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GEOMETRY OF INDEFINITE-METRIC SPACES

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I. Introduction and summary

It has been long time up to now the physicists began searching for a new mathematical structure, sufficiently capacious to contain those important quantities which cannot find enough room in a standard Hilbert space. An indefinite-metric space is one of the possible extensions of the standard structure. Unfortunately, so far one could hardly find a rigorous mathematical theorem dealing with the indefinite metric and known to the physicists. In the present paper we start with a detailed analysis of some aspects of geometry of indefinite-metric spaces. We do not attempt to discuss any possible physical interpretation of the arising structure. It should be clear from the foregoing paper that one cannot hope to provide a physical interpretation to all the vectors. A solution proposed in [6] leads to such phenomena like the dependence of averages not only on states and observables but also on abelian algebras that contain the observables in question. A similar possibility has been discussed by Bell [2] in connection with hidden variables. That is why we consider the results in [6] negative. On the other hand, there is no reason to demand all the states to be physical and also, there is no reason to force all the relevant operators to transform a physical subspace into itself. A mathematical theory of the spectral decomposition of hermitian operators in indefinite-metric spaces may be thus of some interest. This is the main object of the present paper.

The paper consists of five sections. In Section II we singled out, in a concise manner, all the essential properties of general self-dual vector spaces. Section III deals with a very special class of these spaces, J-spaces. Given a complex vector space X with a non-degenerate, sesquilinear hermitian form (x, y) we define a subset $\mathscr{J}(X)$ of the set $\mathscr{L}(X)$ of all continuous, linear operators on X as follows: a hermitian and unitary operator J is in $\mathscr{J}(X)$ if and only if $(x, y)_J=(x, Jy)$ is a Hilbert-space scalar product on X. We show that for each $J\in\mathscr{J}(X)$, $\mathscr{L}(X)$ coincides with $\mathscr{L}(X_J)$ as an algebra. Equipped with any J-norms, $\mathscr{L}(X)$ is a Banach $*$-algebra. We consider the case of $\mathscr{J}(X)$ consisting of at least

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two elements. This is the definition of a J-space. It is shown that there exists a metric d on $\mathscr{J}(X)$ such that $||x||_J \leqslant \exp[d(J, J')] \cdot ||x||_{J'}$, $\mathscr{J}(X)$ is a complete metric space. In fact, our metric is simply related to the perturbation radius of J, J', which in our case is easily seen to be given by $||J-J'||_J$. We show that the unitary group of X acts on $\mathscr{J}(X)$ transitively and preserves the metric. It is of some interest that $\mathscr{J}(X)$ turns out to be highly non-commutative – none two J's commute. For every pair J, J' in $\mathscr{J}(X)$ there exists an "exchange operator" $U(J, J') \in \mathscr{J}(X)$ such that $U(J, J')\,J = J'U(J, J')$. It is possible to show that the classes of Hilbert–Schmidt and trace operators, for different J in $\mathscr{J}(X)$, coincide. The trace is J-independent. Not all the results of this section are new. We preferred, however, to express all the results in a language that seemed to be a most appropriate one for our purpose.

In Section IV we search for necessary and sufficient conditions for a hermitian operator on a J-space to have a spectral decomposition in terms of projections (hermitian idempotents in $\mathscr{L}(X)$). It is firstly shown that for every σ-complete Boolean algebra $\mathscr{B}$ of projections there exists a $J \in \mathscr{J}(X)$ commuting with $\mathscr{B}$. In other words, $\mathscr{B}$ is a Boolean algebra of J-hermitian idempotents on a Hilbert space X_J. It follows that a necessary and sufficient condition for a hermitian operator A to have a spectral decomposition is $\mathscr{J}_A = \{A \in \mathscr{J}(X): JA = AJ\} \neq \emptyset$. An equivalent condition is that the orbits of $\{\exp[iAt]\}_t$ on $\mathscr{J}(X)$ are bounded. We call such a hermitian operator *elliptic*. By the result of Wermer (based on Mackey's idea) it is shown that the sums and products of a finite number of commuting elliptic operators are elliptic. The set of all elliptic operators is thus a partial algebra in the sense of Kochen and Specker [2]. At the end of Section 4 a preliminary discussion of unitary representations of locally compact groups on a J-space is given. A representation $G \to \{V_g\}$ is irreducible if the only projections commuting with $\{V_g\}$ are trivial ones. It is shown that if G is amenable and the orbits of G on $\mathscr{J}(X)$ are bounded, then there is a fixed point in $\mathscr{J}(X)$. In particular, it is shown that every unitary representation of a compact group is reducible.

