# FIBER BUNDLES AND KALUZA-KLEIN THEORY

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ABSTRACT

A new geometrical framework for dimensional reduction is presented. For a G/H Kaluza-Klein theory it predicts a reasonably big effective gauge group $G_{eff}$ which (locally) is a product of $N(H)|H$ and $G|C$, where $N(H)$ is the normalizer of H and C is the center of G.

## 1. INTRODUCTION

The works of Kerner[1], Trautman[2], Cho[3], Cho and Freund[4], laid foundations for a geometrical understanding of a unified description of gravity and (non-Abelian) gauge theory. The idea developed in these papers was to generalize the 5-dimensional Kaluza-Klein theory, and to consider the principal bundle on which gauge fields live not as an auxiliary geometrical object but as a true arena where "real" physics takes place. Thus four-dimensional space-time was replaced by an "extended universe" which locally looked like a product M×G of space-time M and an internal space isomorphic to the group manifold of a compact Lie group G (however, non-compact groups may be also relevant). With an appropriate "Ansatz" or by imposing certain geometrical constraints one could show that gravity in 4+n dimensions (n=dimG) splits into gravity, Yang-Mills field and, possibly, some scalar fields in 4 dimensions. The effective gauge group of the resulting theory in four dimensions is again G (although Higgs mechanisms involving scalar fields could reduce it to a smaller one). That simple method of unification had to come to end with observation that there must be a reasonable compromise between the two opposing tendencies:n should be big enough to accomodate for at least SU(3)×SU(2)×U(1) gauge fields, but n should be small enough not to produce particles of spin higher than 2 in the effective particle spectrum of the dimensionally reduced theory. Witten[5)] analysed the situation in supergravity and found that a viable compromise is possible by assuming that the internal space is a homogeneous space G/H rather than the group G itself[*)]. From that time on quite a number of models of this type has been studied. What was lackig was a clear geometrical understanding of the "Ansatz", understanding comparable to that achieved in the principal bundle case. Percacci and Randjbar-Daemi[7)] attempted to close this gap but their proposal did not seem natural. In [8] Coquereaux and the present Author proposed a simple and natural model of dimensional reduction of gravity. The machinery developed in that paper was later used[9,10)] to improve the results of Forgacs and Manton[11)] (see also [12,13]) on symmetric Yang-Mills fields. The only bad thing with [8] was that the predicted effective gauge group was not G as expected. The "coefficient of effectiveness" as measured by the ratio (dim.of the effective gauge group)/(dim.of the internal space) was in fact smaller than one. In this lecture a possible solution to this dilemma is proposed. I will suggest a new model, more general than that of ref.[8], which allows for the effective gauge group big enough to satisfy the needs: for a G/H internal space the effective gauge group is (locally) a product (N(H)|H)×(G|C).

## 2. AN ILLUSTRATIVE EXAMPLE: THE KLEIN BOTTLE AND THE TORUS

Consider the following two simple models of the extended universe: The torus and the Klein-bottle (see Fig.1). Both, the Klein-bottle and the torus, are $S^1$ fibrations over $S^1$. Internal spaces are circles and space-time (base of the fibration) is a circle in these simplest of the nontrivial examples. Both carry a flat Riemannian metric (ground-state metric) inherited from the piece of $\mathbb{R}^2$ from which they are glued (the gluings of the end circles in Fig.1. are supposed to respect the depicted senses of rotation; that causes well know difficulties when attempted in $\mathbb{R}^3$ ). The <u>local</u> internal symmetry group ("internal" means it does not affect the base points) of the ground state metric is in both

*) Already in 1959 Souriau[6)] proposed to use $S^2$ as an internal space for SU(2) gauge field.



