta^I = 0, \qquad \lambda_I \in \mathbb{K}. \tag{1.37}$$

Given $J \in \mathbb{L}_n$, multiplication from the right by $\theta^{\bar{J}}$ yields, owing to (1.13), $\lambda_J \theta^{12\cdots n} = 0$, whence $\lambda_J = 0$ by (1.31). The $2^n - 1$ elements θ^I are thus a basis of $\mathcal{N}$ showing that one has $\theta \in G_n$. Setting $\alpha(\theta_0^I) = \theta^I$, $I \in \mathbb{L}$, and $\alpha(\mathbb{1}) = \mathbb{1}$, then defines an automorphism of $\mathcal{G}_n$, establishing (ia) $\Rightarrow$ (iia). The converse is obvious, as well as (iia) $\Rightarrow$ (iiia) and (ii) $\Rightarrow$ (iii). Check of (iii) $\Rightarrow$ (i) (resp. (iiia) $\Rightarrow$ (ia)): (1.30) follows from $\theta^I \in \mathcal{G}_n^1$ (resp. is assumed) whilst (1.31) follows from $\theta \in G_n$. Finally (iii) $\Leftrightarrow$ (iv) readily follows from Proposition [1.8].

The next Lemma is a key result for the description of the frames and automorphisms of $\tilde{\mathcal{G}}_n$.

[1.10] Lemma. *Let $\theta = \{\theta_i\}_{i=1,\dots,n} \in \tilde{\Theta}_n$ and consider the decomposition*[n]

$$\theta_i = \theta_i^0 + \theta_i^1, \qquad i = 1, \dots, n \quad \begin{cases} \theta_i^0 \in \mathcal{G}_n^0 \\ \theta_i^1 \in \mathcal{G}_n^1 \end{cases}. \tag{1.38}$$

(i) *$\theta^1 = \{\theta_i^1\}_{i=1,\dots,n}$ is a frame of $\mathcal{G}_n$. In fact one has*

$$\theta_1^1 \theta_2^1 \dots \theta_n^1 = \theta_1 \theta_2 \dots \theta_n \tag{1.39}$$

stemming from

$$\theta_i^0 \theta_j^0 = \theta_i^0 \theta_j^1 + \theta_i^1 \theta_j^0 = 0, \qquad i, j = 1, \dots, n, \tag{1.40}$$

with the consequence

$$\theta_i \theta_j = \theta_i^1 \theta_j^1, \qquad i, j = 1, \dots, n, \tag{1.41}$$

itself implying that the products of an even number of θ_i generate $\mathcal{G}_n^0$ (hence that the even part $\mathcal{G}_n^0$ pertains to $\tilde{\mathcal{G}}_n$).

(ii) *There is a $\mu \in \tilde{\mathcal{G}}_n^1$ such that*

$$\theta_i^0 = \mu \theta_i^1 \quad (\text{hence } \theta_i = (\mathbb{1} + \mu)\theta_i^0), \quad i = 1, \dots, n. \tag{1.42}$$

Proof. (i) We have

$$\theta_i \theta_j = \theta_i^0 \theta_j^0 + \theta_i^0 \theta_j^1 + \theta_i^1 \theta_j^0 + \theta_i^1 \theta_j^1, \tag{1.43}$$

$$0 = \theta_i \theta_j + \theta_j \theta_i = 2(\theta_i^0 \theta_j^0 + \theta_i^0 \theta_j^1 + \theta_i^1 \theta_j^0), \tag{1.44}$$

whence (1.40) and (1.41), which in turn implies

$$\theta_1 \theta_2 \dots \theta_k = \theta_1^1 \theta_2^1 \dots \theta_k^1, \qquad k \text{ even}, \qquad k \le n. \tag{1.45}$$

[n] We identify $\mathcal{G}_n$ and $\tilde{\mathcal{G}}_n$ as algebras.

For n odd, $n = 2p + 1$, we have thus

$$\theta_1\theta_2\dots\theta_{2p+1} = \theta_1^1\theta_2^1\dots\theta_{2p}^1(\theta_{2p+1}^0 + \theta_{2p}^1)$$

$$= \theta_1^1\theta_2^1\dots\theta_{2p}^1\theta_{2p+1}^1\,, \tag{1.46}$$

since $\mathcal{N}^{2p+1} \subset \mathcal{G}_{2p+1}^1$, we proved (1.39).

(ii) For the proof we will need the straightforward°

[1.11] Sublemma. *Let* $\{\theta^i\}_{i=1,\dots,n} \in \tilde{\Theta}_n$, *use the multiindex notation* (1.11) *and let* $K \in \mathbb{L}_n$: *for* $a \in \mathcal{G}_n$, *we have the equivalence*

$$a\theta^k = 0 \Leftrightarrow a = \sum_{\substack{I\in\mathbb{L}\\ I\cap K\neq\varnothing}} \lambda_I\theta^I \qquad \textit{for some } \lambda_I \in \mathbb{K}\,. \tag{1.47}$$

This sublemma shows from the second equation (1.40) for $j = i$ the existence of $\mu_i \in \mathcal{G}_n^1$, $i = 1, \dots, n$, such that

$$\theta_i^0 = \mu_i\theta_i^1\,, \qquad i = 1, \dots, n. \tag{1.48}$$

We prove (1.42) recursively: suppose that, for $k < n$, we have found $\mu^{(k)} \in \tilde{\mathcal{G}}_n^1$ with

$$\theta_i^0 = \mu^{(k)}\theta_i^1\,, \qquad i = 1, 2, \dots, k. \tag{1.49}$$

Owing to (1.40) we have, for each $i \leq k$,

$$0 = \theta_i^0\theta_{k+1}^1 + \theta_i^1\theta_{k+1}^i = \mu^{(k)}\theta_i^1\theta_{k+1}^1 + \theta_i^1\mu_{k+1}\theta_{k+1}^1$$

$$= (\mu^{(k)} - \mu_{k+1})\theta_i^1\theta_{k+1}^1\,, \tag{1.50}$$

and thus, by sublemma [1.11], the existence of λ_k, $\nu_k \in \mathcal{G}_n$ with

$$\mu^{(k)} - \mu_{k+1} = \lambda_k\theta_{k+1}^1 + \nu_k\theta_1^1\theta_2^1\dots\theta_k^1\,. \tag{1.51}$$

Thus our recurrence step is effected by taking

$$\mu^{(k+1)} = \mu^{(k)} - \nu_k\theta_1^1\theta_2^1\dots\theta_k^1 = \mu_{k+1} + \lambda_k\theta_{k+1}^1\,. \tag{1.52}$$

Generalized parities

[1.14] Definition. A **generalized parity** is a subspace $\tilde{\mathcal{G}}_n^1$ of $\mathcal{G}_n$ with the properties

$$\mathcal{G}_n = \mathcal{G}_n^0 \oplus \tilde{\mathcal{G}}_n^1\,, \tag{1.53a}$$

° Observe that $\theta^I\theta^K = 0$ if $I \cap K \neq \varnothing$, and that a set $\{I_k\}$ of distinct multiindices all fulfilling $I_k \cap K = \varnothing$ yields linearly independent elements $\theta^{I_k}\theta^K$. Note also the uniqueness of the sum on the right-hand side of (1.55).

$$\mathscr{G}_n^0\tilde{\mathscr{G}}_n^1 \subset \mathscr{G}_n^0\,, \qquad \tilde{\mathscr{G}}_n^1\tilde{\mathscr{G}}_n^1 \subset \mathscr{G}_n^0\,. \tag{1.53b}$$

We identify the subspace $\tilde{\mathscr{G}}_n^1$ with the corresponding involution $\tilde{\pi}$ of $\tilde{\mathscr{G}}_n$:

$$\tilde{\pi} = \begin{cases} \text{id on } \mathscr{G}_n^0 \\ -\text{id on } \tilde{\mathscr{G}}_n^1 \end{cases}. \tag{1.54}$$

[1.15] Proposition. *Let $\tilde{\mathscr{G}}_n^1$, with associated involution $\tilde{\pi}$, be a generalized parity. Then*
(i) *for $a = a^0 + a^1, b = b^0 + b^1 \in \tilde{\mathscr{G}}_n, a^0, b^0 \in \mathscr{G}_n^0, a^1, b^1 \in \tilde{\mathscr{G}}_n^1$, we have*

$$a^0b^0 = a^0b^1 + a^1b^0 = 0\,, \tag{1.55}$$

$$ab = a^1b^1 = -ba\,. \tag{1.56}$$

Hence $\mathscr{G}_n$ is graded commutative for the grading $\tilde{\pi}$.
(ii) *The generalized parity can be characterized as:*
(a) *the subspaces $\alpha(\tilde{\mathscr{G}}_n^1)$ (or involutions $\tilde{\pi} = \alpha \circ \pi \circ \alpha^{-1}$) with $\alpha \in$ Aut $\tilde{\mathscr{G}}_n^1$;*
(b) *the linear spans of products of an odd number of θ^i with $\theta \in \tilde{\Theta}_n$.*

Proof. (i) We have, from (1.53b),

$$\begin{cases} \mathscr{G}_n^0 \ni ab = a^0b^0 + a^1b^1 + a^0b^1 + a^1b^0 \\ \tilde{\mathscr{G}}_n^1 \ni a^0b + ba^0 = 2a^0b^0 + a^0b^1 + a^1b^0 \end{cases} \tag{1.57}$$

with $a^0b^0 + a^1b^1 \in \mathscr{G}_n^0$, hence $a^0b^1 + a^1b^0 \in \mathscr{G}_n^0 \cap \tilde{\mathscr{G}}_n^1 = \{0\}$; and then $a^0b^0 \in \mathscr{G}_n^0 \cap \tilde{\mathscr{G}}_n^1 = \{0\}$.

(ii) The fact that each $\theta \in \tilde{\Theta}_n$ yields a generalized parity as in (b) was established in [1.10] (i). Conversely, each generalized parity $\tilde{\mathscr{G}}_n^1$ arises as in (b) from any of the $\theta \in \tilde{\Theta}_n$ which it contains. The characterization (b) of generalized parities then follows from that of the type (a) via [1.9] (iia).

Automorphisms

[1.16] Definitions and notation. Besides the already defined groups Aut $\mathscr{G}_n \subset$ Aut $\tilde{\mathscr{G}}_n$, we need the normal subgroup Int $\tilde{\mathscr{G}}_n$ of internal automorphism of $\tilde{\mathscr{G}}_n$, obtained as σ_μ, $\mu \in \mathscr{N}$ where[p]

$$\sigma_\mu(a) = (\mathbb{1} + \mu)a(\mathbb{1} + \mu)^{-1}\,. \tag{1.58}$$

We shall denote by $\sigma(\mathscr{N}^{\text{odd}})$ the set of σ_μ with $\mu \in \mathscr{G}_n^1$.

[1.17] Proposition. (i) *$\sigma(\mathscr{N}^{\text{odd}})$ is an abelian subgroup of* Int $\mathscr{G}_n$: *we have*

$$\sigma_\mu\sigma_{\mu'} = \sigma_{\mu+\mu'}\,, \qquad \mu, \mu' \in \mathscr{N}^1\,. \tag{1.59}$$

[p] cf. [1.4] (iv). Note that for $\alpha \in$ Aut $\tilde{\mathscr{G}}_n$ one has $\alpha \circ \sigma_\mu \circ \alpha^{-1} = \sigma_{\alpha(\mu)}$.

