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Colour and Higgs charges in G/H Kaluza–Klein theory

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Abstract. Geodesics in a multidimensional Universe with G-invariant metric are studied and differential equations describing their space–time projection are derived. It is shown that, when internal spaces are cosets rather than group manifolds, then, in addition to the colour charge through which the particle interacts with the gauge field, a new charge arises which couples to the scalar fields only. This new charge, called the Higgs charge, is shown to be of a nonlinear nature. For certain cosets it may contribute to the colour charge non-conservation.

1. Introduction

Recently a geometrical theory of dimensional reduction was developed based on the concept of G-invariance of a Riemannian metric in a multidimensional Universe [1]. In the present paper geodesics in a multidimensional Universe are studied, and it is shown that the components of the velocity pointing into the extra dimensions give rise to the coloured and higgsonic charges. From the formal point of view our equations (5.1)–(5.3) generalise the well known Kerner–Wong equation (see [2–5], also [6] where motion of a charged string is treated) in two ways: firstly, the internal space is taken to be a homogeneous space G/H rather than a group manifold, and secondly, interaction with the Jordan–Thierry scalars, originating from the metric on G/H, is taken into account. Before going into the details of this paper's results, it is worthwhile analysing the relation between the geometrical framework used in [1], and the, so-called, Kaluza–Klein theory. The Kaluza–Klein mechanism is supposed to work as follows (see e.g. [7, 8], and references therein): one starts with a generally covariant field theory in $(4+n)$ dimensions (e.g. eleven-dimensional supergravity), containing among its fields $\{\phi\}$ the metric field g_{AB} $(A, B=1,2,\ldots,4+n)$. Suppose the theory admits a ground state $\{\mathring{\phi}\}$ with the property that the $(4+n)$-dimensional space E splits into a (local) product $E=M\times S$ with respect to the ground state metric $\mathring{g}_{AB}(x,y)=(\mathring{g}_{\mu\nu}(x), g_{\alpha\beta}(y))$, with $x\in M$, $y\in S$. If M is four-dimensional of signature $(+---)$, and if S is compact, then one says that a *spontaneous compactification* takes place. The next step to be taken is called *dimensional reduction.* It is believed that the lowest excitations from the ground state can be interpreted in terms of massless fields on M alone—the gravity, Yang–Mills fields, scalars. A general geometrical framework taking care of the whole mechanism has not yet been worked out, although much work has been recently done in two directions: (a) constructing particular models of physical interest (e.g. [8–11]),

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and (b) elaborating mathematical apparatus describing selected features of the models (e.g. [12–16]). Of particular interest are recent works on eleven-dimensional supergravity, where it was found that a ground state need not have as its symmetry group G the maximal symmetry group allowed by the topology of the internal space S. For example, the ground state metric of the eleven-dimensional gravity may have only SO(5)×SU(2) (instead of SO(8)) symmetry ('squashed seven-sphere', [17]) or, another possibility, the sphere can be metrically round, but the other fields may have non-zero values $\overset{\circ}{\phi} \neq 0$ in the ground state breaking the symmetry (from SO(8) down to SO(7) in [18], [19]). In such cases it is of interest to study those excitations of the ground state metric which have G as its symmetry group. It is this problem that was solved in [1] (see also [16]). Under assumption that the internal space is a homogeneous space G/H it was shown that the most general G-invariant metric on $M\times(G/H)$ is described in terms of metric $g_{\mu\nu}$ on M, gauge field A_μ for the gauge group $K = N(H)|H$, and charged scalar fields $g_{\alpha\beta}$ describing internal geometry of $S = G/H$. If $H=\{e\}$ (i.e. if S is a group manifold) then $K = G$, but for, say, $G=\mathrm{SO}(8)$, $H=\mathrm{SO}(7)$ one finds $K=Z_2$, so that K looks much smaller than one would expect. In general, therefore, the condition of G-invariance will select only certain and not all zero modes. Thus, the analysis of interaction of point particles with the massless fields $g_{\mu\nu}$, A_μ and $g_{\alpha\beta}$, given in the present paper, should be extended when a geometry of a complete ansatz (see [7], [8]) is fully understood, so that the interaction with the remaining massless modes can be taken into account.

With the above in mind, let us make some more comments on the results of this paper. It is important to observe that the Lie-algebra decomposition corresponding to a homogeneous space G/H is not $\mathcal{G}=\mathcal{H}+\mathcal{S}$, where $\mathcal{S}$ can be interpreted as the space tangent to G/H at the origin, but a more subtle one: $\mathcal{G}=\mathcal{H}+\mathcal{K}+\mathcal{L}$, where $\mathcal{K}$ is the Lie algebra of $K=N(H)|H$ (the effective gauge group), $N(H)$ being the normaliser of H in G. Consequently the metric $g_{\alpha\beta}$ on $S=G/H$ splits into two kinds of Jordan–Thierry-type fields: g_{ab} (a, b—the indices of $\mathcal{L}$), and $g_{\hat{a}\hat{b}}$ ($\hat{a}$, $\hat{b}$—the indices of $\mathcal{K}$). It is shown that the interaction of the particle with both fields is of the form $Z^{\alpha\beta}D_\mu g_{\alpha\beta}$, $Z^{\alpha\beta}$ being composite: $Z^{\alpha\beta}=\lambda^\alpha\lambda^\beta$. $\lambda^{\hat{a}}$ is just the coloured charge, interacting with A_μ, while λ^a is a new charge, called a higgsonic charge. We give evolution equations for all charges and argue that the Higgs charges are nonlinear: they take values in the manifold $\mathcal{L}/\mathrm{Ad}(H)$ of H-orbits in $\mathcal{L}$ (H acting on $\mathcal{L}$ by the adjoint representation). Readers who are not interested in the details may get the idea of the present paper by reading § 5 where the summary of results and a simple example are given.

2. Geometry of a G-Universe

G-space is a pair (E, G), where E is a differentiable manifold and G is a group of transformations of E. We shall assume that G is a compact Lie group acting smoothly on E from the right. (Our discussion can also be applied for non-compact G but in that case, at certain places, extra assumptions are to be made.)

