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ON THE EFFECTIVE GAUGE GROUP FROM G/H SPONTANEOUS COMPACTIFICATION

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ABSTRACT

We discuss two schemes of dimensional reduction: a G-invariant and a non-G-invariant one. The first gives rise to a consistent truncation with N(H)/H gauge bosons, while the second leads to the effective gauge group $G_{eff} \stackrel{loc}{\approx} N(H)/H$ Aut G.

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1. Introduction

The idea of spontaneous compactification is both simple and attractive. One starts here with a field theory in d=m+s dimensions. Let {F} denote the set of primitive fields of the theory. Usually {F} contains a metric tensor or vielbein field, but there are known interesting models in which these fields are composite. After contemplating the Lagrangian and field equations for {F}, and after convincing oneself that the set {F} is reasonably complete (in particular one has to take care of the anomalies), one looks for a "compactifying ground-state solution" $\{F_o\}$. $\{F_o\}$is expected to be a highly symmetric solution of the field equations and it should be stable in an appropriate sense. The symmetry of $\{F_o\}$ need not (and in many cases will not) be the highest possible one - provided that the stability of $\{F_o\}$ is assured. If $\{F_o\}$ gives rise to a splitting of the m+s dimensional world - we shall denote it by E - into a product $E=M\times S$ of m and s dimensions, with S compact, then one says that a spontaneous compactification is taking place. Such a phenomenon may, for instance, easily occur in theories containing in their "menu" antisymmetric tensor fields which acquire non-zero vacuum expectation values. For example, if $F_{AB\ldots C}$ is such a field, and if dF=0, then one has a natural definition of the internal directions X^A as those satisfying $X^A F_{AB\ldots C}=0$. The details of the mechanism of spontaneous compactification may, however, be model-dependent to a great extent. Therefore if we want to draw some *model-independent* conclusions, the natural thing to look at is the symmetry group of the ground state. We shall assume that this group splits in a natural way into a product $G_E=G_M\times G_S$ of "spacetime" symmetry group G_M and "internal" symmetry group G_S. Several comments are to be made at this point. First, we have taken "spacetime" into the quotation marks. Reason for this is the following: one should not be prejudiced and think that a spontaneous compactification occurs necessarily in one step. To the contrary, a realistic scenario may proceed in several steps. For instance, if we think of pure Einstein gravity as the primitive field, then the first compactification will produce (via the Kaluza-Klein mechanism) Yang-Mills fields, while the second compactification may occur around a nontrivial Einstein-Yang-Mills background to produce chiral fermions and spontaneous symmetry breaking via the Higgs potential resulting from the Yang-Mills part of the effective "after-first-step" Lagrangian. One can also think of three or more consecutive steps (M. Duff [1] considers a possibility of such a more-than-one-step scenario for the field theory limit of strings). What is important to notice here is that each step taken separate-

ly can satisfy the appropriate criterium of stability without the final result being a stable compactification.
Next remark concerns the group G_S. It is convenient to assume that G_S is a compact Lie group. On the other hand some non-compact groups may well occur here. There is no reason for G_S to be simple either ... Therefore, when in the following we shall consider mainly the case of G_S compact and/or simple, it will be only for the sake of convenience. Since in the following G_M will not be discussed we shall denote G_S simply by G. And when we will talk of G being the "isometry group of the vacuum", what we will really mean is that "G is some natural part of the symmetry group of $\{F_o\}$ ".
Another remark: we say "isometry group", but what if $\{F_o\}$ has no isometries at all?What if S - the internal space and/or M - the spacetime, is a complicated manifold which is not a homogeneous space and which has no Killing vectors? The answer to this question is not a straightforward one. First remark is that, as we will see, there are interesting manifolds which are not homogeneous spaces but which are "born out" of homogeneous spaces - they are "Kaluza-Klein projections" of homogeneous spaces. These manifolds are the *double cosets* (or parts of double cosets). They carry a finite - parameter family of natural metrics inherited from their homogeneous parents, although they need not to have Killing vectors. (It is also remarkable that, apparently, some of these double-coset manifolds may carry naturally exotic differentiable structures). Therefore, even if S has no isometries, it may happen that there is a still higher dimensional theory, in some $\bar{E}$ containing E, in which the usual Kaluza-Klein mechanism involving isometries works, and such that E is obtained by a projection from $\bar{E}$. We will discuss some of the relevant geometrical constructions later on.
Second remark is that if S has no group acting on it, and if there is no way of using some trick (like that of embedding E into $\bar{E}$ where some G can act), then it becomes really a problem. And this because after spontaneous compactification it is necessary to perform "truncation" and "dimensional reduction", and the only known way of doing that seems to be via "harmonic expansion". But there is no harmonic expansion if there is no group action! This brings us to the next important concepts: harmonic expansion, truncation and dimensional reduction.