(ii) *Each* $\alpha \in \mathrm{Aut}\, \tilde{\mathscr{G}}_n$ *is written uniquely as a product*

$$\alpha = \sigma_\mu \circ \varphi, \qquad \sigma_\mu \in \sigma(\mathscr{N}^{\mathrm{odd}}), \qquad \varphi \in \mathrm{Aut}\, \mathscr{G}_n. \tag{1.60}$$

We first prove

[1.18] Lemma. (i) *Let* $\mu = \mu^0 + \mu^1 \in \mathscr{N}$, $\mu^0 \in \mathscr{G}_n^0$, $\mu^1 \in \mathscr{G}_n^1$: *we have*

$$\begin{aligned} \sigma_\mu(a) &= a + (\mu a - a\mu)(\mathbb{1} + \mu)^{-1}, \qquad a \in \mathscr{G}_n \\ &= \begin{cases} a, & a \in \mathscr{G}_n^0 \\ a - 2a\mu^1(\mathbb{1} + \mu)^{-1}a, & a \in \mathscr{G}_n^1 \end{cases}; \end{aligned} \tag{1.61}$$

in particular, for $\mu \in \mathscr{G}_n^1$,

$$\begin{aligned} \sigma_\mu(a) &= a + (\mu a - a\mu), \qquad a \in \mathscr{G}_n \\ &= \begin{cases} a, & a \in \mathscr{G}_n^0 \\ (\mathbb{1} + 2\mu)a, & a \in \mathscr{G}_n \end{cases}. \end{aligned} \tag{1.61a}$$

(ii) *Let* $\mu, \mu' \in \mathscr{N}$: *we have the equivalences*[q]

$$\begin{cases} \sigma_\mu = \mathrm{id} \Leftrightarrow \mu \in \mathscr{Z}_n \Leftrightarrow \mu \textit{ preserves parity} \\ \sigma_\mu = \sigma_{\mu'} \Leftrightarrow (\mu' - \mu)(\mathbb{1} + \mu)^{-1} \in \mathscr{Z}_n, \end{cases} \tag{1.62}$$

in particular for μ, μ' *odd*

$$\sigma_\mu = \sigma_{\mu'} \Leftrightarrow \mu - \mu' \in \mathscr{Z}_n. \tag{1.62a}$$

Proof. (i) We have, for $\mu \in \mathscr{N}$ owing to (1.26) and $\mu^{n+1} = 0$,

$$\begin{aligned} \sigma_{-\mu}(a) &= (\mathbb{1} - \mu)a(\mathbb{1} - \mu)^{-1} = (\mathbb{1} - \mu)a(\mathbb{1} + \mu + \ldots + \mu^n) \\ &= a + a\mu + a\mu^2 + \ldots + a\mu^n + a\mu^{n+1} - \mu a - \mu a\mu - \ldots - \mu a\mu^{n-1} - \mu a\mu^n \\ &= a + (a\mu - \mu a)(1 + \mu + \ldots + \mu^n) = a + (a\mu - \mu a)(1 - \mu)^{-1}. \end{aligned} \tag{1.63}$$

For μ odd we have $\mu^2 = 0$ and $(\mathbb{1} + \mu)^{-1} = 1 - \mu$: (1.61) then becomes (1.61a).

(ii) $\sigma_\mu = \mathrm{id} \Leftrightarrow \mu \in \mathscr{Z}_n$ is obvious from (1.61).

The equivalence $\sigma_\mu = \sigma_{\mu'} \Leftrightarrow (\mu' - \mu)(\mathbb{1} + \mu)^{-1} \in \mathscr{Z}_n$ then follows from

$$\sigma_{\mu'}\sigma_\mu^{-1} = \sigma_{(\mu' - \mu)(\mathbb{1} + \mu)^{-1}}, \qquad \mu, \mu' \in \mathscr{N}, \tag{1.64}$$

[q] In fact we have the relation (1.64).

in consequence of $(\mathbb{1} + \mu')(\mathbb{1} + \mu)^{-1} = \mathbb{1} + (\mu' - \mu)(\mathbb{1} + \mu)^{-1}$ (multiply from the right by $(1 + \mu)$). For $\mu \cdot \mu'$ odd,

$$(\mu' - \mu)(\mathbb{1} + \mu)^{-1} = (\mu' - \mu)(\mathbb{1} - \mu) = \mu' - \mu - \mu'\mu \tag{1.65}$$

belongs to $\mathscr{Z}_n$ iff $\mu' - \mu$ does. Let now σ_μ be parity-preserving; one has $\sigma_\mu = \pi \circ \sigma_\mu \circ \pi^{-1} = \sigma_{\pi(\mu)}$, hence by (1.62) both $(\mu - \pi\mu)(\mathbb{1} + \pi\mu)^{-1}$ and $(\mu - \pi\mu)(\mathbb{1} + \pi\mu)^{-1}$ must belong to $\mathscr{Z}_n$. This also holds for

$$(\mu - \pi\mu)[(\mathbb{1} + \mu)^{-1} + (\mathbb{1} + \pi\mu)^{-1}] = 4\mu^1(\mathbb{1} + \mu^0)^{-1}, \tag{1.66}$$

causing μ^1, and thus μ, to lie in $\mathscr{Z}_n$ whence $\sigma_\mu = \mathrm{id}$.

Proof of Proposition [1.17]. (i) From the definition (1.58), we have

$$\sigma_\mu \sigma_{\mu'} = \sigma_{\mu + \mu' + \mu\mu'}; \tag{1.67}$$

hence (1.59) follows from the second line (1.62), since we have, owing to $\mu^2 = \mu'^2 = 0$,

$$\mu\mu'(1 + \mu + \mu')^{-1} = \mu\mu'(1 - \mu - \mu') = \mu\mu' \in \mathscr{G}_n^0 \subset \mathscr{Z}_n. \tag{1.68}$$

(ii) Let $\{\theta_i = \alpha(\theta_0^i)\}_{i=1,\ldots,n}$ be the frame of $\tilde{\mathscr{G}}_n$ obtained by applying α to the frame θ_0^i of $\mathscr{G}_n$ (cf. (1.12)). With θ_i^1 the odd part of θ_i, we know from [1.10] (ii) the existence of $\mu \in \mathscr{N}^{\text{odd}}$ with

$$\theta_i = (\mathbb{1} + 2\mu)\theta_i^1 = \sigma_\mu(\theta_i^1) \tag{1.69}$$

(cf. (1.61a)). But $\{\theta_i^1\}_{i=1,\ldots,n}$ is, by [1.10] (i), a frame of $\mathscr{G}_n$ which, according to [1.9] (ii) is obtained as $\theta_i^1 = \varphi(\theta_0^i)$ with $\varphi \in \mathrm{Aut}\,\mathscr{G}_n$. We thus have $\alpha(\theta_0^i) = \sigma_\mu \circ \varphi(\theta_0^i)$, whence (1.60) since $\{\theta_0^i\} \in G_n$. The existence of another decomposition $\alpha = \sigma_{\mu'} \circ \varphi'$, $\mu' \in \mathscr{N}^1$, $\varphi' \in \mathrm{Aut}\,\mathscr{G}_n$ would imply

$$\sigma_{\mu'}^{-1} \circ \sigma_\mu = \sigma_{\mu - \mu'} = \varphi' \circ \varphi^{-1} \in \mathrm{Int}\,\tilde{\mathscr{G}}_n \cap \mathrm{Aut}\,\mathscr{G}_n, \tag{1.70}$$

whence $\mu = \mu'$ and $\varphi = \varphi'$ by (1.62).

2. Derivations. Differential Calculus.

[2.1] Definition. We denote by $\mathbb{L}_n$ the set of **derivations** of $\mathscr{G}_n$ (in the graded sense): $\mathbb{L}_n$ consists of the $\mathbb{K}$-linear operators $\xi = \xi^0 + \xi^1$ of $\mathscr{G}_n$, sums of an **even derivation** ξ^0 (element of $\mathbb{L}_n^0$):

$$\begin{cases} \xi^0 \mathscr{G}^p \subset \mathscr{G}^p, & p \in \mathbb{Z}/2 \\ \xi^0(ab) = \xi^0(a)b + a\xi^0(b) & a, b \in \mathscr{G}_n \end{cases} \tag{2.1}$$

and an **odd derivation** ξ^1 (element of $\mathbb{L}_n^1$):

$$\begin{cases} \xi^1 \mathscr{G}^p \subset \mathscr{G}^{p+1}, & p \in \mathbb{Z}/2 \\ \xi^1(ab) = \xi^1(a)b + (-1)^{\partial a} a \xi^1(b) & b \in \mathscr{G}_n,\ a \in \mathscr{G}_n^{\cdot} \end{cases} . \tag{2.2}$$

The next proposition states, for $\mathbb{L}_n$, known general facts of derivations of super-commutative algebras (subsumed in the concept of "graded Lie Cartan pair" (cf. Appendix C and [2])).

[2.1] Definition. With the graded commutator of derivations as the bracket,

$$[\xi, \eta] = \xi\eta - (-1)^{\partial\xi\partial\eta}\eta\xi, \qquad \xi, \eta \in \mathbb{L}_n^{\cdot}, \tag{2.3}$$

$\mathbb{L}_n$ is a Lie superalgebra. Moreover the bilinear product $a\xi$ given by

$$(a\xi)(b) = a\xi(b), \qquad a, b \in \mathscr{G}_n, \quad \xi \in \mathbb{L}_n \tag{2.4}$$

belongs to $\mathbb{L}_n$, and gives rise to the properties

$$a\xi \in \mathbb{L}_n^{\partial a + \partial\xi}, \qquad a \in \mathscr{G}_n^{\cdot}, \quad \xi \in \mathbb{L}_n^{\cdot}, \tag{2.5}$$

$$\mathbb{1}\xi = \xi, \qquad \xi \in \mathbb{L}_n, \tag{2.6}$$

$$a(b\xi) = (ab)\xi, \qquad a, b \in \mathscr{G}_n, \quad \xi \in \mathbb{L}_n, \tag{2.7}$$

and

$$[\xi, a\eta] = (-1)^{\partial a + \partial\xi} a[\xi, \eta] + \xi(a)\eta \quad \begin{cases} \xi \in \mathbb{L}_n^{\cdot} \\ a \in \mathscr{G}_n^0 \\ \eta \in \mathbb{L}_n \end{cases} . \tag{2.8}$$

These properties characterize the pair $(\mathbb{L}_n, \mathscr{G}_n)$ as so-called injective graded Lie–Cartan pair.[r]

In addition, the pair $(\mathbb{L}_n, \mathscr{G}_n)$ possesses a number of features which we now describe.