A G-Universe is a triplet (E, G, g), where (E, G) is a G-space, and g is a Riemannian metric on E which is G-invariant i.e.

$$g_y(v_1, v_2) = g_{ya}(v_1 a, v_2 a), \tag{2.1}$$

for all $y\in E$, $a\in G$, and $v_1, v_2\in T_yE$. Here va stands for $(R_a)_* v = dR_a(v)$, where

R_a: $y \to ya$ is the right translation of $y \in E$ by $a \in G$. It is assumed that E has only one orbit type (for a compact G the principal orbit theorem [20] asserts that this is always the case outside of a set of measure zero). Thus we assume that the isotropy groups $G_y = \{a \in G | ya = y\}$ are all conjugated to a standard one, denoted by H. One proves then (see the discussion and references in [1]) that E is a fibre bundle with fibre $H \backslash G$† and base $M = E/G$—the space of orbits (observe that $\dim M = \dim E - (\dim G - \dim H)$). Moreover, the submanifold P of E defined by

$$P \overset{\text{df}}{=} \{y \in E | G_y = H\}, \tag{2.2}$$

consisting of all those points $y \in E$ of which the stability group is not only conjugated but exactly equal to H, is a principal bundle with the same base M and structure group $K \doteq N|H$, where N is the normaliser of H in G

$$N \doteq \{n \in G | nHn^{-1} = H\} \tag{2.3}$$

(see figure 1).

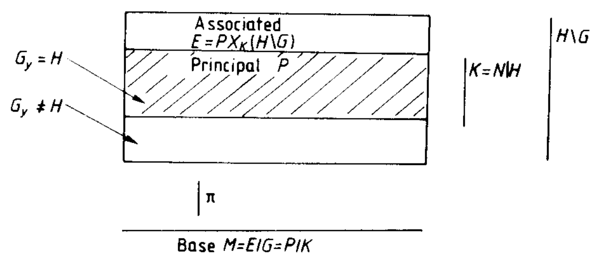

Figure 1. The picture can be misleading since, in general, $\dim P < \dim E$.

One can also show that E can be considered as a bundle associated to P via the canonical representation of K on $H \backslash G$ given by: $[n][a] \doteq [na]$, $[n] \in K = N|H$, $[a] \in H \backslash G$.

It is convenient to split the Lie algebra $\mathscr{G}$ of G as follows (see [1])

$$\mathscr{G} = \underbrace{\overbrace{\mathscr{H} + \mathscr{K}}_{\mathscr{N}}}_{} \!\!\overset{\mathscr{S}}{\overbrace{\phantom{\mathscr{K}} + \mathscr{L}}} \tag{2.4}$$

where $\mathscr{N} = \mathscr{H} + \mathscr{K}$ is a reductive (i.e. $\mathrm{Ad}(H)\mathscr{K} \subset \mathscr{K}$) decomposition of the Lie algebra $\mathscr{N}$ of N, and $\mathscr{G} = \mathscr{N} + \mathscr{L}$ is a reductive (i.e. $\mathrm{Ad}(N)\mathscr{L} \subset \mathscr{L}$) decomposition of $\mathscr{G}$. The subspace $\mathscr{S} \doteq \mathscr{K} + \mathscr{L}$ can be identified with the space tangent to $H \backslash G$ at the origin. We have

$$\begin{aligned} &\text{(i)} && [\mathscr{H}, \mathscr{K}] = 0, \\ &\text{(ii)} && [\mathscr{K}, \mathscr{K}] \subset \mathscr{K}, \\ &\text{(iii)} && \mathrm{Ad}(H)\mathscr{S} \subset \mathscr{S}, \\ &\text{(iv)} && \mathrm{Ad}(N)\mathscr{L} \subset \mathscr{L}, \end{aligned} \tag{2.5}$$

† We denote by $H \backslash G$ the space of right cosets of H in G which is a right G-space, and a left N-space (see also [22, p 112]).

and $\mathcal{K}$ can be identified with the Lie algebra of $K = N|H$ [1]. We shall assume that H is connected†, in which case one also gets

$$\mathcal{K} = \mathcal{S} \cap \mathcal{Z}_H, \tag{2.6}$$

where $\mathcal{Z}_H$ is the commutant (centraliser) of $\mathcal{H}$ in $\mathcal{G}$ [21].

Let ε_i be a basis in $\mathcal{G}$ adapted to the decomposition (2.4). The indices used for basis vectors in a particular subspace of $\mathcal{G}$ are shown in figure 2.

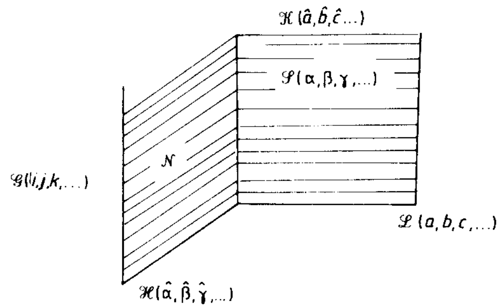

Figure 2.

For every $\xi \in \mathcal{G}$ we denote by $\tilde{\xi}$ the fundamental vector field generated by ξ:

$$\tilde{\xi}(y) = (\mathrm{d}/\mathrm{d}t)(y \exp(t\xi))|_{t=0}. \tag{2.7}$$

The fundamental vector fields corresponding to the basic vectors ε_i are denoted by e_i (they are Killing vectors for g). We have

$$[e_i, e_j] = C^k_{ij} e_k, \tag{2.8}$$

where C^k_{ij} are structure constants of $\mathcal{G}$ in the basis ε_i. The vector fields $e_{\hat{\alpha}}$, corresponding to the action of H, vanish on the submanifold P by (2.2). It follows that (e_α) alone constitute a frame for fundamental vectors on P, and therefore also in a certain neighbourhood U of P. Thus in U

$$[e_\alpha, e_\beta] = f^\gamma_{\alpha\beta} e_\gamma, \tag{2.9}$$

where the structure functions $f^\gamma_{\alpha\beta}(y)$, $y \in U$, are constant on P

$$f^\gamma_{\alpha\beta}(p) \equiv C^\gamma_{\alpha\beta}, \qquad p \in P \tag{2.10}$$

by (2.8).