In Section 5 we single out some unsolved problems. The most important one is: *does there exist a unitary, irreducible representation of the Poincaré group with generators of translations being non-elliptic (or, equivalently, with unbouded orbits of translations on $J(X)$)?* Such a representation, in case it exists, would be very attractive from a physical point of view. The mass-operator may have a non-trivial spectrum in an irreducible representation.

II. Self-dual vector spaces

In this section we give a concise review of the most important, general properties of self-dual vector spaces. For the sake of completeness short proofs of most of the statements are also included.

Let X be a complex vector space, and let (x, y) be a sesquilinear, hermitian and non-degenerate form on X:

(i) (x, y) is linear in y for every $x \in X$;

(ii) $(x, y) = \overline{(y, x)}$ for all x, y in X;

(iii) $(x, y) = 0$ for all $x \in X$ implies $y = 0$.

For every x in X, let f_x be a linear form on X, defined by $f_x(y) = (x, y)$. The *weak topology* T_w is now defined by the family of seminorms

$$p_N(x) = \sup\{|f_y(x)| : y \in N\},$$

where N is any finite sequence of vectors. A sequence $\{x_\alpha\}_{\alpha \in A}$, where A is a directed set of indices, is weakly convergent to x if and only if $(x_\alpha, y) \to (x, y)$ for every $y \in X$. By (iii), a limit, if exists, is unique and we write $\text{w.lim}\, x_\alpha = x$ in this case. All linear forms f_x are now weakly continuous by the very definition. X equipped with the weak topology is a locally convex, Hausdorff vector space.

PROPERTY 2.1. *If f is a weakly continuous linear form on X, then there exists a unique $x \in X$ such that $f = f_x$.*

Proof: See [3], Ch. IV, § 1, sec. 2, Proposition 1. □

By the above property we may identify X with its weak dual. Therefore X is said to be a *self-dual* vector space.

DEFINITION 2.1. A locally convex Hausdorff topology T on X is said to be *compatible with the scalar product* provided every linear form f on X is T-continuous if and only if f is weakly continuous (i.e. $f = f_x$ for some $x \in X$).

PROPERTY 2.2. *If T is compatible with the scalar product then T is stronger than T_w.*

Proof: If $x_i \xrightarrow{T} x$ then $(x_i, y) \to (x, y)$ and thus $x_i \xrightarrow{w} x$. □

DEFINITION 2.2. For every subset C of X, the *orthogonal complement* $C^\perp$ is defined by

$$C^\perp = \{x \in X : (x, y) = 0 \ \ \forall y \in C\}.$$

Clearly, $C^\perp$ is always a weakly closed, linear subspace of X. The condition (iii) above is equivalent to $X^\perp = 0$.

PROPERTY 2.3. *$C \subset D$ implies $D^\perp \subset C^\perp$ and $C^{\perp\perp} \subset D^{\perp\perp}$. For every C, $C \subset C^{\perp\perp}$ and $C^\perp = C^{\perp\perp\perp} = \ldots$ If Y is a linear subspace of X, then $Y^\perp \cap Y^{\perp\perp} = (Y + Y^\perp)^\perp$.*

Proof: Straightforward, from the definition of "$\perp$". □

PROPERTY 2.4. *If Y is a linear subspace of X, then the closure $\bar{Y}$ of Y is the same for every topology compatible with the scalar product. Y is closed if and only if $Y = Y^{\perp\perp}$. Y is dense in X if and only if $Y^\perp = 0$.*

Proof: See [3], Ch. IV, § 2, Sec. 3, Cor. 2. □

We shall say that a linear subspace Y is closed (without referring to the topology) (resp. dense) if $Y = Y^{\perp\perp}$ (resp. $Y^\perp = 0$).

DEFINITION 2.3. A closed linear subspace Y of X is called *non-degenerate* if $Y \cap Y^{\perp} = \emptyset$. A non-degenerate subspace Y is called *regular* if $Y + Y^{\perp} = X$. $\mathscr{B}(X)$ is the set of all regular linear subspaces of X.