[2.3] Proposition. *The $\mathscr{G}_n$-module $\mathbb{L}_n$ is a free module isomorphic to $(\mathscr{G}_n)^n$. Specifically, given $\theta \in \Theta_n$ specified by $\theta^i = \theta(\varepsilon^i) = \alpha \circ \theta_0(\varepsilon^i)$, $\alpha \in$ Aut $\mathscr{G}_n$ (cf. (1.12) and [1.9] (iia)), the definition*

$$\frac{\partial}{\partial\theta^i} = \theta \circ i(e_i) \circ \theta^{-1}, \qquad i = 1, \ldots, n, \tag{2.9}$$

with $\{e_i\} \subset \mathbb{E}$ the dual base of $\{\varepsilon^i\}$, and $i(e_i)$, the interior product

[r] And allow in particular to construct the graded commutative differential algebra $(\Lambda_{\mathscr{G}_n}(\mathbb{L}_n, \mathscr{G}_n), \wedge, \partial, d)$; see below [2.6].

$$\begin{cases} i(e_i)\lambda(x_1,\ldots,x_{n-1}) = \lambda(e_i,x_1,\ldots,x_n) \\ \lambda \in \Lambda^n E^*, \qquad n \geq 1, \qquad x_1,\ldots,\chi_{n-1} \in \mathbb{E} \\ (with\ i(e_i) = 0\ on\ \Lambda^0 E^* = \mathbb{K}) \end{cases} \tag{2.10}$$

yields odd derivations $\frac{\partial}{\partial\theta^i}$ *of* $\mathbb{L}_n$, *building a module basis of the latter; owing to the duality relations*

$$\frac{\partial}{\partial\theta^i}(\theta^k) = \delta_i^k \mathbb{1}, \tag{2.11}$$

each $\xi \in \mathbb{L}_n$ *has the unique decomposition*[s]

$$\xi = \xi(\theta^i)\frac{\partial}{\partial\theta^i}. \tag{2.12}$$

Moreover the $\frac{\partial}{\delta\theta^i}$ *mutually anticommute*:

$$\left[\frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial\theta^j}\right] = 0, \qquad i, j = 1, 2, \ldots, n. \tag{2.13}$$

Proof. The "annihilation operators" $i(x)$, $x \in \mathbb{E}$, are known to be derivations of $\Lambda^*(\mathbb{E})$ (of n-grade -1, thus odd): one has thus $\frac{\partial}{\partial\theta^i} \in \mathbb{L}_n^1$, $i = 1, \ldots, n$. Check of (2.11): we have

$$\begin{aligned} \frac{\partial}{\partial\theta^i}(\theta^k) &= \theta \circ i(e_i) \circ \theta^{-1}\theta(\varepsilon^k) = \theta\{i(e_i)\varepsilon^k\} \\ &= \theta\{\delta_i^k \mathbb{1}_{\Lambda E^*}\} = \delta_i^k \mathbb{1}. \end{aligned} \tag{2.14}$$

The equalities between derivations (2.12), (2.13) need only be checked on a system of generators: now we have, on the one hand,

$$\xi(\theta^i)\frac{\partial}{\delta\theta^i}(\theta^k) = \xi(\theta^i)\delta_i^k \mathbb{1} = \xi(\theta^k), \tag{2.15}$$

and further, since derivations vanish on the unit,

$$\frac{\partial}{\partial\theta^i}\left\{\frac{\partial}{\partial\theta^j}(\theta^k)\right\} = \frac{\partial}{\partial\theta^i}(\delta_j^k \mathbb{1}) = 0. \tag{2.16}$$

[s] Here and in what follows we sum over repeated indices.

[2.4] Remark. The duality relations (2.12) can be interpreted as follows: with the θ^i as above and $\mathbb{L}_n^*$ the dual[t] of the $\mathscr{G}_n$-module $\mathbb{L}_n$, we introduce the following[u] $d\theta^i \in \mathbb{L}_n^*$:

$$d\theta^i(\xi) = (-1)^{\partial\xi}\xi(\theta^i), \qquad \xi \in \mathbb{L}_n^* . \tag{2.17}$$

The $\frac{\partial}{\partial\theta^i}$ and $d\theta^i$, $i = 1, \ldots, n$, are then the dual bases of the dual $\mathscr{G}_n$-modules $\mathbb{L}_n$ and $\mathbb{L}_n^*$ in the sense

$$(d\theta^i)\left(\frac{\partial}{\partial\theta^k}\right) = -\delta_k^i \mathbb{1}, \qquad i, k = 1, \ldots, n. \tag{2.17a}$$

Note also that we have (in analogy with usual differential geometry):[v]

$$da = \frac{\partial a}{\partial\theta^i} \wedge d\theta^i, \qquad a \in \mathscr{G}_n . \tag{2.18}$$

Indeed, using the first relation (C.6) of Appendix C and (1.82), we have

$$\left(\frac{\partial a}{\partial\theta^i} \wedge d\theta^i\right)(\xi) = (-1)^{1+\partial a}\frac{\partial a}{\partial\theta^i} d\theta^i(\xi) = (-1)^{(1+\partial a)\partial\xi} d\theta^i(\xi)\frac{\partial a}{\partial\theta^i}$$

$$= (-1)^{\partial a\partial\xi}\xi(\theta^i)\frac{\partial a}{\partial\theta^i} = (-1)^{\partial a\partial\xi}\xi(a). \tag{2.19}$$

[2.5] Remark. Specifications of a frame $\theta = \{\theta^i\} \in \Theta_n$ of $\mathscr{G}_n$ makes $\mathscr{G}_n$ a "Fock space" with "one particle space" in the linear span of the θ^i. In this aspect, the derivations $\frac{\partial a}{\partial\theta^i}$ appear as "annihilators", whose corresponding "creators" are the product $\theta^i_\cdot$ to the left by the θ^i. One has the familiar anticommutation relations

$$\frac{\partial}{\partial\theta^i}\frac{\partial}{\partial\theta^j} + \frac{\partial}{\partial\theta^j}\frac{\partial}{\partial\theta^i} = 0, \tag{2.13a}$$

$$(\theta^i_\cdot)(\theta^j_\cdot) + (\theta^j_\cdot)(\theta^i_\cdot) = 0, \qquad i, j = 1, \ldots, n, \tag{2.20}$$

$$(\theta^i_\cdot)\left(\frac{\partial}{\partial\theta^j} + \frac{\partial}{\partial\theta^j}\right)(\theta^i) = \delta_j^i, \tag{2.21}$$

respectively following from (2.13), the graded commutativity of $\mathscr{G}_n$, and the derivation property of $\frac{\partial}{\partial\theta^j}$ (which in this particular case also yields a "partial integration" formula).

[t] Set of $\mathscr{G}_n$-linear forms over $\mathbb{L}_n$.
[u] $d\theta^i$ is the image of θ^i under the differential of $\Lambda^*\mathscr{G}_n(\mathbb{L}, \mathscr{G}_n)$, see below.
[v] The wedge product is that of $\Lambda^*\mathscr{G}_n(\mathbb{L}_n, \mathscr{G}_n)$, cf. (C.6) in Appendix C.

Proposition [2.3] assigns to the frame $\{\theta^i\}$ of $\mathscr{G}_n$ odd bases $\frac{\partial}{\partial\theta^j}$ of the free $\mathscr{G}_n$-module $\mathbb{L}^n$. The latter are in fact characterized amongst the odd bases of $\mathbb{L}_n$ by the property (2.13) (in analogy to what happens in usual differential geometry with the bases $\frac{\partial}{\partial x^j}$ coordinate systems). This property will follow from the Koszul formula and the Poincaré Lemma, [2.8] and [2.9], below for the statement, and the proof which we need to describe the equipment will be furnished by the Lie Cartan pair[w] $(\mathbb{L}_n, \mathscr{G}_n)$.

[2.6] Definitions. We consider the space

$$\Lambda_n^* = \Lambda_{\mathscr{G}_n}^*(\mathbb{L}_n, \mathscr{G}_n) \tag{2.22}$$

algebraic direct sum

$$\Lambda_n^* = \bigoplus_{k\in\mathbb{N}} \Lambda_n^k \tag{2.23}$$

of the space

$$\Lambda_n^k = \Lambda_{\mathscr{G}_n}^k(\mathbb{L}_n, \mathscr{G}_n) \qquad (\Lambda_n^0 = \mathscr{G}_n), \tag{2.24}$$

of $\mathscr{G}_n$-valued, graded alternate k-$\mathscr{G}_n$ linear forms over $\mathbb{L}_n$ (the form λ is **graded alternate** if its value changes by a factor $(-1)^{\partial\xi_i\partial\xi_{i+1}}$ upon exchange of consecutive arguments $\xi_i, \xi_{i+1} \in \mathbb{L}_n$, $i = 1, \ldots, k-1$).

The **total grading** of $\lambda \in \Lambda_n^k$ is the sum $\partial\lambda = k + \partial_0\lambda$ of its order and its **intrinsic grading**

$$\partial_0\lambda = \partial\lambda(\xi_1, \ldots, \xi_k) - \sum_{i=1}^{k} \partial\xi_k, \qquad \xi_i \in \mathbb{L}_n^*, \qquad i = 1, \ldots, k. \tag{2.25}$$

Λ_n is equipped with the **graded wedge product** $\wedge$ defined as follows: for $\lambda \in \Lambda_n^{p\cdot}$, $\mu \in \Lambda_n^{q\cdot}$, $p, q \in \mathbb{N}$ we have

$$\lambda \wedge \mu = A_{p+q}(\lambda \otimes \mu), \qquad (2.27)(26)$$

where $\otimes$ is the graded tensor product

$$(\lambda \otimes \mu)(\xi_1, \ldots, \xi_{p+q}) = (-1)^{q\partial_0\lambda}\lambda(\xi_1, \ldots, \xi_p)\mu(\xi_{p+1}, \ldots, \xi_{p+q}) \tag{2.28}$$

and A_{p+q} is the graded alternator defined by

[w] $(\theta^i_\cdot)\frac{\partial}{\partial\theta^i}$ is the corresponding "particle number"—in our case the N-grading of the $\mathscr{G}_n$ determined by the frame θ.

$$A_k = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} [\sigma]. \tag{2.29}$$

Here Σ_k denotes the group of permutations of the k first integers acting on k-linear forms as follows: for $\xi_1, \ldots, \xi_k \in \mathbb{L}_n^0$, we have

$$([\sigma]\lambda)(\xi_1, \ldots, \xi_k) = \chi(\xi, \sigma)\lambda(\xi_{\sigma 1}, \ldots, \xi_{\sigma k}) \tag{2.30}$$

with

$$\chi(\xi, \sigma) = \chi(\sigma) \cdot (-1)^{\sum_{\substack{i<j \\ \sigma i > \sigma j}} (\partial \xi_i)(\partial \xi_j)} . \tag{2.31}$$

(Note that λ is graded alternate iff $\lambda = A\lambda$, and that Λ^* is the "graded exterior algebra" over $\mathbb{L}_n^*$.)

We equip Λ_n^* with a **differential** d, **interior products** $i(\xi)$, and **Lie derivatives** $L(\xi)$, $\xi \in \mathbb{L}_n$ defined as follows: for $\lambda \in \Lambda_n^p$, and $\xi, \xi_1, \ldots, \xi_{p+q} \in \mathbb{L}_n$ one has

(i) $d = d_0 + d\wedge$ with[x]

$$\begin{cases} (d_0\lambda)(\xi_1, \ldots, \xi_{p+1}) = \sum_{i \le i < j \le k} (-1)^{\alpha_{ij}} \lambda([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{p+1}) \\ \text{with } \alpha_{ij} = i + j + (\partial\xi_i + \partial\xi_j) \sum_{k=1}^{i-1} \partial\xi_k + \partial\xi_j \sum_{k=i+1}^{j-1} \partial\xi_k \end{cases} \tag{2.32}$$

and

$$\begin{cases} (d \wedge \lambda)(\xi_1, \ldots, \xi_{p+1}) = \sum_{i=1}^{p+1} (-1)^{\beta_i} \xi_i \{\lambda(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+1}) \\ \text{with } \beta_i = 1 + i + \partial\xi_i \left(\partial_0 \lambda + \sum_{k=1}^{i-1} \partial\xi_k \right) \end{cases} ; \tag{2.33}$$

(ii) $L(\xi) = L_0(\xi) + d(\xi)$, with

$$\begin{cases} \{L_0(\xi)\lambda\}(\xi_1, \ldots, \xi_p) = (-1)^{p\partial\xi+1} \sum_{i=1}^{p} (-1)^{\gamma_i} \lambda(\xi_1, \ldots, [\xi, \xi_i], \ldots, \xi_p) \\ \text{with } \gamma_i = \partial\xi \left(\partial_0 \lambda + \sum_{k=1}^{i-1} \right) \partial\xi_k \end{cases} \tag{2.34}$$

and

$$\{d(\xi)\lambda\}(\xi_1, \ldots, \xi_p) = (-1)^{p\partial\xi} \xi \{\lambda(\xi_1, \ldots, \xi_p); \tag{2.35}$$

[x] The caret $\wedge$ indicates a missing argument; δ_0, $L_0(\xi)$ and $i(\xi)$ vanish on $\Lambda_{\cdot n}^0$.