So far we have discussed only the structure of E resulting from the action of G. We now turn to those important aspects of geometry of E which come from the G-invariant Riemannian metric g in E. First of all each tangent space T_yE decomposes into

$$T_yE = V_y \oplus H_y, \tag{2.11}$$

where $V_y = \{\tilde{\xi}(y) | \xi \in \mathcal{G}\}$, called the vertical space at y, is the space tangent to the orbit of G through y, and H_y (called horizontal) is defined as the orthogonal complement

† If this is not the case one may go to a covering E' of E to get H_0, the connected component of the identity of H, as the isotropy group in E'.

(with respect to the metric g) of V_y in T_yE. The vectors in V_y and H_y are called vertical and horizontal respectively. We fix a coordinate system x^μ in an open domain $W \subset M$, and denote by e_μ the horizontal lifts of vector fields ∂_μ tangent to the coordinate lines. The vector fields $e_A = (e_\mu, e_\alpha)$ now constitute a moving frame in $U \cap \pi^{-1}(W)$—see figure 3.

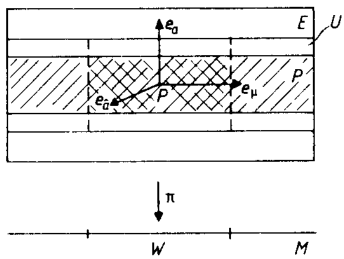

Figure 3. Vector fields $e_{\hat{a}}$ and e_μ are (at $p \in P$) mutually orthogonal and tangent to P. Vector fields e_a are (on P) orthogonal to P.

Owing to the relation (2.6) the restrictions of the adjoint representation of H to $\mathcal{K}$ and $\mathcal{L}$ respectively are disjoint, which implies that the vector fields e_a and $e_{\hat{a}}$ are mutually orthogonal with respect to the invariant metric g [23].

In [1] a reduction theorem was proven giving a complete description of geometry of G-Universes—it was shown that a G-invariant Riemannian metric $g = (g_{AB})$ on E is equivalent to a triple $(\omega^{\hat{a}}, g_{\mu\nu}, g_{\alpha\beta})$, where $\omega^{\hat{a}}$ is a one-form of a principal connection in P, $(g_{\mu\nu})$ is a Riemannian metric on M, and $(g_{\alpha\beta})$ is a cross-section of a certain vector bundle associated to P. This reduction theorem gives thus a theory of the Kaluza–Klein type, for homogeneous fibres and with generalised Jordan–Thierry-type scalars $g_{\alpha\beta}$. The three ingredients $\omega^{\hat{a}}$, $g_{\mu\nu}$, and $g_{\alpha\beta}$ of a G-invariant metric $g = (g_{AB}) = [g(e_A, e_B)]$ can be constructed as follows

$$g_{\mu\nu}(x) = g(e_\mu(y), e_\nu(y)), \qquad \pi(y) = x, \qquad x \in M, \tag{2.12}$$

$$g_{\alpha\beta}(p) = g(e_\alpha(p), e_\beta(p)), \qquad p \in P, \tag{2.13}$$

$$\omega^{\hat{a}}(u_p) = u^{\hat{a}}, \qquad u_p = u^{\hat{a}} e_{\hat{a}}(p) + u^\mu e_\mu(p) \in T_pP. \tag{2.14}$$

Observe that

$$g_{a\hat{a}}(p) = 0, \qquad p \in P, \tag{2.15}$$

owing to the orthogonality of $e_a(p)$ and $e_{\hat{a}}(p)$. For $n \in N$ we have $pn \in P$ for $p \in P$, and

$$e_\alpha(pn) = A(n^{-1})_\alpha^{\alpha'} e_{\alpha'}(p), \tag{2.16}$$

and

$$g_{\alpha\beta}(pn) = A(n^{-1})_\alpha^{\alpha'} A(n^{-1})_\beta^{\beta'} g_{\alpha\beta}(p), \tag{2.17}$$

where $A(n)_\alpha^{\alpha'}$ is the matrix of the adjoint representation

$$\mathrm{Ad}(n)\varepsilon_\alpha = A(n)_\alpha^{\alpha'} \varepsilon_{\alpha'}. \tag{2.18}$$

The scalar fields $g_{\alpha\beta}$ satisfy the constraint of Ad(H)—invariance, infinitesimally

$$g_{\alpha\gamma}(p) C^\gamma_{\hat{a}\beta} + g_{\beta\gamma}(p) C^\gamma_{\hat{a}\alpha} = 0, \qquad p \in P, \tag{2.19}$$

which, owing to the assumed connectedness of H, is also sufficient for H-invariance on an integral level.

We end this section with calculation of coefficients Γ^C_{AB} of the Levi–Civita connection ∇ of g. We introduce the following notation

$$\begin{aligned} &\nabla_A e_B = \Gamma^C_{AB} e_C, \qquad \Gamma_{AB,C} = g_{CD}\Gamma^D_{AB}, \qquad g_{AB} = g(e_A, e_B) = (g_{\mu\nu}, g_{\alpha\beta}), \\ &[e_A, e_B] = f^C_{AB} e_C \qquad f_{AB,C} = g_{CD} f^C_{AB}. \end{aligned} \tag{2.20}$$

Since $e_\mu(p)$ are tangent to P, we get (compare [12])

$$f^\alpha_{\mu\nu}(p) = \begin{cases} 0, & \alpha = a, \\ -F^{\hat{a}}_{\mu\nu}(p), & \alpha = \hat{a}, \end{cases} \tag{2.21}$$

where $F = D\omega$ is the curvature two-form of the connection ω. The coefficients $\Gamma_{AB,C}$ of the Levi–Civita connection can be found from the general formula

$$\Gamma_{AB,C} = \tfrac{1}{2}(e_A(g_{BC}) + e_B(g_{AC}) - e_C(g_{AB})) + \tfrac{1}{2}(f_{AB,C} + f_{CA,B} - f_{BC,A}). \tag{2.22}$$