Let us observe that if Y is non-degenerate, then $Y + Y^{\perp}$ is dense in X. But it does not follow that $Y + Y^{\perp}$ is closed. We also notice that a one-dimensional subspace, spanned by some $x \in X$ is non-degenerate if and only if $(x, x) \neq 0$. It follows immediately from the definition that $Y^{\perp} \in \mathscr{B}(X)$ if and only if $Y \in \mathscr{B}(X)$. Clearly, $\mathscr{B}(X)$ is a partially ordered set ($Y \leqslant Z$ if and only if $x \in Y$ implies $x \in Z$) and $Y \leqslant Z$ if and only if $Z^{\perp} \leqslant Y^{\perp}$. We also have $0 \in \mathscr{B}(X)$ and $X \in \mathscr{B}(X)$. We denote by $\bigvee Y_{\alpha}$ (resp. $\bigwedge Y_{\alpha}$) a least upper bound (resp. a greatest lower bound) for a family $\{Y_{\alpha}\}$ of regular subspaces provided it exists *in* $\mathscr{B}(X)$.

EXAMPLE. We denote by $X_{n,m}$ an n-dimensional complex self-dual space with a signature m. Clearly, if $n = m$, then $\mathscr{B}(X_{n,m})$ is a lattice. Also $\mathscr{B}(X_{2,0})$ is a lattice. Let us consider $\mathscr{B}(X_{3,m})$. If Y and Z are two different elements of $\mathscr{B}(X_{3,m})$, then either $Y \cap Z = 0$ or $Y \cap Z \neq 0$. If $Y \vee Z = 0$ and, say, Y is two-dimensional, then $Y \vee Z = X$. If Y and Z are both one-dimensional, then either $Y \vee Z$ exists and is two-dimensional or there is no two-dimensional regular subspace containing Y and Z. In the last case $Y \vee Z = X$. If $Y \cap Z \neq 0$, then either, say, $Y = Z$ and then $Y \vee Z = Z$ or, say, Y is one-dimensional, and then $Y \vee Z = Z$. Finally, Y and Z may be both two-dimensional. In this case $Y \vee Z = X$. We conclude that $\mathscr{B}(X_{3,m})$ is a lattice.

Let us consider $\mathscr{B}(X_{4,0})$. In a given orthogonal frame (x, y) is of the form $(+, +, -, -)$. Let $a = (1, 0, 0, 0)$, $b = (1, 1, 1, 0)$ and $c = (0, 0, 1, 0)$ $d = (0, 1, 0, 1)$. It is easy to see that a, b, c, d are linearly independent. Let Y_a and Y_b be two linear subspaces spanned by a and b, respectively. Now Y_a and Y_b are both regular. On the other hand, $b - a$ is orthogonal to both Y_a and Y_b. It follows that there is no two-dimensional regular subspace containing Y_a and Y_b. Since Y_1, spanned by a, b, c and Y_2 spanned by a, b, d are both regular, we conclude that $\mathscr{B}(X_{4,0})$ is not a lattice. A similar example can be given for $X_{4,2}$.

It follows that for a self-dual space X which is at least 4-dimensional (complex or real) and contains two vectors y_1, y_2 with $(y_1, y_1) = -(y_2, y_2) \neq 0$, $\mathscr{B}(X)$ is not a lattice.

Let A be a linear operator defined on a dense domain $D(A)$. We set

$$D(A^*) = \{x \in X : \exists x^* \in X \text{ with } (x, Ay) = (x^*, y) \ \forall y \in D(A)\} .$$

It is easy to see that $A^* : x \to x^*$ is a uniquely defined, linear operator on $D(A^*)$. If $D(A) = X$ and A is weakly continuous, then $D(A^*) = X$ and A^* is weakly continuous. It follows that *the set $\mathscr{L}(X)$ of all weakly continuous operators on X is a *-algebra.*

PROPERTY 2.5. *Let A be a linear operator with $D(A) = X$. Then $A \in \mathscr{L}(X)$ if and only if there exists B such that $(Ax, y) = (x, By)$ for all $x, y \in X$. In this case, also $B \in \mathscr{L}(X)$ and $B = A^*$. In particular, if A satisfies $(Ax, y) = (x, Ay)$ for all x, y in X, then $A \in \mathscr{L}(X)$ and $A = A^*$.*

Proof: See [3], Ch. IV, § 4, Sec. 1, Prop. 1. □

Definition 2.4. $A \in \mathscr{L}(X)$ is said to be:

hermitian if $A = A^*$,

unitary if $AA^* = A^*A = 1$,

a *projection* if $A = A^* = A^2$, and $D(A) = X$.