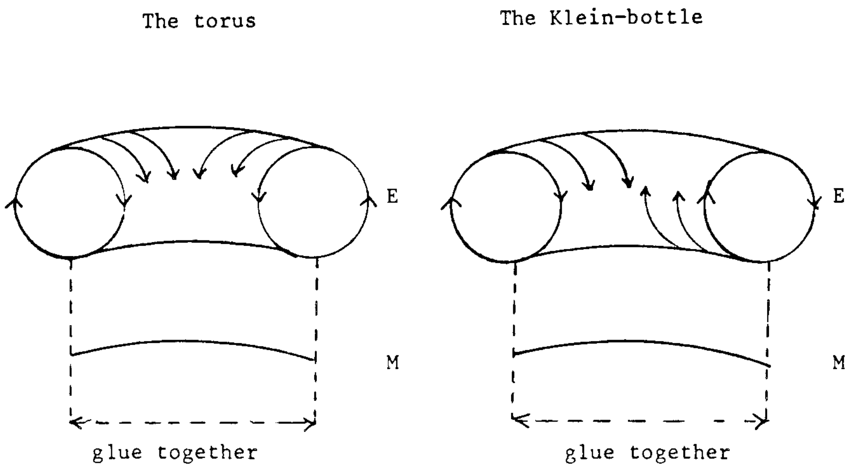


Fig.1.

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cases the same: it is U(1). However, and this is the difference which is crucial for us, *the U(1) acts globally on the torus but can not be made to act globally on the Klein-bottle* (the best proof of this fact is to try and see that it can not be done - see the arrows in the figure). Being locally indistinguishable from the torus, the Klein-bottle seems to be a legitimate candidate for a model of the extended universe. But it does not satisfy the most important assumption of ref.[8]: the requirement of a global group action.

Before we shall formulate assumptions of a scheme more general than that of ref.[8] let us first list some of the properties that both Klein-bottle and the torus have in common. Let E denote one or another and let M be the base circle; x and x′ are points of M, and $E_x$ denotes the fiber of E over x.

- E is a fibration over M

- each fiber $E_x$ has a transitive group of isometries $G_x$
- the groups $G_x$ and $G_{x'}$ acting on different fibers are isomorphic although a natural isomorphism need not to exist; in the Klein-bottle case there is no way to tell whether two rotations, one at x and another at x′, have the same sense.
- it is however possible to set up a local convention and to identify the $G_x$-s with U(1) over an open neighborhood of every point of M; an example of such a local identification is provided by requiring that the U(1) should act locally by isometries of the given ground-state metric of E (e.g. the flat metric).

## 3. A NEW MODEL

Let us now make use of the lesson of the Klein-bottle case and let us try to formulate a general scheme on the basis of the properties listed above.

We assume E to be a fibration $\pi_E : E \to M$, and for each $x \in M$ let $G_x$ be a (compact Lie) group acting transitively on $E_x$ - the fiber of E over x. In application to Kaluza-Klein theory $G_x$ will be the isometry group of a certain metric on $E_x$. All the groups $G_x$, $x \in M$, are assumed to be isomorphic to a standard one, say G, although no particular isomorphism is being distinguished at the beginning. In the Klein-bottle case, for example, there is a $Z_2$-ambiguity in identifying a rotation of a fibre with an element of U(1), and this ambiguity cannot be consistently removed by a smooth global convention. To describe the situation more precisely we introduce the *bundle of groups* $\mathbb{G} = \bigcup_x G_x$, and $\mathbb{G}$ is assumed to be locally trivial (see below). The actions of $G_x$ on $E_x$, when x runs over M, give rise to a bundle map

$$E \times \mathbb{G} \to E$$

$$(y,a) \to ya, \qquad a \in E_x, \quad a \in G_x.$$

Locally E is assumed to look like $E \times (H\backslash G)$. The fact that locally we can describe the situation as acting with G on $H\backslash G$ is expressed by the following fundamental postulate:

*Local Triviality*: There is an open covering $(U_\alpha)$ of M and, for each α, there are

$$\phi_\alpha : \pi_E^{-1}(U_\alpha) \to U_\alpha \times (H\backslash G)$$

$$\psi_\alpha : \pi_{\mathbb{G}}^{-1}(U_\alpha) \to U_\alpha \times G$$

($\phi_\alpha$ and $\psi_\alpha$ are assumed to be diffeomorphisms and are called local trivializations of E and $\mathbb{G}$ respectively) such that $\phi_\alpha$ restricts to group isomorphisms $G_x \to G$ of the fibers, and

$$\psi_\alpha(ya) = \psi_\alpha(y)\phi_\alpha(a) \tag{3.1}$$

for all $y \in E_x$, $a \in G_x$, $x \in U_\alpha$. The product on the right hand side is understood as $(x,[a])(x,b) = (x,[a]b)$, where $[a] = Ha \in (H\backslash G)$ and $b \in G$.

On the intersection of two trivializing neighborhoods $U_\alpha$ and $U_\beta$ we get transition maps $\psi_{\alpha\beta}: (H\backslash G) \to (H\backslash G)$ and $\phi_{\alpha\beta} : G \to G$. By the very definition $\phi_{\alpha\beta}$ is an automorphism of G and $\psi_{\alpha\beta}$ satisfies a condition analogous to (3.1). This motivates the following definition:

<u>Definition</u> A pair $(\phi,\psi)$ of maps $\phi : G \to G$ and $\psi : (H\backslash G) \to (H\backslash G)$ is called a *twisted automorphism* of $H\backslash G$ if

i) $\phi$ is an automorphism of G

ii) $\psi([a]b) = \psi([a])\phi(b)$.

The set of all twisted automorphisms of $(H\backslash G)$ is a group under composition of maps. This group will be denoted $\mathrm{Taut}(H\backslash G)$.

In the following we shall always assume that $H\backslash G$ is an *effective homogeneous space* i.e. that the action of G on $H\backslash G$ is effective. If this is the case then the map $\phi$ is completely determined by $\psi$ and the property ii). According to the definition of a "coordinate bundle" by Steenrod[14), $\mathbb{G}$ and E are such bundles with structure groups AutG and

$Taut(H\backslash G)$ respect. In a canonical way one can construct then the associated principal bundles (see ref.[14,§8.1]) denoted $(P,M,Aut)$ and $(Q,M,Taut)$, so that $\mathbb{G}$ and $E$ can be also considered as bundles associated to $P$ and $Q$ respectively: $\mathbb{G} = P\times_{AutG}G$ , $E = Q\times_{Taut(H\backslash G)}(H\backslash G)$.

These are the rudiments of the new model. In the next Section we shall elaborate a little bit on some of its properties. But first let us see how does the rigid model of ref.[8] is a particular case of the one presented above. Assume $P$ is a *trivial* principal bundle. It means it has a global cross-section $\sigma : M \to P$. This allows us to define the global (right) action $R_\sigma$ of $G$ on $E$ by

$$R_\sigma(a)y \doteq y(\sigma(x)\cdot a), \qquad x\in\pi_E(y),$$

where $\sigma(x)\cdot a\in\mathbb{G} \cong P\times_{AutG}G$. (Of course with a change of $\sigma$ the associated action of $G$ on $E$ will change accordingly).

## 4. THE EFFECTIVE GAUGE GROUP

We have seen how, starting with natural assumptions concerning the structure of $E$, a principal bundle $Q$ with structure group $Taut(H\backslash G)$ can be constructed. It is this kind of analysis that was lacking in ref.[7] where structure group $G$ was guessed*). The group $Taut(H\backslash G)$ introduced in Sec.3 is the biggest possible effective gauge group allowed by geometry. In the following we will denote it also as $G_{eff}$. Let us list the most important properties of the group $G_{eff}$; these are

1° $N(H)|H$ – the effective gauge group of ref.[8] – is an invariant subgroup of $G_{eff}$. In fact we have the exact sequence

$$1 \to Aut(H\backslash G) \to Taut(H\backslash G) \to Aut_{(H)}G \to 1$$

where $Aut(H\backslash G)=N(H)|H$ (see ref.[8]), and $Aut_{(H)}G$ consists of those automorphisms $\phi$ of $G$ for which $\phi(H)$ is conjugated to $H$ (in particular

*) For another guess see e.g. ref.[15], where the Authors suggest the group $G\times H$ ! See however the end remark of this Section.