Because of G-invariance of g_{AB}, the fields e_α are Killing vector fields for g_{AB}:

$$e_\alpha(g_{AB}) = f_{\alpha A,B} + f_{\alpha B,A}. \tag{2.23}$$

Since e_μ are invariant and e_α are fundamental, we have $f_{\alpha\mu,A} = 0$, and therefore

$$e_\alpha(g_{\mu\nu}) = 0. \tag{2.24}$$

It is convenient to introduce the notation

$$e_\mu(g_{\alpha\beta}) = D_\mu g_{\alpha\beta}, \tag{2.25}$$

$$f^\alpha_{\mu\nu} = -F^\alpha_{\mu\nu}, \tag{2.26}$$

also outside P, although it is only on P that $D_\mu g_{\alpha\beta}$ is the covariant derivative (with respect to ω) of the cross-section $g_{\alpha\beta}$, and $F^\alpha_{\mu\nu}$ is the curvature two-form of ω. Taking into account the formulae above we find

$$\begin{aligned} &\Gamma_{\alpha\beta,\gamma} = \tfrac{1}{2}(f_{\alpha\beta,\gamma} - f_{\gamma\alpha,\beta} + f_{\beta\gamma,\alpha}) \\ &\Gamma_{\alpha\beta,\mu} = -\Gamma_{\alpha\mu,\beta} = -\Gamma_{\mu\alpha,\beta} = -\tfrac{1}{2}D_\mu g_{\alpha\beta} \\ &\Gamma_{\alpha\mu,\nu} = \Gamma_{\mu\alpha,\nu} = -\Gamma_{\mu\nu,\alpha} = \tfrac{1}{2}F_{\mu\nu,\alpha} \\ &\Gamma_{\mu\nu,\sigma} = \{\text{the Christoffel symbols of } g_{\mu\nu} \text{ on } M\}. \end{aligned} \tag{2.27}$$

3. Geodesics in G-Universe

Consider a geodesic γ: $t \to \gamma(t)$ in E. Since g_{AB} is G-invariant, it follows that for each $a \in G$ the path γa: $t \to \gamma(t)a$ is a geodesic too. Observe that both γ and γa have the same projection $\pi(\gamma) = \pi(\gamma a)$ on M, and it is this projection which an observer living on M, and blind to extra dimensions, sees. He does not distinguish between the individual elements of the whole bundle $[\gamma] = \{\gamma a\colon a \in G\}$ of geodesics with the same projection on M. Our aim in this section is to identify the set of data which is adequate for a description of the projected motion i.e. the set of quantities which characterise the equivalence class $[\gamma]$ rather than one of its representatives†. In the well known

† The idea of considering the equivalence class $[\gamma]$ was brought to author's attention by M Dubois-Violette.

case of the Kaluza–Klein theory on a principal bundle ($E \equiv P$) this problem does not cause any difficulty since through any point $p \in \pi^{-1}(\pi(\gamma(t)))$ there passes exactly one representative of the class $[\gamma]$. As we will see below this is not the case in general (i.e. when $\gamma(t) \in E \supset P$), in which case a much subtler analysis is necessary. Needless to say the discussion below covers the principal bundle case (see [2, 4, 5, 6]) including also the Higgs charges responsible for interaction with Jordan–Thierry fields.

Consider a class $[\gamma] = \{\gamma a\colon a \in G\}$ of geodesics in E, and let $x_0 = \pi(\gamma(t_0))$ for some fixed t_0. Let $p \in P$ be a point in P above x_0, $\pi(p) = x_0$. Given $\gamma \in [\gamma]$, let

$$\dot{\gamma}(t) = z(t) + v(t) \tag{3.1}$$

be the (orthogonal) decomposition of the vector $\dot{\gamma}(t)$, tangent to γ at t, into its horizontal $z(t) = z^{\mu}(t)e_{\mu}(\gamma(t))$, and vertical $v(t) = v^{\alpha}(t)e_{\alpha}(\gamma(t))$ components. Owing to G-invariance of vector fields e_{μ} it follows that the components $z^{\mu}(t)$ are the same for all members of the equivalence class $[\gamma]$; in fact $z^{\mu}(t)$ are the components of the vector tangent to the projection $\pi[\gamma] \subset M$. We now analyse the information contained in the vertical component $v(t_0)$.

Given $\gamma \in [\gamma]$ we can find $a \in G$ such that $pa = \gamma(t_0)$ (see figure 4). Of course such an a is unique only up to an element h from the isotropy group H of p. Choosing one $a \in G$ with the above property we define $w(p) \in T_p(E)$ by

$$w(p) = v(t_0)a^{-1}. \tag{3.2}$$

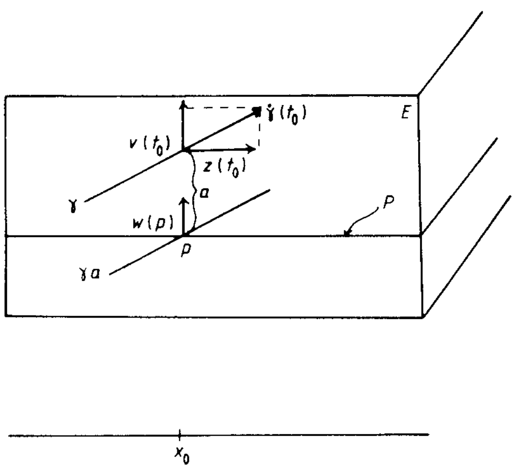

Figure 4. Here P is drawn differently from figure 1 to stress the fact that, in general, a geodesic in E intersects P only once, this is because $\dim P < \dim E$. Observe that angles between γ and the Killing vectors are constants of motion.

Now $w(p)$ is vertical and can be identified with a vector $\eta(p)$ in the vector spaces $\mathcal{S}$,

$$\eta(p) \doteq w^{\alpha}(p)\varepsilon_{\alpha}, \tag{3.3}$$

where

$$w(p) = w^{\alpha}(p)e_{\alpha}(p). \tag{3.4}$$

The vector $\eta(p)$ depends both, on a choice of $\gamma \in [\gamma]$, and $a \in G$. If γ is replaced by

$\gamma' = \gamma b$, $b \in G$, and a is replaced by a' satisfying $pa' = \gamma'(t_0)$, then

$$w' = v'a'^{-1} = vba'^{-1} = waba'^{-1}, \tag{3.5}$$

and, since $pa' = \gamma'(t_0) = \gamma(t_0)b = pab$, it follows that $aba'^{-1} = h \in H$, and consequently

$$\eta' = \mathrm{Ad}(h)\eta, \qquad h \in H. \tag{3.6}$$

It follows that the class $[\gamma]$ determines an $\eta \in \mathcal{S}$ up to a transformation $\mathrm{Ad}(h)$, $h \in H$.

