Frames of second order

Arkadiusz Jadczyk

November 2, 2003

1 Frames of second order

1.1 A. The bundle $P^2(M)$

Let M be a smooth n-dimensional manifold. Two maps from open neighbourhoods of the origin $0 \in \mathbb{R}^n$ to M define the same 2-jet at 0 if and only if their partial derivatives up to the second order coincide. The 2-jet determined by such a map e is denoted $j_0^2(e)$. If e is a diffeomorphism, then $j_0^2(e)$ is called a second order frame at the point $p = e(0)$. The set of of all second-order frames is denoted by $P^2(M)$.

Let (x^μ) be a local chart of M, and let (t^a) be the standard coordinates on $\mathbb{R}^n$. Given $j_0^2(e)$ such that p is in the domain of the chart, a set of coordinates of $j_0^2(e)$ is defined by:

$$\begin{cases} e^\mu & \doteq & x^\mu(p) \\ e^\mu{}_a & \doteq & \frac{\partial (x \circ e)^\mu}{\partial t^a}\big|_{t=0} \\ e^\mu{}_{ab} & \doteq & \frac{\partial^2 (x \circ e)^\mu}{\partial t^a \partial t^b}\big|_{t=0} \end{cases}$$

If (x^μ) is replaced by $(x^{\mu\prime})$, the coordinates of $j_0^2(e)$ change:

$$\begin{cases} e^{\mu\prime} & = & x^{\mu\prime}(p) \\ e^{\mu\prime}{}_a & = & \frac{\partial x^{\mu\prime}}{\partial x^\mu}(p) e^\mu{}_a \\ e^{\mu\prime}{}_{ab} & = & \frac{\partial x^{\mu\prime}}{\partial x^\mu}(p) e^\mu{}_{ab} + \frac{\partial^2 x^{\mu\prime}}{\partial x^\mu x^\nu} e^\mu{}_a e^\mu{}_b \end{cases}$$

It follows that $(e^\mu{}_a$ may be considered as an ordinary (i.e. first order) frame at p. A natural projection $P^2(M) \to P^1(M)$ exists, and is given by $j_0^2(e) \mapsto j_0^1(e)$ or, in coordinates, by $(e^\mu, e^\mu{}_a, e^\mu{}_{ab}) \mapsto (e^\mu, e^\mu{}_a)$. A simple interpretation can be given to $e^\mu{}_{ab}$. First notice that the matrix $e^\mu{}_a$ is always invertible. Let $e^a{}_\mu$ denote the inverse matrix, so that we have $e^\mu{}_a e^a{}_\nu = \delta^\mu_\nu$ and $e^a{}_\mu e^\mu{}_b = \delta^a_b$. Define "connection coordinates of e by

$$e^\mu{}_{\rho\sigma} \doteq -e^r{}_\rho e^s{}_\sigma e^\mu{}_{\rho\sigma}.$$

It follows from the transformation properties of the coordinates of e above that $e^\mu{}_{\rho\sigma}$ transform as connection coeeficients at p. Therefore each section of $P^2(M)$ determines a pair: a section of $P^1(M)$ (i.e. a frame) and a torsion-free affine connection on M, the correspondence being bijective. In particular, if $P^1(M)$ is reduced to the orthogan or pseudo-orthogonal group, the Hilbert-Palatini principle for General Relativity can be considered as a functional on the space of sections of $P^2(M)$ Also notice that the diffeomorphisms group of M acts on $P^2(M)$ and on the space of its sections in a natural way. If e is a map from an open neighbourhood of the origin $0 \in \mathbb{R}^n$ to M, and if $\phi : M \to M$ is a local diffeomorphism defined at $p = e(0)$, then $\phi \circ e$ is another map from an open neighbourhood of the origin $0 \in \mathbb{R}^n$ to M. If e_1 and e_2 define the same second order frame: $j^2_0(e_1) = j^2_0(e_2)$, then the composed maps define the same second order frame as well: $j^2_0(\phi \circ e_1) = j^2_0(\phi \circ e_2)$.

1.2 B. The structure group $G^2(n)$.

Let $G^2(n)$ denote the set of all second-order frames at $0 \in \mathbb{R}^n$. $G^2(n)$ is a group with the group multiplication law given by

$$j^2_0(h)j^2_0(h) \doteq j^2_0(h \circ k).$$

The group $G^2(n)$ acts on $P^2(M)$ from the right

$$j^2_0(e)j^2_0(h) \doteq j^2_0(e \circ h).$$

Corresponding to the canonical coordinates in $\mathbb{R}^n$, there are natural coordinates in $G^2(n)$: $(h^a{}_b, h^a{}_{bc})$, and each $j^2_0(h)$ can be uniquely represented by the map $\mathbb{R}^n \to \mathbb{R}^n$ given by

$$t^a \mapsto h^a{}_r t^r + \frac{1}{2} h^a{}_{rs} t^r t^s.$$

In terms of natural coordinates the group composition law in $G^2(n)$ can be written as

$$(h^a{}_b, h^a{}_{bc})(k^a{}_b, k^a{}_{bc}) = (h^a{}_r k^r{}_b, h^a{}_{rs} k^r{}_b k^s{}_c + h^a{}_r k^r{}_{bc})$$

While the group $G^2(n)$ acts on $P^2(M)$ from the right, and $P^2(M)$ is a principal bundle over M with $G^2(n)$ as its structure group, the group $Diff(M)$ of diffeomorphisms of M acts on $P^2(M)$ from the left, by fibre preserving transformations, commuting with the right action of $G^2(n)$ - thus as an automorphism group of $P^2(M)$. An affine connection can be considered as a section of a bundle associated to $P^2(M)$ via an appropriate representation of $G^2(n)$ by affine transformations.
Exercise: Find this representation.