(iii) finally

$$\{i(\xi)\lambda\}(\xi_1,\ldots,\xi_{p-1}) = (-1)^{(p+\partial_0\lambda)\partial\xi}\lambda(\xi,\xi_1,\ldots,\xi_{p-1}). \tag{2.36}$$

[2.7] Proposition. *With the notation and definitions in* [2.6] *we have that* (i) $(\Lambda_n^*, \partial, \wedge, d)$ *is a bigraded differential algebra: namely an associative complex algebra with mutually commuting* $\mathbb{N}$*- and* $\mathbb{Z}/2$ *gradings, and a grade one,* ∂*-odd differentiation (i.e., d is a* ∂*-odd derivation of* $\mathbb{N}$*-grade one with vanishing square). Furthermore* Λ_n^* *is generated in grade zero (as a differential algebra), i.e., in each frame* $\{\theta^i\}$ *of* $\mathcal{G}_n$*, the* $\lambda \in \Lambda_n^p$ *are representable as sums of products*

$$\lambda = \lambda_{i_1\ldots i_p} d\theta^{i_1} \wedge \ldots \wedge d\theta^{i_p} \tag{2.37}$$

(note that the graded wedge product of the $d\theta^i$ *are symmetric owing to* $\partial\theta^i = 1$)

$$d\theta^i \wedge d\theta^j = -d\theta^j \wedge d\theta^i, \qquad i, j = 1,\ldots,n.$$

(ii) $L(\xi)$ *and* $i(\xi)$ *are derivations of* Λ^* *(w.r.t.* ∂) *of* ∂*-grade* $\partial\xi$ *and of respective* $\mathbb{N}$*-grades* 0 *and* -1.

(iii) *One has the following identities, where* $\xi, \eta \in L$*, and* $[\,,]$ *denotes graded commutators (w.r.t.* ∂):

$$[i(\xi), i(\eta)] = 0, \tag{2.38}$$

$$[\theta(\xi), \theta(\eta)] = \theta([\xi,\eta]), \tag{2.39}$$

$$[d, i(\xi)] = L(\xi), \tag{2.40}$$

$$[i(\xi), L(\eta)] = i([\xi,\eta]). \tag{2.41}$$

[2.8] Remark. We note the following expressions of the wedge product and of the differential in low grade: we have, for $a \in \Lambda_n^{0\cdot} = A$ and $\alpha \in \Lambda_n^1 = \mathcal{G}_n^*$:

$$\begin{cases} (a \wedge \varphi)(\xi) = (-1)^{\partial a}\varphi(\xi), & \begin{cases} a \in \Sigma^0(L,A)^{\cdot} = A^{\cdot},\ \varphi \in \Lambda^1(L,A) \\ \xi \in L \end{cases}, \\[2ex] (-1)^{\partial_0\varphi_1}(\varphi_1 \wedge \varphi_2)(\xi_1,\xi_2) = (-1)^{\partial_0\varphi_2\partial\xi_1}\varphi_1(\xi_1)\varphi_2(\xi_2), & \begin{cases} \varphi_1, \varphi_2 \in \Lambda^1(L, A^{\cdot}) \\ \xi_1, \xi_2 \in L^{\cdot} \end{cases} \\[2ex] \qquad -(-1)^{\partial\xi_2(\partial\xi_1+\partial_0\varphi_2)}\varphi_1(\xi_2)\varphi_2(\xi_1) & \end{cases} \tag{2.42}$$

and

$$\begin{cases} \delta_0 a = 0 \\ \delta_0 \alpha(\xi_1, \xi_2) = -\alpha([\xi_1, \xi_2]) \end{cases} \tag{2.43}$$

$$\begin{cases} (d \wedge a)(\xi_1) = (-1)^{\partial\xi_1 \partial a}\xi_1(a) \\ (d \wedge \alpha)(\xi_1, \xi_2) = (-1)^{\partial\xi_1 \partial_0 \alpha}\xi_1(\alpha(\xi_2)) - (-1)^{\partial\xi_2 \partial_0\alpha + \partial\xi_1}\xi_2(\alpha(\xi_1)) \end{cases} \tag{2.44}$$

[2.9] Lemma. *Let $\{\eta_i\}_{i=1,\ldots,n}$ be a base of the $\mathscr{G}_n$-module $\mathbb{L}_n$ consisting of n odd elements, with dual base $\{\omega^k\}_{k=1,\ldots,n}$ in $\mathbb{L}_n^*$:*

$$\omega^k(\eta_i) = \delta_i^k \mathbb{1}_{\mathscr{G}_n}, \qquad i, k = 1, \ldots, n. \tag{2.45}$$

The following are equivalent for elements

$$f_{lk}^i = f_{kl}^i \quad \text{of } \mathscr{G}_n, \qquad i, k, l = 1, 2, \ldots, n:$$

(i) *one has*

$$[\eta_k, \eta_l] = f_{kl}^i \eta_i \qquad k, l = 1, \ldots, n, \tag{2.46}$$

(ii) *one has*

$$d\omega^i - \tfrac{1}{2} f_{kl}^i \omega^k \wedge \omega^l, \qquad i = 1, \ldots, n. \tag{2.47}$$

Proof. Let $\eta_1, \eta_2 \in \mathbb{L}_n$ of respective grades $\partial\eta_1, \partial\eta_2$; $\omega_1, \omega_2 \in \Lambda_n^1$ of respective grades $\partial_0\omega_1$, $\partial_0\omega_2$ (thus of respective total grades $\partial\omega_1 = 1 + \partial_0\omega_1$, $\partial\omega_2 = 1 + \partial_0\omega_2$). We have according to (2.42), (2.43)

$$\begin{aligned}(-1)^{\partial_0\omega_1}(\omega_1 \wedge \omega_2)(\eta_1, \eta_2) = (-1)^{\partial_0\omega_2 \partial\eta_1}\omega_1(\eta_1)\omega_2(\eta_2) \\ -(-1)^{\partial\eta_2(\partial\eta_1 + \partial\omega_2)}\omega_1(\eta_2)\omega_2(\eta_1)\end{aligned} \tag{2.48}$$

and, for $\omega \in \Lambda'(L_n, \mathscr{G}_n)$ of intrinsic grade $\partial_0\omega$,

$$d\omega(\eta_1, \eta_2) = (-1)^{\partial\eta_1(\partial_0\omega)}\eta_1\{\omega(\eta_2) - (-1)^{\partial\eta_2(\partial_0\omega + \partial\eta_1)}\eta_2\{\omega(\eta_1)\} - \omega([\eta_1, \eta_2]). \tag{2.49}$$

Application of these formulae to η_i and ω^k yield (observing $\delta\eta_i = 1$, $\delta_0\omega^k = 1$)

$$(\omega^k \wedge \omega^l)(\eta_\alpha, \eta_\beta) = \delta_\alpha^k \delta_\beta^l + \delta_\beta^k \delta_\alpha^l, \qquad k, l, \alpha, \beta = 1, \ldots, n, \tag{2.50}$$

$$(d\omega^i)(\eta_\alpha, \eta_\beta) = -\omega^i([\eta_\alpha, \eta_\beta]), \qquad i, \alpha, \beta = 1, \ldots, n. \tag{2.51}$$

For elements $f^i_{kl} \in \mathscr{G}_n$ symmetric in their lower indices, (2.50) implies that we have

$$\tfrac{1}{2} f^i_{kl}(\omega^k \wedge \omega^l)(\eta_\alpha, \eta_\beta) = f^i_{kl}, \qquad i, \alpha, \beta = 1, \dots, n. \tag{2.52}$$

Assume that (2.46) holds (implying symmetries of the f^i_{kl} in their lower indices since $\mathbb{L}_n$ is a Lie superalgebra) then (2.51) reads to (2.45)

$$d\omega^i(\eta_\alpha, \eta_\beta) = -\omega^i(f^j_{\alpha\beta}\eta_j) = -f^j_{\alpha\beta}\delta^i_j \mathbb{1}_{\mathscr{G}_n} = -f^i_{\alpha\beta}, \tag{2.53}$$

yielding (2.47) by comparison with (2.52).

Conversely, if we assume (2.47), comparison of (2.51) with (2.50) yields

$$\omega^i([\mu_\alpha, \mu_\beta]) = -f^i_{\alpha\beta}, \tag{2.54}$$

whence, by the completeness relation of dual basis,

$$\eta_i \omega^i([\eta_\alpha, \eta_\beta]) = [\eta_\alpha, \eta_\beta] = f^i_{\alpha\beta}\eta_i. \tag{2.55}$$

[2.10] Proposition (the Poincaré Lemma). *The differential d is acyclic.*

The proof follows from

[2.11] Lemma. *With $\{\theta^i\}_{i=1,\dots,n} \in \tilde{\Theta}$, let*

$$Y_\theta = -(\theta^j_\wedge) i\left(\frac{\partial}{\partial\theta^j}\right). \tag{2.56}$$

Then Y_θ is an odd derivation of Λ^*_n. Moreover,

$$\chi_\theta = dY_\theta + Y_\theta d = (\theta^j_\wedge) L\left(\frac{\partial}{\partial\theta^j}\right) - (d\theta^j_\wedge) i\left(\frac{\partial}{\partial\theta^j}\right) \tag{2.57}$$

is an even derivation of Λ^*_n of $\mathbb{N}$-grade zero, commuting with d and invertible on all Λ^p_n with $p \geq 1$. In restriction to the latter, one thus gets the homotopy connection

$$dY_\theta X_\theta^{-1} + Y_\theta X_\theta^{-1} d = \mathbb{1}. \tag{2.58}$$

Proof. The derivation property of d entails

$$d(\theta^j_\wedge) + (\theta^j_\wedge)d = (d\theta^j_\wedge)_\wedge, \qquad j = 1, \dots, n, \tag{2.59}$$

whilst we have, by (2.40),

$$di\left(\frac{\partial}{\partial\theta^j}\right) - i\left(\frac{\partial}{\partial\theta^j}\right)d = L\left(\frac{\partial}{\partial\theta^j}\right), \qquad j = 1, \dots, n. \tag{2.60}$$

Relation (2.57) then follows:

$$dY_\theta = -d(\theta^j_\wedge)i\left(\frac{\partial}{\partial\theta^j}\right) = [(\theta^j_\wedge)d - (d\theta^j)_\wedge]i\left(\frac{\partial}{\partial\theta^j}\right)$$

$$= (\theta^j_\wedge)i\left[\left(\frac{\partial}{\partial\theta^j}\right)d + L\left(\frac{\partial}{\partial\theta^j}\right)\right] - (d\theta^i_\wedge)i\left(\frac{\partial}{\partial\theta^j}\right). \tag{2.61}$$

Now X_θ commutes with $d: dX_\theta = X_\theta d = dX_\theta d$: hence, in order to prove (2.58) we need only to show the claimed inversibility. This will result from the fact that *the second term on the right-hand side of* (2.57) *is nothing but the* $\mathbb{N}$*-grading of* Λ^*_n, whilst the first is positive. Indeed, we have, on the one hand, for a p-form λ and $\xi_1, \ldots, \xi_p \in \mathbb{L}^0_n$,

$$\left\{i\left(\frac{\partial}{\partial\theta_j}\right)\lambda\right\}(\xi_1,\ldots,\xi_{p-1}) = (-1)^{p+\partial_0\lambda}\lambda\left(\frac{\partial}{\partial\theta_j}(\xi_1,\ldots,\xi_{p-1})\right). \tag{2.62}$$