Corresponding to the decomposition $\mathcal{S} = \mathcal{K} + \mathcal{L}$ of $\mathcal{S}$, we decompose η into

$$\eta = q + \lambda, \qquad q \in \mathcal{K}, \qquad \lambda \in \mathcal{L}. \tag{3.7}$$

Then, because of (2.5i), we find that the $\mathcal{K}$-component q of η is $\mathrm{Ad}(H)$ invariant, and therefore $q = (q^{\hat{a}})$ does not depend on the choice of γ in the class and of the connecting element a.

The above considerations give the following result: the class $[\gamma]$ of geodesics determines at $p \in P$ two quantities: a vector $q(p) = (q^{\hat{a}}(p)) \in \mathcal{K}$, and an orbit $[\lambda(p)] = [\lambda^a(p)] \in \mathcal{L}/\mathrm{Ad}(H)$ of H acting on $\mathcal{L}$ by the restriction of the adjoint representation. Now, suppose p is replaced by pn, $n \in N$. It is easy to see that then

$$q(pn) = \mathrm{Ad}(n^{-1})q(p), \qquad \lambda(pn) = \mathrm{Ad}(n^{-1})\lambda(n), \tag{3.8}$$

the transformations of q and $[\lambda]$ being dependent only on the class $[n]$ of n in $K = N|H$. Therefore $q(p)$, considered as a vector $q(x_0) \doteq p \cdot q(p)$ in the fibre of the vector bundle associated to P via the adjoint representation of K on $\mathcal{K}$, is nothing but a coloured charge, at x_0, of the particle described by the geodesic class $[\gamma]$†. Similarly $[\lambda(p)]$ can be considered as a coordinate of the point $[\lambda](x_0) = p \cdot [\lambda(p)]$ in the nonlinear fibre of the associated bundle $P \times_K (\mathcal{L}/\mathrm{Ad}(H))$. The charge $[\lambda]$, taking values in the orbit manifold rather than in a vector space, describes the slope of $[\gamma]$ with respect to P. It is straightforward to show that, conversely, the set of data consisting of $z^\mu(x_0)$, $q(x_0)$ and $[\lambda(x_0)]$ determines geodesic class $[\gamma]$, with $x_0 \in \pi[\gamma]$, uniquely. Thus the initial data $x^\mu(t_0)$, $\dot{x}^\mu(t_0)$, $q^{\hat{a}}(t_0)$, and $[\lambda^a(t_0)]$ determine $x^\mu(t)$, $q(t)$, and $[\lambda(t)]$, at least in a neighbourhood of t_0.

4. Equations of motion for a particle carrying coloured and higgsonic charges

According to the discussion given in § 3 the adequate set of initial data describing the motion of a particle in external fields, $A^{\hat{a}}_\mu$, $g_{\mu\nu}$, $g_{\alpha\beta}$ consists of x^μ_0, $\dot{x}^\mu_0$, q_0, and $[\lambda_0]$, where q_0 and $[\lambda_0]$ are elements of the associated bundles $P \times_K \mathcal{K}$ and $P \times_K (\mathcal{L}/\mathrm{Ad}(H))$. It remains only to find differential equations governing the time evolution of x^μ, q, and $[\lambda]$. Whereas the differential equations for x^μ and q can be derived rather easily, it is much harder to get one for $[\lambda]$—the reason being that $[\lambda]$ takes values in a manifold rather than in a vector space.

Let $[\gamma]$ be a geodesic class as discussed in § 3, and let $x^\mu(t)$, $q(t)$, and $\lambda(t)$ describe the projection of $[\gamma]$ on M and the time evolution of q and λ. Our strategy for deriving differential equations for x^μ, q, and λ is as follows

(1) choose t_0, $x_0 = (x^\mu(t_0))$, and $p_0 \in P$ with $\pi(p_0) = x_0$),

† To make contact with [4, 6], where Wong equations on a principal bundle are discussed and the charge takes values in $\mathcal{K}^*$ rather than in $\mathcal{K}$ observe that $g_{\hat{a}\hat{b}}$ gives us an (x-dependent) isomorphism between $\mathcal{K}$ and $\mathcal{K}^*$.

(2) choose a representative $\gamma \in [\gamma]$ such that $\gamma(t_0) = p_0$
(3) choose a convenient gauge (i.e. a local section of P)
(4) choose a convenient connecting function $a(t)$
(5) find the relation between x^μ, $q^{\hat{a}}$, λ^a, and their rates of change (time derivatives) at t_0, for the particular choices made before
(6) interpret the resulting equations in an invariant way.

The first two choices being made let us discuss the choice (3) of a convenient gauge. We define $p(t)$ to be the unique horizontal curve in P with the same projection $x^\mu(t)$ as $\gamma(t)$, and such that $p(t_0) = p_0$. We take σ: $x \to \sigma(x)$ to be any local cross-section of P which contains $p(t)$. Such a cross-section σ containing a given horizontal curve always exists since curvature gives no local obstructions along a one-dimensional manifold. This choice of gauge can be described also in terms of the gauge potential:

$$A_\mu^{\hat{a}}(x(t))\, \mathrm{d}x^\mu/\mathrm{d}t \equiv 0, \tag{4.1}$$

i.e. $A_\mu(x(t))$, vanishes on vectors tangent to $x(t)$.

Let now $a(t)$ be a smooth function $t \to a(t) \in G$ connecting $p(t)$ with $\gamma(t)$†

$$p(t)a(t) = \gamma(t), \qquad a(t_0) = e. \tag{4.2}$$

Differentiating with respect to t, and taking into account the decomposition (3.1) we find that

$$v(t) = \tilde{\zeta}(t)(\gamma(t)),$$

where

$$\zeta(t) = (\mathrm{d}/\mathrm{d}s)a^{-1}(t)a(s)|_{s=t} = a^{-1}\dot{a}(t).$$

From (3.2) we then get

$$w(t) = \tilde{\eta}_1(t)(p(t)),$$

where

$$\eta_1(t) = (\mathrm{d}/\mathrm{d}s)a(s)a^{-1}(t)|_{s=t} = \dot{a}a^{-1}(t).$$

If $a(t)$ is replaced by $a'(t) = h(t)a(t)$, $h(t) \in H$, then

$$\eta_1' = h\eta h^{-1} + \dot{h}h^{-1}.$$

It follows that there is a unique function $t \to h(t)$ such that $\eta_1'(t) \in \mathcal{S}$ for all t. Indeed, $h(t)$ must be a solution of the equation

$$\dot{h}h^{-1} = -h\eta_{1,\mathcal{H}}h^{-1},$$

where $\eta_{1,\mathcal{H}}$ is the $\mathcal{H}$-component of $\eta_1 \in \mathcal{G} = \mathcal{H} + \mathcal{S}$, and such a solution, satisfying $h(t_0) = e$, is unique. In other words: there exists a unique connecting function $a(t)$ with the property that

$$(\mathrm{d}/\mathrm{d}s)a(s)a^{-1}(t)|_{s=t} = \eta(t) \in \mathcal{S} \tag{4.3}$$

for all t.