Suppose $\{F_o\}$ is given, is stable, and the d=m+s dimensional world splits into M×S. Let also G be the isometry group of $\{F_o\}$, with S acting transitively on S. The next thing to do is to analyse the fluctuations $F_o+\delta F$ around F_o. However, most of these fluctuations will completely destroy

the product structure M×S - they will hopelessly mix the "internal" with the "external". If we want to have an effective description of phenomena from the point of view of the m-dimensional base manifold M, then we have to use: harmonic expansion, dimensional reduction and, eventually, some or other truncation. The first step - harmonic expansion - has as its aim to select a basis in the space of all fluctuations δF, a basis δF_n such that every δF can be expressed as a series $\delta F = c_n \delta F_n$, and every δF_n can be interpreted as a finite-component field on M. The number of components of δF_n will, in general, increase with n and will be related to the consecutive dimensions of irreducible representations of G. If this process of harmonic expansion is induced by geometry, then there is a good chance that it will not destroy the gauge invariance of the theory. Such a geometrical scheme of harmonic expansion has been proposed in Ref. [2].
The next step, if possible, consists of truncation of the infinite tower of fields so as to get a theory on M with only a finite number of fields. Here one may wish to truncate the theory in such a way that masses of the Planck order do not appear in the effective m-dimensional theory and the truncated theory deals with the massless modes done. The important point which should be observed here is that *neither the procedure of harmonic expansion nor that of truncation is, in general, unique*. This fact is neither "good" nor "bad" - it is a reality one has to live with. A simple recipe for truncation can be given: let G be a subgroup of the isometry group of S, and suppose G is transitive on S. Then consider only those δF's which are G-invariant - this defines a truncation scheme. We get an effective theory on M with only finite number of fields and no Planck masses in the effective mass spectrum. A source of non-uniqueness is clearly seen: there may be more than one choice of G, and the smaller G is, the richer is the spectrum of the effective m-dimensional theory. A well known example comes from 11-dimensional supergravity compactified on $S=S^7$ - the seven-sphere. If we take for G the group SO(8) then *the SO(8)-invariant ansatz gives no gauge fields at all.* On the other hand the group $U(2;\mathbb{H})$ is a subgroup of SO(8) which is also transitive on S^7, and the $U(2;\mathbb{H})$-invariant ansatz produces gauge bosons of SU(2). Of course, this second ansatz is more natural for the "squashed" ground state rather than the "round" one; nevertheless there are no good reasons why it should not be applied in the latter case too. Here it is important to stress that the G-invariant ansatz is, as a rule, consistent.