$Aut_{(H)}G$ contains all inner automorphisms). In fact the bundle $P$ of Sec. 3 has structure group $Aut_{(H)}G$, and can be also constructed as the quotient of $Q$ by $N(H)|H$.

2° Let $\bar{G}$ be the semidirect product of $AutG$ and $G$

$$\bar{G} \doteq Aut\,G \circledS G$$

and let $\bar{H} \doteq I\times H\subset\bar{G}$. Then we have the following natural isomorphisms

i) $Taut(H\backslash G) \cong N(\bar{H})|\bar{H}$

ii) $Aut\ G \times (H\backslash G) \cong \bar{H}\backslash\bar{G}$

iii) $H\backslash G \cong H\backslash\bar{G}/AutG$ – a double coset

3° Locally we have

$$G_{eff} \cong Taut(H\backslash G) \overset{loc}{=} (N(H)|H) \times (G|C)$$

where $C$ is the center of $G$

The following comments are relevant:

*Remark 1* There is a simple method which allows us to obtain the bundle $E$ as discussed above with the help of constructions known already from ref.[8]. Nemely, having $\bar{G}$, $\bar{H}$, and $Q$ which(by i) is a principal bundle with structure group $N(\bar{H})|\bar{H}$), we can construct, with the methods of ref.[8], the associated bundle $\bar{E}$ with typical fiber $\bar{H}\backslash\bar{G}$. The group $\bar{G}$ acts globally on $\bar{E}$ (from the right). This is the situation known from ref.[8]. Now, $AutG$ is a subgroup of $\bar{G}$, and we can take the quotient bundle $\bar{E}/\bar{G}$. Then, by iii), this quotient bundle is naturally isomorphic to $E$:

$$E \cong \bar{E}/Aut\,G\,.$$

This suggests us the class of interesting metrics on E: those which are images of $\bar{G}$-invariant metrics on $\bar{E}$. It is clear then from the results of ref.[8] that this class of metrics give rise to gauge fields with gauge group $N(\bar{H})|\bar{H} \equiv G_{eff}$ (recall that E is a bundle associated to Q)*). Thus $G_{eff}$ is not only geometrically allowed, but also kinematically available.

*Remark 2* If G is semisimple then G is (locally) contained in $G_{eff}$ (see 3°). But even if G has central U(1) factors, if they are not in H (which is anyway excluded by the assumption of effectiveness of $H\backslash G$) then they contribute to $G_{eff}$ via N(H)|H. Below are three simple examples of G, H and $G_{eff}$ (all equalities are local)

a) G = U(1), H trivial, the case of 5-dimensional Kaluza-Klein theory. Then Aut(G) is trivial(=$Z_2$), N(H)|H = U(1). Thus $G_{eff}$ = U(1)

b) G = SO(8), H = SO(7), $H\backslash G = S^7$. Aut(G) = SO(8), N(H)|H is trivial (=$Z_2$). Thus $G_{eff}$ = SO(8) - the isometry group of the round seven-sphere

c) G = U(2;H), H = U(1;H), $H\backslash G = S^7$, AutG = U(2;H) = SO(5), N(H)|H = = U(1;H) = SU(2). Thus $G_{eff}$ = SO(5)×SU(2) - the isometry group of the squashed seven-sphere (see ref.[17-18]).

These few examples show us that the formula for $G_{eff}$ given by 2° i) and 3° predicts a reasonable effective gauge group even without any particular dynamical model. As we have already remarked, for concrete models the Higgs mechanism can break a part of $G_{eff}$.