Having chosen a convenient gauge (4.1), and a canonical connecting function $a(t)$ defined by (4.3) we proceed to derive differential equations for x^μ, $q^{\hat{a}}$, and λ^a.

† To guarantee the existence of a smooth connecting function one uses local triviality of the fibration $G \to H\backslash G$ ([22, p 83]).

Denoting by $u^A=(z^\mu, v^\alpha)$ the components of the vector $\dot\gamma$ tangent to γ, the geodesic equations are

$$du^A/dt+\Gamma^A_{BC}u^Bu^C=0.$$

Substituting the connection coefficients (2.20), and specifying $A=\mu$ and $A=\alpha$, we get

$$dz^\mu/dt+\Gamma^\mu_{\nu\rho}(x(t))z^\nu z^\rho+g^{\mu\rho}F_{\nu\rho,\alpha}(\gamma(t))z^\nu v^\alpha-\tfrac{1}{2}D^\mu g_{\alpha\beta}(\gamma(t))v^\alpha v^\beta=0, \qquad (4.3)'$$

$$dv^\alpha/dt+g^{\alpha\delta}(D_\mu g_{\beta\delta})z^\mu v^\beta+g^{\alpha\beta}f_{\gamma\beta,\delta}(\gamma(t))v^\gamma v^\delta=0. \qquad (4.4)$$

Since $\gamma(t)=p(t)a(t)$, and taking into account the invariance properties of $F^\alpha_{\mu\nu}$ and $g_{\alpha\beta}$, we have

$$F_{\nu\rho,\alpha}(\gamma(t))v^\alpha=F_{\nu\rho,\alpha}(p(t)a(t))(w(t)a(t))^\alpha$$
$$=F_{\nu\rho,\alpha}(p(t))w^\alpha(t)=F_{\nu\rho,\hat a}(x(t))q^{\hat a}(t),$$

the last term being expressed in the gauge σ. Similarly

$$D^\mu g_{\alpha\beta}(\gamma(t))v^\alpha v^\beta=D^\mu g_{\hat a\hat b}(x(t))q^{\hat a}(t)q^{\hat b}(t)+D^\mu g_{ab}(x(t))\lambda^a(t)\lambda^b(t).$$

Consequently, with $z^\mu=\dot x^\mu$, (4.3) can be written, in the canonical gauge σ defined by (4.1) as

$$D\dot x^\mu/dt\equiv\ddot x+\Gamma^\mu_{\nu\rho}\dot x^\nu\dot x^\rho=g^{\mu\rho}g_{\hat a\hat b}F^{\hat a}_{\rho\nu}\dot x^\nu q^{\hat b}+\tfrac{1}{2}(D^\mu g_{\hat a\hat b})q^{\hat a}q^{\hat b}+\tfrac{1}{2}(D^\mu g_{ab})[\lambda^a][\lambda^b]. \qquad (4.5)$$

Notice that the last term depends on the class $[\lambda]$ of λ only—this owing to the $\mathrm{Ad}(H)$-invariance of g_{ab} (2.19) and, consequently, $D_\mu g_{ab}$.

The equation (4.5) is a generalised Kerner–Wong equation describing the trajectory of a particle in gravitational, gauge, and scalar fields $g_{\mu\nu}$, $A^{\hat a}_\mu$, and $g_{\alpha\beta}$. We proceed now to find the time evolution of the charges q and λ from geodesic equations (4.4). The relation between v^α appearing in (4.4) and $\eta^\alpha=q^\alpha+\lambda^\alpha$ is given implicitly by (3.2)–(3.4), but what we need is an explicit formula. To derive such a formula observe that the components v^α of v with respect to $e_\alpha(\gamma(t))$ are the same as the components of $w=va^{-1}$ with respect to $e_\alpha(\gamma(t))a^{-1}$. Now

$$e_\alpha(\gamma(t))a^{-1}(t)=A^\beta_\alpha(t)e_\alpha(p(t)), \qquad (4.6)$$

where $A^\beta_\alpha(t)$ are the elements of the matrix of the adjoint representation of G:

$$\mathrm{Ad}(a(t))\varepsilon_i=A^\alpha_i(t)\varepsilon_\alpha+A^{\hat\alpha}_i(t)\varepsilon_{\hat\alpha}. \qquad (4.7)$$

Therefore we obtain

$$\eta^\alpha(t)=A^\alpha_\beta(t)v^\beta(t). \qquad (4.8)$$

Differentiating (4.7) with respect to t, and taking into account (4.3), we get

$$\dot A^\alpha_\beta=(f^\alpha_{\gamma\delta}A^\delta_\beta+f^\alpha_{\gamma\hat\delta}A^{\hat\delta}_\beta)A^\gamma_{\gamma'}v^{\gamma'}, \qquad (4.9)$$

and therefore

$$\dot\eta^\alpha=\dot A^\alpha_\beta v^\beta+A^\alpha_\beta\dot v^\beta=A^\alpha_\beta\dot v^\beta+f^\alpha_{\gamma\hat\delta}A^{\hat\delta}_\beta v^\gamma v^\beta. \qquad (4.10)$$