Before discussing briefly this problem of the consistency let us first try to make more precise what is meant by the term "ansatz". Once a stable ground state solution $\{F_o\}$ of the classical field equation in m+s dimensions has been selected, and once a spontaneous compactification $E \to M \times S$ induced by $\{F_o\}$ has taken place, then we have still to decide on the form of the fluctuations δF-s which will define the effective quantum theory in m dimensions. Usually one selects some finite set of fields $\{f_i\}$ on M and considers δF's as built out of the δf_i-s. In the following we will not split F into F_o and δF but, instead, we will describe how are the field configurations F in E built out of the field configurations f in M. This is called "ansatz", and one should not confuse this "ansatz" with a method of finding topologically interesting solutions of the field equations. Here we are not that much interested in *solutions* but rather in restricting the space of *field configurations* (which defines a domain of the functional integral). For instance, what we call "G-invariant ansatz" is defined as follows: assuming G is a symmetry group of the ground state $\{F_o\}$, we consider only these field configurations {F} in E which are G-invariant (G-singlets). One then finds that every such F can be expressed in terms of a certain number of fields f on M. Solving out these constraints of G-invariance in terms of f's gives then the explicit form of the ansatz. Now, consider the problem of a "consistency" of a given ansatz. Suppose that we have given an explicit expression for F[f], where {f} are fields on M. It is then an easy matter to put F[f] into the action $A_E = \int_{M\times S} L[F[f]]$, to integrate it over S, and to obtain in this way an "effective action" A_M for the fields {f} on M. However, there is no guarantee whatsoever that the field theory on M obtained in this way will be *consistent* with the original one. The requirement of *consistency* is similar to that of *stability*. A truncation obtained by an ansatz F[f] is called *consistent* if the extrema $\{f_o\}$ of the effective action A_M determine extrema $F[f_o]$ of the original action. Or, even simpler, if every solution of the reduced theory is a solution of the original one (see [3-4]). There are many ansatze which are inconsistent. The G-invariant ansatz, which will be discussed in more details later, can be shown to be consistent [3,5]. On the other hand the most popular non-G-invariant ansatz involving Killing vectors is, in general, inconsistent. However, it is to be noticed that an ansatz which is inconsistent with one set of fields may well become consistent with another. This situation apparently happens with the "Killing-vectors-ansatz" used in 11 dimensional supergravity (see the discussion in [3]).

2. The G-invariant ansatz

We will now discuss the geometrical "milieu" of the G-invariant ansatz. The fact that we will strictly adhere to the pure geometrical aspects will make much of the discussion *model-independent*. The drawback of using geometrical methods in the particular case which interests us, but as well in any other case, is that the results depend on satisfying the assumptions, and the assumptions may happen to be too restrictive to accomodate some interesting models. After discussing the G-invariant scheme we will later on weaken our assumptions. But it must be understood that even these weaker assumptions are arbitrarily imposed - they seem to constitute a natural description of today's models, but tomorrow... But even if this is going to happen the "*tool*" of the G-invariant scheme will remain to be useful.

Observe that transformation properties and dynamics of gauge fields are most naturally expressed in geometrical terms when gauge fields are represented by connections on principal fibre bundles. Therefore if in a theory of a Kaluza-Klein type one believes that the dimensional reduction scheme leads to gauge fields with a certain gauge group G, then the natural question to ask is: "where is the principal bundle on which the gauge field is supposed to live?". Answering this question is not a problem *if one starts with assuming* that the universe is a principal bundle to begin with... However such a position seems to be not quite what one wants; and indeed there exists a more natural and more general framework. This is the framework of G-invariant dimensional reduction. This framework is conceptually simple and it has a nice geometrical interpretation. It is well adopted for harmonic expansion and for reduction of all kinds of geometrical objects and matter fields. Last but not least, it leads as a rule to a consistent truncation of massive modes. The method has many advantages but, at the same time, it is certainly not the key to all the enigmas of the Universe. It should be considered rather as a *powerful and convenient mathematical tool*, which it is good to have at hand when it is needed. After describing first this tool of G-invariant dimensional reduction, we will next consider a more general setting, covering it, and we will see how this universal tool can be applied to produce another scheme of dimensional reduction which is more subtle than the G-invariant one.

Let us begin with the following remark: our tool will work and will do its job *whenever there is a group acting on some manifold*. It need not be in the context of dimensional reduction, for instance one can look for solutions of some field equations having certain symmetry, or one can think of the infinite dimensional group of (x-dependent) gauge transformations or diffeomerphisms acting on an infinite dimensional manifold of field configurations... It is therefore for convenience and in order to have some concrete picture in mind that, while describing this tool, we will use a terminology which is adapted to the problems of dimensional reduction in Kaluza-Klein theories.