Hence, by (4.33) in [2], (2.17a) and (2.12),

$$\left\{(d\theta^j_\wedge)i\left(\frac{\partial}{\partial\theta^j}\right)\right\}\lambda(\xi_1,\ldots,\xi_p)$$

$$=(-1)^{1+\partial_0\lambda}\sum_{i=1}^{p}(-1)\partial\xi_i^{\left(\partial_0\lambda+1+\sum\limits_{k=1}^{i-1}\right)\partial\xi_i}[d\theta^j(\xi_i)=(-1)^{\partial\xi_i}\xi_i(\theta^j)]\lambda\left(\frac{\partial}{\partial\theta^j}\hat{\xi}_1,\ldots,\hat{\xi}_i,\ldots,\xi\right)$$

$$=(-1)^{1+\partial_0\lambda}\sum_{i=1}^{p}(-1)^{\partial_0\lambda+\partial\xi_i\sum\limits_{k=1}^{i-1}\partial\xi_k}\lambda(\xi_i,\xi_1,\ldots,\hat{\xi}_i,\ldots,\xi_p)$$

$$= -p\lambda(\xi_1,\ldots,\xi_p). \tag{2.63}$$

The first term on the right-hand side of (2.57) on the other hand consists of two contributions corresponding to $L = L_0 + d$. We have for λ as above, on the one hand, by (3.6) in [2],

$$\left\{d\left(\frac{\partial}{\partial\theta_j}\right)\lambda\right\}(\xi_1,\ldots,\xi_p) = (-1)p\frac{\partial}{\partial\theta^j}\{\lambda(\xi_1,\ldots,\xi_p)\}, \tag{2.64}$$

yielding for the d-contribution the $\mathbb{N}_\theta$-graded of $\mathscr{G}_n$ acting on the value of the form

$$\left\{(\theta^j_\wedge)d\frac{\partial}{\partial\theta_j}\lambda\right\}(\xi_1,\ldots,\xi_p) = \left(\theta^j\frac{\partial}{\partial\theta^j}\right)\lambda(\xi_1,\ldots,\xi_p), \tag{2.65}$$

and on the other hand, plugging into (3.5) of [2] the expression

$$\left[\frac{\partial}{\partial\theta^j},\xi\right] = \frac{\partial\xi(\theta^j)}{\partial\theta^k}\frac{\partial}{\partial\theta^j}, \tag{2.66}$$

the relations

$$\left\{L_0\left(\frac{\partial}{\partial\theta_j}\right)\lambda\right\}(\xi_1,\ldots,\xi_p) = (-1)^{p+1}\sum_{i=1}^{p}(-1)^{\partial_0\lambda+\sum_{i=1}^{p}\partial\xi_k} \lambda\left(\xi_1,\ldots,\xi_{i-1},\frac{\partial\xi_i(\theta^k)}{\partial\theta^j}\frac{\partial}{\partial\vartheta^k},\ldots,\xi_p\right) \tag{2.67}$$

and, by (4.32) in [2],

$$\left\{(\theta^j_\wedge)L_0\left(\frac{\partial}{\partial\theta_j}\right)\lambda\right\}(\xi_1,\ldots,\xi_p) = -\sum_{i=1}^{p}{}^{\partial_0\lambda+\sum_{k=1}^{i-1}\partial\xi_k}\theta^j\lambda\left(\xi_1,\ldots,\xi_{i-1},\frac{\partial\xi_i(\theta^k)}{\partial\theta^j}\frac{\partial}{\partial\theta^k},\ldots,\xi_p\right)$$

$$= -\lambda\left(\xi_1,\ldots,\frac{\partial\xi_i(\theta^k)}{\partial\theta^j}\xi_i(\theta^k),\ldots,\xi_p\right). \tag{2.68}$$

The first operator on the right-hand side of (2.57) is thus the difference of N_θ acting on the values $- N_\theta$ acting on the arguments expressed in (2.12). This together with (2.65) implies the claimed invertibility of X_θ. The situation is perhaps more transparent if looked at via the forms of the type (2.37), noting that all operators at hand, Y_θ, resp. X_θ and its two parts on the right-hand side of (2.57) are (an odd, resp. even) derivation (as composition of a derivation of a graded commutative algebra by wedging times an algebraic element). Property (2.65) then means vanishing on N-grade zero, and the following straightforward expression in N-grade one:

$$(d\theta^j_\wedge)i\left(\frac{\partial}{\partial\theta_j}\right)d\theta^\mu = -d\theta^\mu, \qquad \mu = 1,\ldots,n. \tag{2.63}$$

On the other hand, the first derivation on the right-hand side of (2.57) is easily checked to vanish on the $d\theta^k$, $k = 1,\ldots,n$, whilst reducing on $\mathscr{G}_n$ to the $d\wedge$ part easily checked to yield $\mathbb{N}_\vartheta$. We are now in a position to prove

[2.12] Proposition. *Let ξ_i, $i = 1,\ldots,n$ be an odd base of $\mathbb{L}_n$. The following are equivalent*:

(i) *there is a $\theta \in \tilde{\Theta}$ with $\xi_i = \dfrac{\partial}{\partial\theta^i}$*;

(ii) *the ξ^i-mutually anticommute*: $[\xi^i,\xi^j] = 0$ *for* $i, j = 1,\ldots,n$.

Proof. Straightforward from [2.9] and [2.10].

3. The Berezin Integral

[3.1] Definition. Let θ be a frame of $\tilde{\mathscr{G}}_n$, $\theta \in \tilde{\Theta}_n$. The corresponding **Berezin integral** I_θ is the element of $\tilde{\mathscr{G}}_n^*$ given by

$$I_\theta(a) = \frac{\partial}{\partial\theta^n}\frac{\partial}{\partial\theta^{n-1}}\cdots\frac{\partial}{\partial\theta^1}a, \tag{3.1}$$

(the right-hand side is an element of $\mathbb{K}$, since the derivations $\frac{\partial}{\partial\theta^k}$, $k = 1, \dots, n$ are of grade -1 for $\mathbb{N}$-grading of $\tilde{\mathscr{G}}_n$ determined by θ and since they vanish on $\mathbb{K}\mathbb{1}$). In fact we have

[3.2] Lemma. *Let $a \in \tilde{\mathscr{G}}_n$ be expressed as*

$$a = \sum_{I \in \mathbb{L}_n} a_I \theta^I \tag{3.2}$$

along the lexicographic base θ^I of $\tilde{\mathscr{G}}_n$. We have

$$I_\theta(a) = a_{1,2,\dots,n} \tag{3.3}$$

(*coefficient of the "top element" $\theta^1\theta^2 \dots \theta^n$*).

Proof. One has

$$\frac{\partial}{\partial\theta^n}\frac{\partial}{\partial\theta^{n-1}}\cdots\frac{\partial}{\partial\theta^1}\theta^I = \begin{cases} 0 \text{ if the first index of } I \neq 1 \\ \theta_{I\setminus\{1\}} \text{ otherwise} \end{cases}, \tag{3.4}$$

where $I\setminus J \in \mathbb{L}_n$ denotes a relative complement for $J \in \mathbb{L}_n$; further, if $\{1\ 2 \dots p\} \cap I = \emptyset$,

$$\frac{\partial}{\partial\theta^n}\frac{\partial}{\partial\theta^{n-1}}\cdots\frac{\partial}{\partial\theta^1}\theta^I = \begin{cases} 0 \text{ if the first index of } I \neq p+1 \\ \theta_{I\setminus\{p+1\}} \text{ otherwise} \end{cases}, \tag{3.5}$$

whence our proof after n steps.

[3.3] Definitions and notation. (i) We denote by $\tilde{\mathscr{I}}_n$ the set of Berezin integral of $\tilde{\mathscr{G}}_n$, $n \in \mathbb{N}$:

$$\tilde{\mathscr{I}}_n = \{I_\theta;\ \theta \in \tilde{\Theta}_n\}, \tag{3.6}$$

by $\mathscr{I}_n$ the subset of the latter obtained with frames of $\mathscr{G}_n$:

$$\mathscr{I}_n = \{I_\theta;\ \theta \in \Theta_n\}. \tag{3.7}$$

(ii) We denote by $\mathbf{\Gamma}_n$ the set of generators of $\tilde{\mathscr{G}}_n^*$ (as a $\mathscr{G}_n$-module, cf. [1.4] (ii)):

$$\mathbf{\Gamma}_n = \{\varphi \in \mathscr{G}_n^*;\ \mathscr{G}_n^* = \mathscr{G}_n\varphi(\text{or } \varphi\mathscr{G}_n)\} = \{\varphi \in \mathscr{G}_n^*;\ \varphi|_{\mathscr{N}^n} \neq 0\}, \tag{3.8}$$

and by $\mathbf{\Gamma}_n^0$ the set of the latter vanishing on the unit:

$$\mathbf{\Gamma}_n^0 = \mathbf{\Gamma}_n \cap \{\mathbb{1}\}^\perp. \tag{3.9}$$

(iii) We denote by $\mathbb{K}^* \oplus \mathscr{N}_n$ the set of invertibles of $\mathscr{G}_n$ ($\mathbb{K}^* = \mathbb{K}\setminus 0$, cf. [1.4] (iv).)

(iv) We make $\mathbf{\Gamma}_n \times \mathbf{\Gamma}_n$ into a groupoid with the convention

$$\begin{cases} \exists(\varphi_1,\varphi_2)(\varphi_2',\varphi_3) \Leftrightarrow \varphi_2 = \varphi_2', \qquad \varphi_1,\varphi_2,\varphi_2',\varphi_3 \in \mathbf{\Gamma}_n \\ \text{and if so, } (\varphi_1,\varphi_2)(\varphi_2,\varphi_3) = (\varphi_1,\varphi_3). \end{cases} \tag{3.10}$$

(v) We make $\mathbf{\Gamma}_n \times (\mathbb{K}^* \oplus \mathcal{N}_n)$ into a groupoid with the convention

$$\exists(\varphi_1, \mathbb{1} + \nu_1)(\varphi_2, \mathbb{1} + \nu_2) \Leftrightarrow \varphi_1(\mathbb{1} + \nu_1) = \varphi_2,\ \varphi_1,\varphi_2 \in \mathbb{L},\ \nu_1,\nu_2 \in \mathcal{N}_n \tag{3.11}$$

and if so,

$$(\varphi_1, \mathbb{1} + \nu_1)(\varphi_2, \mathbb{1} + \nu_2) = (\varphi_1, \mathbb{1} + \nu_1 + \nu_2 + \nu_1\nu_2).$$

(vi) With $\varphi_1, \varphi_2 \in \mathbf{\Gamma}_n$ we define $\delta_{\varphi_1,\varphi_2} \in \mathbb{K}^* \oplus \mathcal{N}_n$ by requiring

$$\varphi_2 = \varphi_1\delta_{\varphi_1,\varphi_2}, \qquad \text{i.e.,} \qquad \varphi_2(x) = \varphi_1(\delta_{\varphi_1,\varphi_2}x). \tag{3.12}$$

[3.4] Lemma. *With the definition and notation in* [2.3] *we have*

(i) $\mathbf{\Gamma}_n \supset \mathbf{\Gamma}_n^0 \supset \tilde{\mathcal{I}}_n \supset \mathcal{I}_n$;

(ii) *the map*

$$\mathbf{\Gamma}_n \times \mathbf{\Gamma}_n \ni \varphi_1,\varphi_2 \to \delta_{\varphi_1,\varphi_2} \tag{3.13}$$

is a character of the groupoid $\mathbf{\Gamma}_n \times \mathbf{\Gamma}_n$ *with values in* $\mathbb{K}^* \oplus \mathcal{N}_n$ *(in fact covering the latter). We have*

$$\begin{cases} \delta_{\varphi_1,\varphi_2}\cdot\delta_{\varphi_2,\varphi_3} = \delta_{\varphi_1,\varphi_3} \\ \delta_{\varphi_2,\varphi_1} = \delta^{-1}_{\varphi_1\varphi_2} \end{cases} \qquad \varphi_1,\varphi_2,\varphi_3 \in \mathbf{\Gamma}; \tag{3.14}$$

(iii) *the map*

$$(\varphi_1,\varphi_2) \to (\varphi_1, \delta_{\varphi_1,\varphi_2}) \tag{3.15}$$

is an isomorphism $\mathbf{\Gamma}_n \times \mathbf{\Gamma}_n \to \mathbf{\Gamma}_n \times (\mathbb{K}^* \oplus \mathcal{N}_n)$ *of groupoid mapping the subgroupoid* $\mathbf{\Gamma}_n^0 \times \mathbf{\Gamma}_n^0$ *onto the subgroupoid*

$$\{(\varphi, \mathbb{1} + \nu);\ \varphi \in \mathbf{\Gamma},\ \nu \in \mathcal{N},\ \varphi(\nu) = 0\} \tag{3.16}$$

of $\mathbf{\Gamma}_n \times (\mathbb{K}^* \oplus \mathcal{N}_n)$.