The first term of (4.9) gives no contribution to (4.10) because of the antisymmetry of $f_{\gamma\delta}$. We could now use the geodesic equations (4.4) to get the evolution equation for η^α if not for the second term in (4.10) containing the unknown functions $A^{\hat\delta}_\beta$ which cannot be eliminated. At this place the following observation should be made: we know that $x^\mu(t)$, $\dot x^\mu(t)$, $q^{\hat a}(t)$ and $[\lambda^a(t)]$ determine the trajectory, true, but we have no *a priori* guarantee that the evolution of $q^{\hat{a}}(t)$ and $[\lambda^a(t)]$ can be described by first-order differential equations. In fact, often, after projecting down of a differential equation, one gets an integrodifferential equation instead. Fortunately the difficulty we met can be dealed with as follows. First of all notice that $\dot{\eta}^\alpha$ are the components of a vector tangent to $\mathscr{S}$ at $\eta=(\eta^\alpha)$. Passing to the quotient $\mathscr{S}/\mathrm{Ad}(H)$ all directions obtained by infinitesimal transformations of η by elements of H become equivalent. In other words $\dot{\eta}$ and $\dot{\eta}+[\chi,\eta]$, $x\in\mathscr{H}$, determine the same vector $[\dot{\eta}]$ tangent to $\mathscr{S}/\mathrm{Ad}(H)$ at $[\eta]$. Now, the second term of (4.10) is precisely of the form $[\chi,\eta]$ for $\chi=A^{\hat{\delta}}_\beta v^\beta\varepsilon_{\hat{\delta}}$. Therefore

$$[\dot{\eta}^\alpha]=A^\alpha_\beta\dot{v}^\beta,$$

and taking into account the covariance of $g_{\alpha\beta}$ and $f_{\gamma\beta,\delta}$, we get at $p(t_0)$ the following equation

$$[\dot{\eta}^\alpha]+g^{\alpha\delta}D_\mu g_{\beta\delta}\dot{x}^\mu[\eta^\beta]+g^{\alpha\beta}C_{\gamma\beta,\delta}[\eta^\gamma][\eta^\delta]=0. \tag{4.11}$$

Since the path $p(t)$ is horizontal (see also (4.1)), the derivative coincides with the covariant derivative $\mathrm{D}\eta/\mathrm{d}t$. Specifying $\alpha=\hat{a}$, and introducing $q_{\hat{a}}=g_{\hat{a}\hat{b}}q^{\hat{b}}$, we get from (4.6)

$$\mathrm{D}q_{\hat{a}}/\mathrm{d}t=C_{\hat{a}\hat{b},\hat{c}}q^{\hat{b}}q^{\hat{c}}+C_{\hat{a}b,c}[\lambda^b][\lambda^c]=0, \tag{4.12}$$

where the last term depends on the equivalence class of λ only because of (2.5i) and owing to the $\mathrm{Ad}(H)$-invariance of $g_{\alpha\beta}$. One has to bear in mind that the indices are being lowered and raised with the help of $g_{\alpha\beta}$—the reason why $C_{\alpha\beta,\gamma}$ are, in general, not antisymmetric with respect to the last two indices (unless $g_{\alpha\beta}$ is $\mathrm{Ad}(G)$-invariant). Finally, for $\alpha=a$, (4.11) gives us

$$\mathrm{D}[\lambda_a]/\mathrm{d}t=C_{ab,c}[\lambda^b][\lambda^c], \tag{4.13}$$

with understanding that the equation tells us that both sides describe the same element of $T^*_{[\lambda]}(\mathscr{L}/\mathrm{Ad}(H))$.

The equations (4.5), (4.12) and (4.13) constitute the final set of equations of motion for a particle carrying a coloured charge $q_{\hat{a}}$ and a higgsonic charge $[\lambda_a]$.

5. Summary and example

To every homogeneous space G/H there corresponds Lie algebra decomposition $\mathscr{G}=\mathscr{H}+\mathscr{K}+\mathscr{L}$, where $\mathscr{K}$ consists of all $\mathrm{Ad}(H)$ singlets in $\mathscr{G}/\mathscr{H}$ and is the Lie algebra of the group $K:=N(H)/H$, $N(H)$ being the normaliser of H in G.

We considered a multidimensional universe E which is (locally) a product $E=M\times(G/H)$. A G-invariant metric in E can be described in terms of fields on M. We get in this way gravity $g_{\mu\nu}(x)$, gauge field $A^{\hat{a}}_\mu(x)$ (the indices $\hat{a},\hat{b},\hat{c},\ldots$ are those of $\mathscr{K}$), and two kinds of scalar fields: non-singular $g_{\hat{a}\hat{b}}(x)$, and non-singular $\mathrm{Ad}(H)$-invariant $g_{ab}(x)$ (the indices $a,b,c,\ldots$ are those of $\mathscr{L}$).

We studied geodesics $\gamma(t)$ in E and their projections $x(t)$ in M. We proved that a complete description of the projected trajectory is given by the equations

$$\mathrm{D}\dot{x}_\mu/\mathrm{d}t=q_{\hat{a}}F^{\hat{a}}_{\mu\nu}\dot{x}^\nu \quad+\quad \tfrac{1}{2}q^{\hat{a}}q^{\hat{b}}D_\mu(g_{\hat{a}\hat{b}}) \quad+\quad \tfrac{1}{2}\lambda^a\lambda^bD_\mu g_{ab} \tag{5.1}$$

Non-Abelian Lorentz force	Type I Higgs force	Type II Higgs force

$$\mathrm{D}q_{\hat{a}}/\mathrm{d}t = C_{\hat{a}\hat{b},\hat{c}}q^{\hat{b}}q^{\hat{c}} \qquad + \qquad C_{\hat{a}b,c}\lambda^{b}\lambda^{c} \tag{5.2}$$

Type I charge non-conservation — Type II charge non-conservation

$$\mathrm{D}\lambda_a/\mathrm{d}t = C_{ab,c}\lambda^{b}\lambda^{c}. \tag{5.3}$$

Higgs charge non-conservation

The indices are lowered and raised with $g_{\mu\nu}$, $g_{\hat{a}\hat{b}}$, g_{ab}, and their inverses. The covariant derivatives are gauge covariant derivatives, except for the left-hand side of (5.1) which is gravitational (Levi–Civita). All charged fields are in the adjoint representation of $K = N(H)/H$ in $\mathcal{G}$. Thus, for example

$$\mathrm{D}\lambda_a/\mathrm{d}t = \frac{\mathrm{d}\lambda_a}{\mathrm{d}t} - \dot{x}^{\mu}A^{\hat{a}}_{\mu}C_{\hat{a}a}{}^{b}\lambda_b, \tag{5.4}$$