Let therefore G be a group acting on a manifold E. To make things regular and easy we will assume that G is a compact Lie group which acts smoothly from the right on a d-dimensional smooth manifold E. Given $y \in E$ we denote by G_y the stabilizer of y. The manifold E decomposes now into several *strata* according to the type of the stabilizer. We choose one of these strata and call it E in the following. All the stabilizers G_y, $y \in E$ are now conjugated to a standard one, say H. We now define M to be the space of orbits: $M=E/G$, so that locally $E=M\times(H\backslash G)$ (we write $H\backslash G$ and not G/H since we have chosen *right* action of G). Thinking of some dynamical theory with gauge fields as an output it is now natural to ask: "what principal fibre bundle over M can be seen in the structure we have?". The answer reads: the only potentially non-trivial principal fibre bundle over M which can be constructed out of the ingredients we have put into the game is a principal bundle P with structure group N(H)/H, N(H) being the normalizer of H in G. P is constructed as a subset of E

$$P = \{y \in E \; : \; G_y = H\}$$

The point to be stressed is that no non-trivial fiber bundle with structure group G can be seen emerging. This was the surprising result of [6] , where we have found N(H)/H as the effective gauge group, instead of the expected G. It was also shown that what is geometrically allowed and natural is also dynamically available, i.e. one really gets an N(H)/H gauge field and its Lagrangian from dimensional reduction of G-invariant metric and Einstein-Hilbert action on E.

The effective gauge group from G-invariant dimensional reduction is therefore $G_{eff}=N(H)/H$. In many cases this group N(H)/H can be considered as the biggest subgroup K of G such that $H\times K \subset G$. Here "in many" does not mean "in all"! One must be particularly careful if G is non-compact or non-semisimple. For instance, if G is the Poincaré group and H is the translation group then N(H)/H is the Lorentz group which is not the *direct* factor of the translation group in G.

In [6] the following result has been proven: there is 1-1 correspondence between G-invariant metrics g_E on E and triples (g_M, A, ϕ) of fields on M, where g_M is a metric on M, A is a gauge field with gauge group N(H)/H and ϕ is a multiplet of scalar fields. The metric g_M induced by g_E is called "the Kaluza-Klein projection of g_E". The following comments can be given to this result:

i) The projection $E \to M$ is an example of what is called in the mathematical literature a "Riemannian submersion". There are many results in the mathematical papers dealing with what is called "totally geodesic" case. The case we have to deal with is not of that kind unless the scalar fields are switched off. (There are also mathematical results dealing with the case of gauge fields switched off).

ii) The results and the formulae of [6] form up a *tool*. It can be applied whenever one has a group acting on some manifold, and whenever one is interested in geometrical objects invariant under this group action. A general theory of dimensional reduction of geometrical objects has been initiated in [2].

iii) It is convenient to introduce a concept of a "dimensionally reducible geometrical object". This is an object on E which can be also interpreted as a finite-component field on M. As a rule all objects on E which are G-invariant (G-singlets) are dimensionally reducible. But also objects whose values transform under a finite dimensional representation of G are dimensionally reducible. Sometimes it may be, however, convenient to consider objects transforming under an infinite dimensional representation of G as dimensionally reducible too.

iv) The assumption of the global action of G on E is used in the process of harmonic expansion of fields. As we shall discuss it later it is not necessary to assume that much for the harmonic expansion scheme to work.

3. The non-G-invariant scheme of dimensional reduction

Let us start with an example which will illustrate the idea of "non-invariant" dimensional reduction. The example will at the same time introduce the concept of a double coset, the concept which may prove to be useful for building model manifolds with interesting geometrical properties. Consider a homogeneous space $H\backslash G$, on this space there is a finite parameter family of G-invariant metrics. Indeed, since G acts transitively on the coset