Proof. (i) is obvious. (ii) Remember (cf. [1.4] (v)) that, for $\varphi \in \mathbf{\Gamma}_n$, the map $x \in \tilde{\mathcal{G}}_n \to \varphi x \in \mathcal{G}_n^*$ is a linear isomorphism. Hence, for $\varphi_1, \varphi_2 \in \mathbf{\Gamma}_n$, (3.12) uniquely determines $\delta_{\varphi_1,\varphi_2} \in \tilde{\mathcal{G}}_n$, with, for $\varphi_1, \varphi_2, \varphi_3 \in \mathbf{\Gamma}_n$, and $x \in \tilde{\mathcal{G}}_n$,

$$\varphi_3(x) = \varphi_2(\delta_{\varphi_2,\varphi_3}x) = \varphi_1(\delta_{\varphi_1,\varphi_2}\delta_{\varphi_2,\varphi_3}x), \tag{3.17}$$

proving the first line of (3.14). The second line follows from the choice $\varphi_3 = \varphi_1$, combined with $\delta_{\varphi_1,\varphi_1} = \mathbb{1}$.

The map (3.13) is onto the irreversible of $\tilde{\mathscr{G}}_n$, since, for one of the latter and $\varphi \in \mathbf{\Gamma}_n$, one has obviously $\varphi a \in \mathbf{\Gamma}_n$: this entails the fact that (3.15) is an isomorphism: $\mathbf{\Gamma}_n \times \mathbf{\Gamma}_n \to \mathbf{\Gamma}_n \times (\mathbb{K}^* \oplus \mathscr{N}_n)$. Finally, $\mathbf{\Gamma}_n^0 \times \mathbf{\Gamma}_n^0$ is obviously a subgroupoid of $\mathbf{\Gamma}_n \times \mathbf{\Gamma}_n$; and $(\varphi, \lambda + v)$, $\varphi \in \mathbf{\Gamma}_n$, $v \in \mathscr{N}_n$, is the image of the latter under the map (3.15) iff $\varphi \in \mathbf{\Gamma}_n$ and $\varphi(1 + v) \in \mathbf{\Gamma}_n$, i.e., $\varphi(1 + v) = \varphi(v) = 0$.

The next result both identifies the general Berezin integrals with the generators of $\tilde{\mathscr{G}}_n$ vanishing on $\mathbb{1}$ for even n; and describe the change of variables between restricted Berezin integrals—we combine these two results because they have a common root.

[3.4] Theorem. *With the definitions and notation of* [3.3] *we have that*

(i) $\tilde{\mathscr{I}}_{2p} = \mathbf{\Gamma}_{2p}^0$, $p \in \mathbb{N}$;

(ii) *for* $\theta, \bar{\theta} \in \Theta_n$, $n \in \mathbb{N}$, *we have the "Jacobian"*

$$\delta_{I_\theta, I_{\bar\theta}} = \mathrm{Det}\left[\frac{\partial\theta^i}{\partial\bar\theta^j}\right]\left(\textit{denoted}\left|\frac{\partial\theta}{\partial\bar\theta}\right|\right), \tag{3.18}$$

(*in other terms, the rule*

$$I_{\bar\theta} = I_\theta\left|\frac{\partial\theta}{\partial\bar\theta}\right| \tag{3.18a}$$

where the determinant is well-defined owing to mutual commutation of the $\dfrac{\partial\theta^i}{\partial\bar\theta^j}$).

For the proof we use

[3.5] Lemma. *With* $\theta \in \tilde\Theta_n$, $\tilde\pi$ *the corresponding generalized parity, and* $v \in \mathscr{N}$, $\tilde\pi$*-odd or* $\tilde\pi$*-even, defining* $(\mathbb{1} + v)\theta \in \tilde\Theta_n$ *as*

$$\{(\mathbb{1} + v)\theta\}^i = (1 + v)\theta^i, \qquad i = 1, \ldots, n, \tag{3.19}$$

we have, for $p \in \mathbb{N}$, $2p + 1 < n$,

$$I_{(1+\lambda\theta^1\ldots\theta^{2p+1})\theta} = \begin{cases} I_\theta(\mathbb{1} - \lambda\theta^1\ldots\theta^{2p+1}), & n \text{ even} \\ I_\theta, & n \text{ odd} \end{cases}, \tag{3.20}$$

and, for $p \in \mathbb{N}$, $2p \leq n$,

$$I_{(1+\lambda\theta^1\ldots\theta^{2p+1})\theta} = I_\theta(\mathbb{1} - (n - 2p)\lambda\theta^1\ldots\theta^2 p). \tag{3.21}$$

Proof. It follows from the fact that $(\mathbb{1} + v)\theta \in \tilde\Theta_n$ is known for v odd (essentially from (1.61a)) and for v even, from the relations

$$\begin{cases} [(\mathbb{1} + v)\theta^i, (\mathbb{1} + v)\theta^j] = (\mathbb{1} + v)[\theta^i, \theta^j] = 0, \qquad i, j = 1, \ldots, n \\ \prod_{i=1}^{n} (\mathbb{1} + v)\theta^i = (\mathbb{1} + v)^n \prod_{i=1}^{n} \theta^i \neq 0. \end{cases} \tag{3.22}$$

Let $\bar{\theta} = (1 + v)\theta$: we have (cf. footnote s)

$$\frac{\partial}{\partial \theta^i} = \{\mathbb{1} + \tilde{\pi}(v)\} \frac{\partial^i}{\partial \bar{\theta}^i} + \frac{\partial v}{\partial \theta^i}\left(\theta^j \frac{\partial^i}{\partial \bar{\theta}^j}\right); \tag{3.23}$$

hence, for $v = \lambda\theta^1 \ldots \theta^k$, $k \leq n$, for $i > k$,

$$\frac{\partial}{\partial \theta^i} = (\mathbb{1} + (-1)^k \lambda\theta^1 \ldots \theta^k) \frac{\partial^i}{\partial \bar{\theta}^i}, \tag{3.24}$$

and, for $i \leq k$, the caret indicating a missing argument,

$$\begin{aligned} \frac{\partial}{\partial \theta^i} &= (\mathbb{1} + (-1)^k \lambda\theta^1 \ldots \theta^k) \frac{\partial^i}{\partial \bar{\theta}^i} + (-1)^{i-1} \lambda\theta^1 \ldots \hat{\theta}^i \ldots \theta^k \left(\theta^j \frac{\partial}{\partial \bar{\theta}^j}\right) \\ &= (\mathbb{1} + (-1)^k \lambda\theta^1 \ldots \theta^k) \frac{\partial^i}{\partial \bar{\theta}^i} + (-1)^{k+1} \lambda\theta^1 \ldots \theta^k \frac{\partial}{\partial \bar{\theta}^i} \\ &\quad + (-1)^{i-1} \lambda\theta^1 \ldots \hat{\theta}^i \ldots \theta^k \left(\sum_{j \neq i} \theta^j \frac{\partial}{\partial \bar{\theta}^j}\right) \\ &= \frac{\partial}{\partial \bar{\theta}^i} - (-1)^i \theta^1 \ldots \hat{\theta}^i \ldots \theta^k \sum_{j=k+1}^{n} \theta^j \frac{\partial}{\partial \bar{\theta}^j} \end{aligned} \tag{3.25}$$

(we isolated the term $j = i$ in the summation over j and took account of the fact that the latter gives vanishing contributions for $j \leq k$, $j \neq i$). Rewriting (3.24) as

$$(\mathbb{1} - (-1)^k \lambda\theta^1 \ldots \theta^k) \frac{\partial}{\partial \theta^i} = \frac{\partial}{\partial \bar{\theta}^i}, \qquad i > k, \tag{3.24a}$$

we get for k even, $k = 2p$,

$$\prod_{i=2p+1}^{n} \frac{\partial}{\partial \bar{\theta}^i} = \left(\prod_{i=2p+1}^{n} \frac{\partial}{\partial \theta^i}\right)(\mathbb{1} - \lambda\theta^1 \ldots \theta^{2p})^{4-2p} = \left(\prod_{i=2p+1}^{n} \frac{\partial}{\partial \theta^i}\right)(\mathbb{1} - (n - 2p)\theta^1 \ldots \theta^{2p}), \tag{3.26}$$

whilst, for k odd, $k = 2p + 1$,

$$\prod_{i=2p+2}^{n} \frac{\partial}{\partial\bar{\theta}^i} = \begin{cases} \left(\prod_{i=2p+2}^{n} \frac{\partial}{\partial\theta^i}\right)(\mathbb{1} - (-1)^k\lambda\theta^1 \ldots \theta^{2p+1}) \\ \prod_{i=2p+2}^{n} \frac{\partial}{\partial\theta^i} \end{cases}, \qquad n \text{ even}. \tag{3.27}$$

In the subsequent termwise multiplication of (3.26), resp. (3.27) by (3.25), where, successively, $i = k, \ldots, i = k-1, \ldots, i = 1$, the second terms on the right-hand side of the last line in (3.25) are ineffective, owing to (2.13). This establishes (3.21), resp. (3.20).

Proof of Theorem [3.4]. (i) Let $\varphi \in \Gamma^0_{2p}$ and let $\theta \in \tilde{\Theta}_{2k}$. We have, by [1.4] (iv),

$$\varphi = I_\theta a \qquad \text{for some } a \in \mathbb{K}^* \oplus \mathcal{N}_{2p}, \tag{3.28}$$

where (e.g., through the change $\theta^1 \to \tau(a)^{-1}\theta^1$, all other θ^j unchanged) we can arrange that $a \in \mathbb{1} + \mathcal{N}_{2p}$. Our proof then proceeds as follows: starting from (3.8) with such a couple (θ, a), we shall exhibit, successively for $q = 1, \ldots, 2p$, other couples (θ, a) satisfying (3.8) with $a \in \mathbb{1} + \bigoplus_{j>q} E^q$, E the linear span of $\{\theta^i\} i = 1, \ldots, 2p$. Once $q = 2p - 1$ we will have $a = 1$ from $\varphi(\mathbb{1}) = 0$, whence $\varphi = L_\theta$.

We thus start from (3.28) with $\theta \in \tilde{\Theta}_{2p}$ and

$$a = \mathbb{1} + \sum_{i=1}^{2p} \lambda_i \theta^i \bmod \left(\bigoplus_{j>q} E^j\right), \qquad \lambda^i \in \mathbb{C}, i = 1, \ldots, 2k. \tag{3.29}$$

Owing to (2.20) we have

$$\varphi = I_\theta a = I_{(1-\lambda_1\theta^1)}(\mathbb{1} - \lambda_1\theta^1)a \tag{3.30}$$

with

$$(\mathbb{1} - \lambda_1\theta^1)a = \sum_{i=2}^{2p} \lambda_i \theta^i \bmod \left(\bigoplus_{j\le 2} E^j\right). \tag{3.31}$$

We removed the monomial $\lambda_1\theta^1$ without altering the other monomials of first order. Removing the latter successively in the same fashion effects our step $q = 2$.