*Remark 3* The bundle P will be, in general, non-trivial. Nevertheless one can expect that the ground-state metric of E will give rise to a *flat* principal connection in P. In that case although a *global* cross-section σ of P does not exist, there are *distinguished local* cross-sections: the *horizontal* sections. Transition functions between these sections are then *constant*. Thus different actions of G on E as defined by

*) In the particular case of trivial H, E becomes a "weak principal bundle", and the induceed $G_{eff}$-connection becomes a "weak principal connection", according to the terminology of ref.[16].



these cross-sections (see end of Sec.3) are related by *constant* group automorphisms. This enables one to apply the equivariant techniques of ref.[19] also to the present, more general, case.

*Remark 4* Castellani, Romans and Warner [20] analysed Killing vectors on coset spaces and indicated N(H)|H×G (with a correction for common U(1)-factors) as a maximal gauge group for a given realization of a coset. This statement essentially agrees with $G_{eff}$ as derived above by a careful analysis of fiber bundle structure of the Kaluza-Klein theory. To see the relation between the two groups observe that every pair (n,r) with n∈N(H)|H and r∈G determines an element (ϕ,ψ)∈Taut($H\backslash G$) given by $\phi(a)=rar^{-1}, \psi([a])=[nar^{-1}]$, and in the connected neighborhood of the identity every element of Taut($H\backslash G$) is of this form.

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## 5. REFERENCES

[1] Kerner, R. Ann.Inst.H.Poincaré 9,143(1968)
[2] Trautman, A. Rep.Math.Phys.1,29(1970)
[3] Cho, Y.M. J.Math.Phys.16,2029(1975)
[4] Cho, Y.M.,Freund, P.G.O. Phys.Rev.D12,1711(1975)
[5] Witten, E. Nucl.Phys.B186,412(1981)
[6] Souriau, J.M. in: Les Theories relativistes de la gravitation. Colloque de Royaumont 1959, Paris: CNRS 1962.
[7] Percacci, R., Randjbar-Daemi, S. J.Math.Phys.24,807(1983)
[8] Coquereaux, R., Jadczyk, A. Commun.Math.Phys.90,79(1983)
[9] Jadczyk, A., Pilch, K. Lett.Math.Phys.8,97(1984)
[10] Coquereaux, R., Jadczyk, A. Commun.Math.Phys.98,79(1985)
[11] Forgacs, P., Manton, N.S. Commun.Math.Phys.72,15(1980)
[12] Harnad, J., Schnider, S., Vinet, L. J.Math.Phys.21,2719(1980)
[13] Harnad, J., Schnider, S., Tafel, J. Lett.Math.Phys.4,107(1980)
[14] Steenrod, N. "The topology of fibre bundles", Princeton University Press, Princeton, New Jersey, 1951

[15] Volkov, D.V., Sorokin, D.P., Tkach, V.I. Theor.and Math.Phys.56, 171(1983), in Russian.

[16] Kolář, I. in: Global Differential Geometry and Global Analysis, Colloquium Berlin 1979, Lecture Notes in Mathematics, 838, Springer-Verlag, Berlin-Heidelberg-New York 1981

[17] Duff, M.J., Nilsson, B.E.W., Pope, C.N., Phys.Rev.Lett.50,2043 (1983), see also Preprint ICTP /82-83/29

[18] Bais, F.A., Nicolai, H., van Nieuwenhuizen, P.,CERN preprint TH 3577/83

[19] Coquereaux, R., Jadczyk, A. Preprint CERN TH 4023/84

[20] Castellani, L., Romans, L.J., Warner, N.P. Nucl.Phys.B241,429 (1984), see also Caltech preprint CALT-68-1057(1983)



# RELATION BETWEEN GROUP CONTRACTION AND NON-LINEAR REALIZATIONS

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ABSTRACT

Non linear realizations of the invariance group are recovered by means of contraction of group representations in spontaneously broken symmetry theories. A detailed analysis of Goldstone model, SO(n) vector model and SU(2) doublet model is given. Extension to gauge theories is finally presented.