$H\backslash G$, a G-invariant metric on $H\backslash G$ is completely determined by knowing it at one point; the number of G-invariant metrics on $H\backslash G$ is therefore equal to the number of AdH-invariant scalar products at the origin of the coset space. Let now K be another closed Lie subgroup of G, we can form then the double coset space $H\backslash G/K$. The (right) action of K on $H\backslash G$ will have, in general, more than one orbit type. In such cases we can restrict ourselves to an open dense submanifold of $H\backslash G$ which constitutes the principal stratum of K-action on $H\backslash G$. With this understanding $H\backslash G/K$ becomes a manifold. Observe the analogy: $E \sim H\backslash G$, $M \sim H\backslash G/K = E/K$. Every G-invariant metric on $H\backslash G$ is now, a fortiori, K-invariant and therefore, according to the G-invariant scheme of dimensional reduction, determines its Kaluza-Klein projection on $H\backslash G/K$. In this way we obtain a finite-parameter family of metrics on $H\backslash G/K$ which, in general, have no isometries at all. Observe that the group which survives the double quotient and still acts on $H\backslash G/K$ is $N(H)/H \times N(K)/K$ (it is however not automatically guaranteed that the action of this group on $H\backslash G/K$ is effective). It may be instructive to consider a concrete example. Let us therefore take for G the group $U(2;\mathbb{H})$, and for H and K the following two subgroups of $U(2;\mathbb{H})$, each isomorphic to $U(1;\mathbb{H}) \sim SU(2)$:

$$H = \left\{ \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \; : q \in \mathbb{H}, \quad q \neq 0 \right\}$$

$$K = \left\{ \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \; : q \in \mathbb{H}, \quad q \neq 0 \right\}$$

The coset G/K is isomorphic to a seven-sphere S^7. The coset $H\backslash G/K$ is S^4 and $G/K \to H\backslash G/K$ is nothing but the Hopf fibration of S^7. Observe that the residual group which still acts on S^4 is $N(H)/H \times N(K)/K = O(2) \times SU(2)$.

Remark. The groups H and K are both naturally isomorphic to $U(1;\mathbb{H})$, therefore we can take first a partial quotient of G by the "diagonal" $U(1;\mathbb{H})$. The resulting manifold of orbits of the diagonal $U(1;\mathbb{H})$ acting (on both sides) on $U(2;\mathbb{H})$ is diffeomorphic to an exotic seven-sphere Σ^7. The group $O(2)\times SU(2)$ acts therefore on Σ^7. The principal stratum of this action projects onto an open dense subset of S^4. It would be interesting to know whether there is any relation between this construction and exotic $\mathbb{R}^4$ recently investigated (see [7] for Σ^7 and [8] for a review on exotic $\mathbb{R}^4$-s).

After discussing the double-coset example let us discuss a similar construction which will generate a class of non-G-invariant, dimensionally reducible, metrics on a G-space E. Let therefore E, G and M be as in the discussion of the G-invariant ansatz. We already know the form of the most general G-invariant metric on E. We will enlarge now this class of metrics so as to include some dimensionally reducible non-G-invariant ones. To this end we will use the following recipe: first, replace E with a bigger space $\bar{E}=E\times G$. Then, on $\bar{E}$ we have right action of the group G×G: $(y,a)(b,c)=(yb,c^{-1}a)$, and E is isomorphic to the quotient $\bar{E}/G^d$, where G^d is the diagonal of G×G. Indeed, the isomorphism of $\bar{E}/G^d$ onto E is given by $(y,a)\rightarrow ya$. Observe that, in fact, we have action on $\bar{E}$ of the product (G×G)×G, the last factor being the right action of G on itself; it goes to the quotient $E=\bar{E}/G^d$ to coincide with the right action of G on E. Now, consider the class of all G×G-invariant metrics on $\bar{E}$. Since G×G acts on $\bar{E}$ with the stability group $\bar{H}=H\times id$, it follows (by application of the tool of G-invariant dimensional reduction, with $G\rightarrow G\times G$) that these metrics can be described in terms of fields on M, and that they give rise to gauge fields of $N(\bar{H})/\bar{H}=N(H)/H\times G$. But each of these metrics, being G×G-invariant is, a fortiori, G^d-invariant, and therefore it defines, by the Kaluza-Klein projection, a metric on E.The class of metrics on E induced that way contains the class of G-invariant ones, as a subclass. But it contains much more: it contains also those metrics on E which give rise to gauge bosons of G, which degrees of freedom are not contained in the G-invariant ansatz. The receipt given above may seem to be unnatural, this is not, however, so. We will describe now a framework for dimensional reduction which does not require the assumption of a global G-action. And in this framework the receipt above will find its natural place. But before discussing the technical side of the extended framework, let us first analyse the following simple illustrative example: the two-torus contra the Klein-bottle. Both are S^1 fibrations over S^1. Both carry a flat Riemannian metric which correspond to the vacuum configuration $\{F_o\}$ discussed at the beginning. Both are candidates for E with $M=S^1$, $G=U(1)$, H trivial. But the internal U(1) acts globally on the two-torus but does not act globally on the Klein bottle. It is this Klein bottle example which is an archetype for the extended model. This model can be defined by the following axioms