We now proceed by induction with respect to q: we assume that the situation (3.28) has been obtained with $\theta \in \tilde{\Theta}_{2p}$ and

$$a = \mathbb{1} + \sum_{k=1}^{q!} \lambda_{I_k} \theta^{I_k} \bmod \left(\bigoplus_{j>q} E^j\right), \tag{3.32}$$

where the I_k are multiindices of length $q < 2p$, and we show that the same situation prevails for $q = 1$ by successive removal of the monomials θ^{I_k}. Owing to (3.20), (3.21) we have, indeed, for $\mu = \lambda_{I_k}$ for p odd, resp. $\mu = (2p - 2q)^{-1}\lambda_{I_k}$, for p even,

$$\varphi = I_\theta a = I_{(1-\mu\theta^{I_1})}(\mathbb{1} - \mu\theta^{I_1})a \tag{3.33}$$

with

$$(\mathbb{1} - \mu\theta^{I_1})a = \mathbb{1} + \sum_{k=2}^{p!} \lambda_{I_k}\theta^{I_k} \bmod\left(\bigoplus_{j>q+1} E^j\right). \tag{3.34}$$

We removed the monomial θ^{I_1}: doing the same successively for the I_k, $k = 2, \ldots, p!$ will achieve our recurrence step.

(ii) Consider the previous situation for $\varphi = I_{\bar\theta}$: we have (2.28) with $a \in \mathscr{G}_n^0$. The previous reasoning will allow one to successively remove the (even) monomials of a (now without having to assume n even, since we need only (3.21)). This procedure yields a decomposition of the change of frame $\theta \to \bar\theta$ in a succession of changes of the type

$$\theta \to \theta(\mathbb{1} + \lambda\theta^1 \ldots \theta^{2p}), \qquad \lambda \in \mathbb{K}. \tag{3.35}$$

Now owing to the property (cf. footnote s)

$$\frac{\partial\bar\theta^k}{\partial\bar\theta^1} = \frac{\partial\bar\theta^k}{\partial\theta^i}\frac{\partial\theta^i}{\partial\bar\theta^1}, \qquad \theta, \bar\theta \in \tilde\Theta_n, \tag{3.36}$$

combined with multiplicativity of the determinant, it suffices to check (2.18) for changes of frame of the type (2.35), or, in view of (2.21), to show that

$$\mathrm{Det}\left[\partial\frac{\theta^i(\mathbb{1} + \lambda\theta^1 \ldots \theta^{2p})}{\partial\theta^k}\right] = \mathbb{1} + (n - 2p)\lambda\theta^1 \ldots \theta^2 p. \tag{3.37}$$

Now we have

$$\partial\theta^i(\mathbb{1} + \lambda\theta^1 \ldots \theta^{2p}) \begin{cases} \delta^i_k, & i \le 2p \\ \delta^i_k(\mathbb{1} + \lambda\theta^1 \ldots \theta^{ip}), & 2p < k, i \\ \delta^i_k(\mathbb{1} + \lambda\theta^1 \ldots \theta^{2p}) + (-1)^k\theta^1 \ldots \hat\theta^k \ldots \theta^{2p}, & k \le 2p < i, \end{cases} \tag{3.38}$$

showing that the Jacobian matrix is triangular, hence its determinant equals the product of its diagonal elements, yielding (3.37).

4. Non-Commutative Geometry of Grassmann Algebras

4.1. *Hochschild cochains as multilinear forms*

Let A be an associative $\mathbb{Z}/2$-graded algebra. A Hochschild[y] cochain Φ of degree n is defined as a multilinear form Φ $(a_0, a_1, \ldots, a_n)$ of order $n+1$ on A. The space of Hochschild cochains of degree n will be denoted $\mathscr{C}^n$.

4.2. *The Hochschild coboundary operator and Hochschild cohomology*

The **Hochschild coboundary** $b: \mathscr{C}^n \to \mathscr{C}^{n+1}$ is defined as

$$[b\Phi](a_0, \ldots, a_n, a_{n+1}) = \sum_{j=0}^{n} (-1)^j \Phi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1})$$

$$-(-1)^{n+1}(-1)^{\partial a_n \sum_{i=0}^{n-1} \partial a_i} \Phi(a_{n+1} a_0, a_1, \ldots, a_n),$$

where ∂a_i denotes the intrinsic grade (0 or 1) of the element a_i. It is not difficult to show that $b^2 = 0$. Defining $Z^n = \{\Phi/\Phi \in \mathscr{C}^n$ and $b\Phi = 0\}$ and $B^n = [\Phi/\Phi \in \mathscr{C}^n$ and $\psi \in \mathscr{C}^{n+1}$ s.t. $\Phi = b\psi\}$, the Hochschild cohomology groups are[z]: $H^n = Z^n/B^n$.

4.3. *Cyclicity operator and cyclic cohomology*

The **cyclicity operator** $\lambda: \mathscr{C}^n \to \mathscr{C}^n$ is defined as

$$[\lambda\Phi](a_0, a_1, \ldots, a_n) = (-1)^n(-1)^{\partial a_{n+1} \sum_{i=0}^{n} \partial a_i} \Phi(a_n, a_0, \ldots, a_{n-1}).$$

The multilinear form Φ is called **cyclic** whenever $\lambda\Phi = \Phi$.

One shows [5] that if Φ is cyclic, so is $b\Phi$: this yields a subcomplex $\mathscr{C}^*_\lambda$ of $\mathscr{C}^*$ built out of cyclic cochains. Defining the space of **cyclic cocycles** as $Z^n_\lambda = \{\Phi \in \mathscr{C}^n_\lambda$ and $b\Phi = 0\}$ and the space of **cyclic coboundaries** as $B^n_\lambda = b(Z^{n-1}_\lambda)$, the **cyclic cohomology groups** are the $H^n_\lambda = Z^n_\lambda/B^n_\lambda$.

4.4. *The periodicity operator S*

Let Φ be a cyclic cocycle of order $n(\Phi \in Z^n_\lambda)$, following [5] the periodicity operator S is defined as follows:

[y] Here we identify Hochschild cochains of degree n with values in the dual of the algebra A with multilinear forms of order $n+1$ on A.

[z] Here and below we describe the Z_2-*graded* Hochschild and cyclic cohomologies.

$$\frac{1}{2i\pi}[S\Phi](a_0, a_1, \ldots, a_n, a_{n+1}, a_{n+2})$$

$$= \Phi(a_0 a_1 a_2, a_3, \ldots, a_{n+2}) + \sum_{j=2}^{n+1} [\Phi(a_0, \ldots, a_{j-1} a_j a_{j+1}, a_{j+2}, \ldots, a_{n+2})$$

$$+ \sum_{i=0}^{j-2} (-1)^{j-i+1} \Phi(a_0, \ldots, a_i a_{i+1}, a_{i+2}, \ldots, a_j a_{j+1}, a_{j+2}, \ldots, a_{n+2})].$$

S maps cyclic cocycles (resp. coboundaries) of order n into cyclic cocycles (resp. coboundaries) of order $n + 2$ and consequently maps H^n_λ into H^{n+2}_λ. The inductive limit of cyclic cohomology groups under the operator S is called **periodic cyclic cohomology** $\left(H^n_{\text{per}} = \lim_{p\to\infty} S^p(H^n_\lambda)\right)$.

4.5. *An example of a Grassmann algebra with two generators*

Let $\mathscr{G}_2$ the algebra generated by 1, a, b with $a^2 = b^2 = 0$ and $ab = -ba$. In order to illustrate the previous definitions, we will explicitly compute the Hochschild cohomology and the cyclic cohomology of $\mathscr{G}$ up to degree 2.

$p = 0$ As a vector space, $\mathscr{G}$ is of dimension 4 (base $\{1, a, b, ab\}$), therefore the space of Hochschild cochains $\mathscr{C}^0$ is just the dual $\mathscr{G}^*$ of $\mathscr{G}$ and is generated by the dual basis $\{\varphi_1, \varphi_a, \varphi_b, \varphi_{ab}\}$. If $\varphi \in \mathscr{C}^1$, then $b\varphi \in \mathscr{C}^2$ and the condition $b\varphi = 0$ means that φ has to vanish on graded commutators (since $[b\varphi](a_0, a_1) = \varphi(a_0, a_1) - (-1)^{\partial a_0 \partial a_1}\varphi(a_1 a_0)$). This will be automatically satisfied for any φ since all graded commutators vanish (by definition of $\mathscr{G}$). Therefore $\mathscr{C}^0 = Z^0 = \mathscr{G}^*$. It is also clear that all coboundaries are trivial and that the cyclicity condition is trivially satisfied, therefore $H^0 = H^0_\lambda = \mathscr{G}^*$.

$p = 1$ As a vector space, the space of Hoschshild cochains is of dimension $4 \times 4 = 16$. Let us directly compute what the cyclic cocycles are. Cyclicity condition ($\varphi\lambda = \varphi$) imposes the following constraints: $\varphi(1,1) = -\varphi(1,1)$, $\varphi(1,a) = -\varphi(a,1)$, $\varphi(1,b) = -\varphi(b,1)$, $\varphi(1,ab) = -\varphi(ab,1)$, $\varphi(a,b) = \varphi(b,a)$, $\varphi(a,ab) = -\varphi(ab,a)$, $\varphi(ab,ab) = -\varphi(ab,ab)$. This implies in particular that $\varphi(1,1) = \varphi(ab,ab) = 0$. Also the Hochschild condition ($b\varphi = 0$) implies $\varphi(1,ab) = 0$ (indeed $0 = [b\varphi](1,a,b) = \varphi(a,b) - \varphi(1,ab) - \varphi(b,a) = -\varphi(1,ab)$), also $\varphi(1,a) = 0$ (this comes from $[b\varphi](1,1,a) = 0$). In the same way $\varphi(1,b) = 0$. The space of cyclic cocycles is therefore generated by the five following cocycles: $\varphi_{a,a}$, $\varphi_{b,b}$, $\varphi_{a,b}$, $\varphi_{a,ab}$, $\varphi_{b,ab}$ with

$$\varphi_{a,a}(a,a) = 1 \quad \text{and} \quad \varphi_{a,a}(x,y) = 0 \quad \text{if } (x,y) \neq (a,a),$$

$$\varphi_{b,b}(b,b) = 1 \quad \text{and} \quad \varphi_{b,b}(x,y) = 0 \quad \text{if } (x,y) \neq (b,b),$$

$$\varphi_{a,b}(a,b) = \varphi_{a,b}(b,a) = 1 \quad \text{and} \quad \varphi_{a,b}(x,y) = 0 \quad \text{in other cases},$$

$$\varphi_{a,ab}(a,ab) = 1 = -\varphi_{a,ab}(ab,a) \quad \text{and} \quad \varphi_{a,ab}(x,y) = 0 \quad \text{in other cases},$$

$$\varphi_{b,ab}(b,ab) = 1 = -\varphi_{b,ab}(ab,b) \quad \text{and} \quad \varphi_{b,ab}(x,y) = 0 \quad \text{in other cases}.$$

Notice that the space of cyclic cocycles is itself Z_2-graded and that $\varphi_{a,a}$, $\varphi_{b,b}$, $\varphi_{a,b}$ are even whereas $\varphi_{a,ab}$, $\varphi_{b,ab}$ are odd. Since the space of cyclic coboundaries is obviously trivial, we have

$$Z^1_\lambda = H^1_\lambda = \mathbb{C}^3 \hat{\oplus} \mathbb{C}^2.$$

4.6. *Tables of results for $H^p_\lambda(\mathscr{G}_N)$ and $H^p_\lambda(\mathscr{G}_N)$*

For big values of N (the number of Grassman generators) and p, the list of algebraic constraints imposed by the cyclicity and Hochschild conditions becomes rather large. (More precisely, one gets: $2^{N(p+1)} + 2^{Np}$ conditions.) We will just give a few explicit results (for $N = 1, 2, 3, 4$ and $p = 0, 1, 2$). The cases $H^0_\lambda(\mathscr{G}_2)$ and $H^1_\lambda(\mathscr{G}_2)$ have been analyzed in detail in the previous paragraph. In order to present these results it is useful to introduce the following notation. Let $\theta = \{\theta^i\}_{i\in\{1,\ldots,N\}}$ denote a frame of $\mathscr{G}_N$ and θ^I denote the corresponding lexicographic base (I is a multiindex running from 0 to 2^N). Then we will denote by ε_I the dual basis of $(\mathscr{G}_N)^*$, i.e., $\varepsilon_I(\theta^J) = \delta^J_I$. Moreover, it is clear that $\mathscr{G}^*_N \times \mathscr{G}^*_N \times \ldots \times \mathscr{G}^*_N$ (q factors) is spanned by $\varepsilon_{I_1,I_2,\ldots,I_q}$, where $\varepsilon_{I_1,I_2,\ldots,I_q}(\theta^{J_1},\theta^{J_2},\ldots,\theta^{J_q}) = \delta^{J_1}_{I_1}\delta^{J_2}_{I_2}\ldots\delta^{J_q}_{I_q}$.