$C_{..}$ being the structure constants of G. The $q_{\hat{a}}(\lambda^a)$ are essentially cosines of the angles between the $\mathcal{K}$-Killing vectors ($\mathcal{L}$-Killing vectors) and the geodesic in question. Whereas the q_a describe the colour charge (more precisely: charge/mass ratio) of the particle and couple it to the gauge field and to the Higgs field of type I, we have also found the new charges λ_a which have geometric interpretation of giving the slope of the geodesic with respect to the submanifold P of E on which the principal K-bundle lives (see figure 4). We call λ_a the Higgs charge since it interacts with the scalar field g_{ab} only. The Higgs charge takes values in $\mathcal{L}$ or, rather, in the quotient space $\mathcal{L}/\mathrm{Ad}(H)$ of H-orbits in $\mathcal{L}$). Indeed, the couplings of λ in (5.1) and (5.2) are of such a nature that they allow us to determine λ only modulo an arbitrary $\mathrm{Ad}(H)$ transformation. This essentially nonlinear nature of the Higgs charge was shown to follow from the fact that the non-trivial isotropy group H rotates a geodesic γ into γh, with γ and γh intersecting in E, but having both the same projection onto M.

The following commentaries deal with special cases of (5.1)–(5.2):

(i) If $g_{\hat{a}\hat{b}}$ is $\mathrm{Ad}(K)$-invariant, in particular if it is the Killing metric of K, then the type I Higgs force and the type I colour charge non-conservation term disappear.

(ii) If $G/N(H)$ is isotropy irreducible then the type II colour charge non-conservation term vanishes.

(iii) If g_{ab} is $\mathrm{Ad}(K)$-invariant then the type II Higgs force vanishes.

(iv) If $G/N(H)$ is a symmetric space then the Higgs charge non-conservation term vanishes.

Consider now the following example, which in a simple way illustrates general features of the theory.

Example.

$$G = \mathrm{U}(2, \mathbb{H}) \cong \mathrm{SO}(5)$$

$$H = \mathrm{U}(1, \mathbb{H}) = \mathrm{SU}(2) \cong \mathrm{SO}(3).$$

We find $N(H) = \mathrm{U}(1, \mathbb{H}) \times \mathrm{U}(1, \mathbb{H})$,

$$K = N(H)/H = \mathrm{U}(1, \mathbb{H}).$$

The internal space $S=G/H$ is $\mathrm{U}(2,\mathbb{H})/\mathrm{U}(1,\mathbb{H})=S^7$. $L=G/N=\mathrm{U}(2,\mathbb{H})/\mathrm{U}(1,\mathbb{H})\times\mathrm{U}(1,\mathbb{H})=\mathbb{H}P^1$ which is a symmetric space. Therefore $[\mathscr{L},\mathscr{L}]\subset\mathscr{N}$ and thus $C^c_{ab}=0$. The Lie algebra decomposition $\mathscr{G}=\mathscr{H}+\mathscr{K}+\mathscr{L}$ can be written explicitly as

$$\begin{pmatrix} x & -z^* \\ z & y \end{pmatrix}=\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}+\begin{pmatrix} 0 & -z^* \\ z & 0 \end{pmatrix}$$
$$\mathscr{G} \quad = \quad \mathscr{H} \quad + \quad \mathscr{K} \quad + \quad \mathscr{L},$$

where $x, y, z\in\mathbb{H}$ are quaternions, x and y being pure imaginary. The charge q can be written as $\mathbb{q}=q^{\hat{a}}e_{\hat{a}}$, $e_{\hat{a}}$ being the three imaginary quaternionic units, whereas $\lambda\in\mathbb{H}$. The adjoint representations of H and K on $\mathscr{L}=\mathbb{H}$ are

$$\begin{aligned}
&\mathrm{Ad}(a)z=az, && a\in K=U(1,\mathbb{H}),\\
&\mathrm{ad}(k)z=kz, && k\in\mathscr{K}=u(1,\mathbb{H}),\\
&\mathrm{Ad}(a)z=za^*, && a\in H=U(1,\mathbb{H}),\\
&\mathrm{ad}(h)z=-zh, && h\in\mathscr{H}=u(1,\mathbb{H}).
\end{aligned}$$

Since the action of H on $\mathscr{L}$ is irreducible, any two $\mathrm{Ad}(H)$-invariant scalar products are proportional. Thus we must have

$$g_{ab}=\phi\delta_{ab}. \tag{5.5}$$

It is convenient to introduce $\Lambda=\phi\lambda$. Instead of discussing a general $g_{\hat{a}\hat{b}}$—the metric on our gauge group, let us consider a particular case of $g_{\hat{a}\hat{b}}=\delta_{\hat{a}\hat{b}}$—the Killing metric. The equations (5.1)–(5.3) read then

$$\mathrm{D}x_\mu/\mathrm{d}t=(\mathbb{q}\cdot\mathbb{F}_{\mu\nu})\dot{x}^\nu+\tfrac{1}{2}\partial_\mu\phi\Lambda^2/\phi^2, \tag{5.6}$$

$$\mathrm{d}\mathbb{q}/\mathrm{d}t+\mathbb{q}\times\mathbb{A}=0, \tag{5.7}$$

$$\mathrm{d}\Lambda/\mathrm{d}t+\mathbb{A}\Lambda=0, \tag{5.8}$$

where

$$\mathbb{A}=(\mathrm{d}x^\mu/\mathrm{d}t)\mathbb{A}_\mu(x(t)), \tag{5.9}$$

and the term $\mathbb{A}\Lambda$ in (5.8) should be understood as the quaternionic product of $\mathbb{A}$ and Λ. From (5.8) it follows that $\Lambda^2=\Lambda^*\Lambda=$ constant. The nonlinear Higgs charge $[\Lambda]$ takes values in $\mathscr{L}/\mathrm{Ad}(H)=R_+$—the half-line, and indeed it is only Λ^2 that enters (5.6), all other information about the direction of Λ is lost. Thus the equation (5.8) carries no information whatsoever except that Λ^2 is a constant of motion. The second term in (5.6) generalises the well known classical Wong equation [3].

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