1. There are two fibrations $\mathbb{G}$ and E over M.
2. The fibers G_x $(x\in M)$ are groups which act transitively (from the right) on the fibers E_x of E.

3. There is an open covering (U_α) of M and, for each α, there are maps

$$\phi_\alpha : \pi_E^{-1}(U_\alpha) \to U_\alpha \times (H\backslash G)$$

$$\psi_\alpha : \pi_{\mathbb{G}} (U\ ) \to U_\alpha \times G$$

(ϕ_α and ψ_α are assumed to be diffeomorphisms and are called local trivializations of E and $\mathbb{G}$ respectively; G is a (compact) Lie group, and H is a closed Lie subgroup of G) such that ϕ_α restricts to group isomorphisms $G_x \to G$ on the fibers, and ψ_α satisfies

$$\psi_\alpha(ya) = \psi_\alpha(y)\phi_\alpha(a)$$

for all $y \in E_x$, $a \in G_x$, $x \in U_\alpha$.

The Klein bottle example is a particular case of such a structure. The group G is U(1) here, and the bundle of groups $\mathbb{G}$ coincides with E,i.e. with the Klein bottle itself, in this case. The model of a global G-action considered in Sect.2. is a particular case of the above situation corresponding to the case of $\mathbb{G}$ being the global product $\mathbb{G}=M\times G$.

The important question to be answered reads as follows: what is the natural class of metrics on E? We will answer this question later on, where we will see that the effective gauge group, resulting from the class of metrics we will describe, consists of gauge bosons of the group $G_{eff} \approx N(H)/H \times G$ (modulo the common central factors). Here, anticipating the final result, we will first concentrate on the group theoretical structure arising from the discussed scheme .

Let us start with giving the precise definition of the group G_{eff}. To construct this group we will have to introduce the groups AutG and $\mathrm{Aut}_H G$. The group AutG of all automorphisms of G is a Lie group. This group, however, need not be compact even if G is such. Indeed, the group of automorphisms of the torus U(1)×U(1) contains the non-compact group SL(2,Z) (the map $(u,v) \to (u^m v^n, u^k v^l)$, with $u,v \in U(1)$, is 1-1 if and only if $ml-nk=\pm 1$). If H is a Lie subgroup of G then $\mathrm{Aut}_H G$ will denote the subgroup of AutG consisting of those automorphisms ϕ of G for which $\phi(H)$ is conjugated to H:

$$\mathrm{Aut}_H G = \{\phi \in \mathrm{Aut}\, G : \exists a \in G,\ \ \phi(H) = aHa^{-1}\} .$$

Recall that an automorphism ϕ of G is called inner if there exists $a \in G$ such that $\phi(b)=aba^{-1}$ for all $b \in G$. The group of all inner automorphisms of G is an invariant subgroup of Aut G . Observe that all inner automorphisms belong automatically to $\mathrm{Aut}_H G$. Later we will be interested in the Lie algebra of $\mathrm{Aut}_H G$. Locally (i.e. in a neighborhood of the identity) the group $\mathrm{Aut}_H G$ is isomorphic to $G/Z(G)$, $Z(G)$ being the center of G. On the Lie algebra level therefore we have $\mathrm{Lie}(\mathrm{Aut}_H G)=\mathrm{Lie}(G) - \mathrm{Lie}(Z(G))$.