Using this notation, we see for example that the two odd cocycles generating $Z^1_\lambda(\mathscr{G}_2)$ obtained at the end of Sec. 4.5 read respectively $\varphi_{a,ab} = \varepsilon_{1,12} - \varepsilon_{12,1}$ and $\varphi_{b,ab} = \varepsilon_{2,12} - \varepsilon_{12,2}$.

Using this notation, we obtain the following results. Here $(\varepsilon_0, \varepsilon_1)$ denotes the dual basis of $(1,\theta)$, and we list a set of generators for $Z^*_\lambda(\mathscr{G}_N)$:

N = 1

$p = 0$	ε_0	even
	ε_1	odd
$p = 1$	$\varepsilon_{1,1}$	even
$p = 2$	$\varepsilon_{0,0,0}$	even
	$\varepsilon_{0,0,1} + \varepsilon_{0,1,0} + \varepsilon_{1,0,0}$	odd

We already mentioned the fact that $Z^p_\lambda(\mathscr{G}_N)$ is itself Z_2-graded; it is therefore natural to write $Z^p_\lambda = (Z^p_\lambda)^+ \oplus (Z^p_\lambda)^-$. The parity (even, odd) was indicated in the above table. It is easy to show that the above cyclic cocycles are not cyclic coboundaries and, therefore, each one defines a non-trivial cyclic cohomology class. When a cyclic cocycle happens to be also a cyclic coboundary (as it happens in the case $N = 2$, $p = 2$ below), it is explicitly written:

$N = 2$ (Here, $\mathcal{G}$ is generated by $\{1, \theta^1, \theta^2, \theta^1\theta^2\}$).

Generators of Z^p_λ

$p=0$	ε_0	even	
	ε_1	odd	$Z^0_\lambda = H^0_\lambda = \mathbb{C}^2 \oplus \mathbb{C}^2$
	ε_2	odd	
	ε_{12}	even	
$p=1$	$\varepsilon_{1,1}$	even	
	$\varepsilon_{2,2}$	even	
	$\varepsilon_{1,2} + \varepsilon_{2,1}$	even	$Z^1_\lambda = H^1_\lambda = \mathbb{C}^3 \oplus \mathbb{C}^2$
	$\varepsilon_{1,12} - \varepsilon_{12,1}$	odd	
	$\varepsilon_{2,12} - \varepsilon_{12,2}$	odd	

$p=2$	$\varepsilon_{0,0,0} \qquad = (2i\pi)^{-1}S(\varepsilon_0)$	even
	$\varepsilon_{0,0,1} + \varepsilon_{0,1,0} + \varepsilon_{0,0,1} = b(\varepsilon_{0,1} - \varepsilon_{1,0}) = (2i\pi)^{-1}S(\varepsilon_1)$	odd
	$\varepsilon_{0,0,2} + \varepsilon_{0,2,0} + \varepsilon_{2,0,0} = b(\varepsilon_{0,2} - \varepsilon_{2,0}) = (2i\pi)^{-1}S(\varepsilon_2)$	odd
	$\varepsilon_{0,0,12} + \varepsilon_{0,12,0} + \varepsilon_{12,0,0} = b(\varepsilon_{0,12} - \varepsilon_{12,0}) = (2i\pi)^{-1}S(\varepsilon_{12})$	even
	$\varepsilon_{1,1,2} + \varepsilon_{1,2,1} + \varepsilon_{2,1,1}$	odd
	$\varepsilon_{2,2,1} + \varepsilon_{2,1,2} + \varepsilon_{1,2,2}$	odd
	$\varepsilon_{1,1,12} + \varepsilon_{12,1,1} - \varepsilon_{1,12,1}$	even
	$\varepsilon_{2,2,12} + \varepsilon_{12,2,2} - \varepsilon_{2,12,2}$	even
	$\varepsilon_{1,1,1}$	odd
	$\varepsilon_{2,2,2}$	odd
	$\varepsilon_{1,2,12} + \varepsilon_{12,1,2} + \varepsilon_{2,1,12} + \varepsilon_{12,2,1} - \varepsilon_{1,12,2} - \varepsilon_{2,12,1}$	even

$$Z^2_\lambda = \mathbb{C}^5 \hat{\oplus} \mathbb{C}^6 \quad \text{but} \quad B^2_\lambda = C \hat{\oplus} \mathbb{C}^2 \quad \text{so that} \quad H^2_\lambda = \mathbb{C}^4 \hat{\oplus} \mathbb{C}^4.$$

Notice that Z^0_λ can be identified with the dual of $\mathcal{G}$ (this is true for any number of generators) and that the hierarchy associated with each linear form of $\mathcal{G}$ using the periodicity operator S is essentially trivial at the cohomological level: the only non-trivial hierarchy is obtained by using the action of S on the unit 1 of $\mathcal{G}$, and this shows that the even cyclic cohomology of $\mathcal{G}$ contains the cyclic cohomology of algebras of complex numbers.

$N = 3$ $\mathcal{G}$ is generated by $\{1, \theta^1, \theta^2, \theta^3, \theta^1\theta^2, \theta^2\theta^3, \theta^1\theta^3, \theta^1\theta^2\theta^3\}$. Here again $Z^0_\lambda = H^0_\lambda$ can be identified with $\mathcal{G}^*_3$, so that dim $Z^0_\lambda = 2^3 = 8$ and is generated by $\{\varepsilon_0 = 1^*, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{123}\}$. We will only give the generators of Z^1_λ. The space of even cocycles is of dimension 9 and generated by

$$\varepsilon_{1,1}, \varepsilon_{2,2}, \varepsilon_{3,3}, \varepsilon_{1,2} + \varepsilon_{2,1}, \varepsilon_{2,3} + \varepsilon_{3,2}, \varepsilon_{3,1} + \varepsilon_{1,3},$$

$$\varepsilon_{1,123} + \varepsilon_{123,1} - \varepsilon_{12,13} + \varepsilon_{13,12},$$

$$\varepsilon_{2,123} + \varepsilon_{123,2} - \varepsilon_{12,23} + \varepsilon_{23,12},$$

$$\varepsilon_{3,123} + \varepsilon_{123,3} + \varepsilon_{13,23} - \varepsilon_{23,13}.$$

The space of odd cocycles is of dimension 8 and is generated by

$$\varepsilon_{1,12} - \varepsilon_{12,1}$$

$$\varepsilon_{1,13} + \varepsilon_{13,1}$$

$$\varepsilon_{2,23} - \varepsilon_{23,2}$$

$$\varepsilon_{2,12} - \varepsilon_{12,2}$$

$$\varepsilon_{3,13} + \varepsilon_{13,3}$$

$$\varepsilon_{3,23} - \varepsilon_{23,3}$$

and by *two* other independent generators Ψ which have to be chosen in such a way that

$$\Psi(\theta^1, \theta^2\theta^3) + \Psi(\theta^1\theta^2, \theta^3) - \Psi(\theta^1\theta^3, \theta^2) = 0.$$

One can choose $\Psi = \alpha\varepsilon_{1,23} + \beta\varepsilon_{12,3} + \gamma\varepsilon_{13,2}$ with $-\alpha + \beta - \gamma = 0$, for instance $(\alpha, \beta, \gamma) = (1, 2, 1)$ and $(1, 1, 0)$, but there is no "privileged" choice. For $N = 3$ and $p = 1$, one therefore finds $Z_\lambda^1 = H_\lambda^1 = \mathbb{C}^9 \oplus \mathbb{C}^8$.

4.7. *Cyclic cohomology of Grassmann algebras* (General results)

The techniques illustrated in the previous paragraph allow one to compute explicitly a set of generators for the cyclic cohomology of the Grassman algebra $\Lambda\mathbb{C}^r$. However, if we are not interested in obtaining such a set of generators, there is another method studied in [6] which uses the known result for Z_2-graded cyclic cohomology of $\Lambda\mathbb{C}$ along with a Künneth formula. Before stating the general result, let us note that, if $V = \sum_n V_n$ is a Z-graded and Z_2-graded vector space ($V_n = V_n^+ \oplus V_n^-$), one introduces a Z_2-graded Poincaré polynomial $P(V)(t) = \sum_n [(\dim V_n^+) + \theta(\dim V_n^-)]t^n$, where θ is the generator of Z_2. The complex H_λ^* being Z_2-graded, one finds, for the cyclic cohomology of $\Lambda\mathbb{C}^r$, that

$$H_\lambda^*(\Lambda\mathbb{C}^r) = H_\lambda^*(\mathbb{C}) \oplus V^*,$$

where $H_\lambda^*(\mathbb{C})$ denotes the cyclic cohomology of the algebra of complex numbers (which is periodic modulo 2) and where V^* is a Z_2-graded complex, whose Poincaré polynomial is

$$P(V)(t) = [2^{r-1}(1 + \theta) - (1 - t)^r]/[(1 + t)(1 - t)^r].$$

For instance, in the particular case of $\Lambda\mathbb{C}^2$ we get

$$P(V)(t) = (1 + 2\theta) + (3 + 2\theta)t + (3 + 4\theta)t^2 + (5 + 8\theta)t^3 + \dots$$

and therefore

$$H_\lambda^*(\Lambda\mathbb{C}^2) = \mathbb{C} \oplus \mathbb{C} \hat{\oplus} \mathbb{C}^2,$$

$$H_\lambda^1(\Lambda\mathbb{C}^2) = \mathbb{C}^3 \hat{\oplus} \mathbb{C}^2,$$

$$H_\lambda^2(\Lambda\mathbb{C}^2) = \mathbb{C} \oplus \mathbb{C}^3 \hat{\oplus} \mathbb{C}^4,$$

$$H_\lambda^3(\Lambda\mathbb{C}^2) = \mathbb{C}^5 \oplus \mathbb{C}^8.$$

These results agree with our previous explicit calculations.

In the particular case of cyclic cocycles of degree one on $\mathscr{G}_n$, it is possible to express a set of generators in terms of Berezin integration and graded derivations on the algebra. For instance, in the case $N = 2$, setting

$$X = X_0 + X_1\theta^1 + X_2\theta^2 + X_{12}\theta^1\theta^2,$$

$$Y = Y_0 + Y_1\theta^1 + Y_2\theta^2 + Y_{12}\theta^1\theta^2,$$

one can write the cyclic cocycle $\varepsilon_{1,2} + \varepsilon_{2,1}$,

$$(\varepsilon_{1,2} + \varepsilon_{2,1})(X, Y) = X_1 Y_2 + X_2 Y_1,$$

as

$$(\varepsilon_{1,2} + \varepsilon_{2,1})(X, Y) = I_\theta(XDY),$$

where

$$D = -\theta^1 \frac{\partial}{\partial\theta^1} + \theta^2 \frac{\partial}{\partial\theta^2}.$$

This property was noticed and used in [7] but has not been generalized yet to cyclic cocycles of degrees higher than one.

As already mentioned in the introduction, we only touched here the problem of non-commutative geometry (in the spirit of [5]) for graded commutative algebras. Much more should be done: relations between cyclic cohomology and graded derivations of $\mathscr{G}_n$, relation between Hochschild cocycles and supersymmetric "De Rham currents" on $\mathscr{G}_n$, non-commutative connections on modules over $\mathscr{G}_n$, generalization of the above to more general examples of graded commutative algebras in relation with the geometry of "supermanifolds", etc. We shall return to this in a later publication.

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