We will describe now the structure of the group G_{eff} - the effective gauge group arising from the non-invariant scheme. We will first describe the construction of G_{eff}, and only later justify it. The first step is to build the semidirect product $\bar{G}=G \otimes \mathrm{Aut}_H G$. The group $\bar{G}$ consists of pairs (a,ϕ), $a \in G, \phi \in \mathrm{Aut}_H G$, with the semidirect product multiplication law:

$$(a,\phi)(a',\phi') = (a\phi(a'),\phi\phi').$$

The group G_{eff} is then defined as

$$G_{eff} = N(\bar{H})/\bar{H},$$

where $\bar{H}=H \otimes id$ is the subgroup of $\bar{G}$ which is isomorphic to H, and $N(\bar{H})$ is the normalizer of $\bar{H}$ in $\bar{G}$.

<u>Remark</u>. A similar construction of the effective gauge group appeared in studying symmetric Yang-Mills fields [9], with the difference that $\bar{G}$ there was the direct product $\bar{G}=G\times R$ of G and the initial gauge group R, and $\bar{H}$ was the diagonal $H\times\lambda(H)$, $\lambda : H \to R$ being a group homomorphism characterizing the action of G on a principal bundle carrying the initial gauge fields.

Some relevant information concerning the group G_{eff} is contained in the following diagram whose rows and columns are exact

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & H & \to & N(H) & \to & N(H)/H & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \bar{H} & \to & N(\bar{H}) & \to & N(\bar{H})/\bar{H} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & \to & \mathrm{Aut}_H G & \to & \mathrm{Aut}_H G & \to & 1
\end{array}$$

Of particular interest is the last column which tells us that $G_{eff}=N(\bar{H})/\bar{H}$ is an extension of $Aut_H G$ by $N(H)/H$. In the compact case we therefore locally have $N(\bar{H})/\bar{H} \overset{loc}{\simeq} N(H)/H \times Aut_H G \overset{loc}{\simeq} (N(H)/H) \times (G/Z(G))$. To describe the Lie algebra of G_{eff} it is convenient to decompose the Lie algebra of G as follows (we assume that the action of G on G/H is effective, what implies that $H \cap Z(G)$ is trivial).

$$Lie(G) = Lie(H) + S ,$$

$$S = K + L ,$$

$$K = Z + K_1$$

where S is a reductive complement of Lie(H) in Lie(G), K is the subset of S consisting of H - singlets of the adjoint representation, Z is the Lie algebra of the center Z(G) of G, K_1 is a complement of Z in K , and L is a reductive complement of K in S.
The Lie algebra of G_{eff} is then given by

$$Lie(G_{eff}) = Z + K_1 + H + K_1 + L .$$

Observe that K_1 , which is the Lie algebra of $N(H)/(H \times Z)$, enters twice. For instance, if G is simple and H is trivial, then the effective gauge group is $G \times G$. At this place it is to be stressed again that the extended, non-G-invariant scheme of dimensional reduction will, in general, lead to an *inconsistent* ansatz, unless one retains *all* the modes in the harmonic expansion of fields. Before, however, commenting on the problem of a harmonic expansion in the absence of global G-action, let us first show the relation of G_{eff} defined above to the extended scheme based on a pair $(E,\mathbb{G})$, $\mathbb{G}$ being a bundle of groups. The axioms for $(E,\mathbb{G})$ deal with local trivializations ϕ_α, ψ_α so that, on the intersections of the domains $U_\alpha \cap U_\beta$, we get transition functions $\phi_\alpha \circ \phi_\beta^{-1}$ and $\psi_\alpha \circ \psi_\beta^{-1}$. The functions $\phi_\alpha \circ \phi_\beta^{-1}$ take value in the group $Aut_H G$ - the group we already know. The functions $\psi_\alpha \circ \psi_\beta^{-1}$ take value in another group - the group $Taut(H\backslash G)$ of *twisted automorphisms* of $(H\backslash G)$. The group $N(H)/H$ can be identified with the group of automorphisms $Aut(H\backslash G)$ of the homogeneous space $H\backslash G$. Recall that an automorphism of $H\backslash G$ is a map $\psi: H\backslash G \to H\backslash G$ such that $\psi([a]b)=\psi([a])b$ for all $a,b \in G$. To every $n \in N(H)$ there corresponds an automorphism $\psi_n:[a]\to[na]$ of $H\backslash G$, and the map $n \to \psi_n$ defines an isomorphism between $N(H)/H$ and $Aut(H\backslash G)$. The group $Taut(H\backslash G)$ of twisted automorphisms of $H\backslash G$ contains $Aut(H\backslash G)$, and is defined as follows:

Definition A twisted automorphism of $H\backslash G$ is a pair of diffeomorphisms $\phi : G \to G$, $\psi : H\backslash G \to H\backslash G$, where ϕ is an automorphism of G, and ψ satisfies

$$\psi([a]b) = \psi([a])\phi(b)$$

for all $a,b \in G$. The set of all twisted automorphisms $\{(\phi,\psi)\}$ of $H\backslash G$ is a group under the composition of maps, and is denoted by $\mathrm{Taut}(H\backslash G)$.

Remark. We always assume that $H\backslash G$ is an *effective* homogeneous space. In this case the map ϕ in the formula above is completely determind by ψ.

Let us now show that $G_{eff}=N(\bar{H})/\bar{H}$ is indeed isomorphic to $\mathrm{Taut}(H\backslash G)$. First of all observe that $(a,\phi) \in N(\bar{H})$ if and only if $\phi(H)=a^{-1}Ha$. Now, given $(\phi,\psi) \in \mathrm{Taut}(H\backslash G)$ let $[a]=\psi([e])$. Then $[a]=\psi([e])=\psi([e]h)=\psi([e])\phi(h)=[a]\phi(h)$, and so $(\phi,a) \in N(\bar{H})$. It is easy to see that this gives rise to the required isomorphism.

Transition functions allow one to construct a principal bundle. Therefore, as a corollary to the above considerations we find: given an extended Kaluza-Klein scheme $(E,\mathbb{G})$ one can automatically construct two principal bundles: a principal bundle Q with structure group G_{eff}, and another principal bundle, with structure group $\mathrm{Aut}_H G$. In fact the second bundle is the quotient of Q by $N(H)/H$ which is an invariant subgroup of G_{eff}. The group bundle $\mathbb{G}$ is a bundle associated to this quotient bundle. Knowing the above structure one can easily distinguish now a class of dimensionally reducible metrics on E which give rise to G_{eff} gauge boson on M. Namely, having the principal bundle Q over M, with structure group $G_{eff}=N(\bar{H})/\bar{H}$, we can construct an associated bundle $\bar{E}$ with fiber $\bar{H}\backslash\bar{G}$. The group $\bar{G}$ acts now globally on $\bar{E}$ from the right, and E can be identified with a quotient of $\bar{E}$ by $\mathrm{Aut}_H G$, which is a subgroup of $\bar{G}$. The class of metrics on E which interests us is now defined as the Kaluza-Klein projections of $\bar{G}$-invariant metrics on $\bar{E}$. Indeed, every $\bar{G}$-invariant metric on $\bar{E}$ is, a fortiori, $\mathrm{Aut}_H G$-invariant, and therefore projects onto $E=\bar{E}/\mathrm{Aut}_H G$ by the Kaluza-Klein projection. This ansatz produces automatically gauge bosons of G_{eff} on H.

Let us finally comment on the problem of harmonic expansion in the absence of global G-action. A particular example we may keep in mind is that of harmonic expansion of fields defined on the Klein bottle. The important point to observe in this connection is that the harmonic expansion scheme is well defined provided the bundle $Q/(N(H)/H)$ is equipped with a *flat connection*. It does not mean it has to be trivial - by a *flat* connection we mean a connection with zero curvature but possibly non-trivial global holonomy group. (The ground state metric should be a natural source of such a connection). This flat connection will distinguish a class of trivializations of the associated group bundle $\mathbb{G}$, which will be related one to another by *constant* transition functions. Or, in other words, there is a restricted class of *local* G-actions of G on E, related one to another by constant automorphisms of G. And, as one can easily see, the method of harmonic expansion developed in Ref. [2] , can be applied to each local G-action with the results being independent of the choice of a local G-actions in the restricted class.

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