Property 2.6. *If E is a projection, then $Y = EX$ is regular. Conversely, if $Y \in \mathscr{R}(X)$, then there is a unique projection E such that $Y = EX$. In this case $Ex = x$ if and only if $x \in Y$ and $Ex = 0$ if and only if $x \in Y^{\perp}$. We have $(I-E)X = Y^{\perp}$ and $I-E$ is also a projection.*

Proof: $Y = EX$ is a closed linear subspace as a range of a continuous idempotent. Now, $x \in Y$ if and only if $x = Ey$ for some $y \in Y$, i.e. if and only if $x = Ex$. Since $E = E^*$, it follows that $x \in Y^{\perp}$ if and only if $Ex = 0$. For $x \in Y \cap Y^{\perp}$, $x = Ex = 0$ and so Y is non-degenerate. Clearly, $(I-E)X = Y^{\perp}$ and thus $Y \in \mathscr{R}(X)$. Conversely, if $Y \in \mathscr{R}(X)$, then for every $x \in X$ we have $x = x_1 + x_2$ with $x_1 \in Y$ and $x_2 \in Y^{\perp}$. Let $E: x \to x_1$. It is easy to see that $EX = Y$, $E^2 = I$ and for all $x, y \in X$, $(Ex, y) = (x, Ey)$. Thus $E = E^* \in \mathscr{L}(X)$. The rest of the statement is obvious. □

Property 2.7. *If E, F and $E+F$ are projections, then $EF = FE = 0$.*

Proof: Let $G = E+F$. By $G^2 = G$ we obtain $EF + FE = 0$, or $EF = -FE = -(EF)^*$. On the other hand, $EF = EEF = -EFE = -(EF)(EF)^*$. Thus $(EF)^* = -(EF)(EF)^* = EF$. It follows that $EF = (EF)^* = 0$. □

Property 2.8. *Let E, F be projections, $Y = EX$ and $Z = FX$. The following statements* (i)–(v) *and* (a)–(d) *are respectively equivalent*:

(i) $Y \perp Z$,	(a) $Y \subset Z$,
(ii) $EF = 0$,	(b) $FE = E$,
(iii) $FE = 0$,	(c) $EF = E$,
(iv) $EZ = 0$,	(d) $F-E$ *is a projection.*
(v) $FY = 0$;	

Proof: We restrict ourselves to the implication (d)⇒(a) in this proof. If $F-E = G$ is a projection, then $EG = GE = 0$ by Proposition 2.7. Now, $x \in Y$ implies $Ex = x$ and so, by (i)–(v), $Gx = 0$. But $G = F-E$ and therefore $Fx = x$. □

By the properties above we may identify the set of all projections with the set $\mathscr{R}(X)$ of all regular subspaces. *We shall use the same symbol $\mathscr{R}(X)$ for both.*

The next two properties follow directly from the definitions.

Property 2.9. *If $E_1, \ldots, E_n$ are mutually orthogonal projections, then $E = \sum_{i=1}^{n} E_i$ is a projection and $E = \bigvee E_i$.* □

Property 2.10. *Let $E \in \mathscr{R}(X)$ and $A \in \mathscr{L}(X)$; then $Y = EX$ is A-invariant if and only if $AE = EAE$. If Y is also A^*-invariant, then $AE = EA$. In particular, if A is hermitian or unitary, then Y is A-invariant if and only if $AE = EA$.* □

PROPERTY 2.11. *If $E, F \in \mathscr{B}(X)$, then $P=EF\in\mathscr{B}(X)$ if and only if $EF=FE$. In this case, $P=E\wedge F$.*

Proof: The first part of the assertion is obvious. If $P=EF\in\mathscr{B}(X)$, then $PE=P$ and $PF=P$. If $GE=G$ and $GF=G$, then $GP=G$ and so $P=E\wedge F$. □

It follows from the above that $\mathscr{B}(X)$ is a non-trivial example of a partial Boolean algebra in the sense of Kochen and Specker (see [7], p. 183). A complementary statement to Property 2.11 is

PROPERTY 2.12. *If $E, F\in\mathscr{B}(X)$, then $P=E+F-EF$ is in $\mathscr{B}(X)$ if and only if $EF=FE$. In this case, $P=E\vee F$.* □

Let us now observe that for $x\in X$ with $(x, x)\neq 0$, a projection onto the subspace spanned by x is given by

$$E_x y=\frac{(x, y)}{(x, x)}\cdot x .$$

Now, if Y is a non-degenerate, finite-dimensional subspace of X, then it is always possible to span Y by a finite sequence of vectors satisfying $(x_i, x_j)=0$ if and only if $i\neq j$. A projection onto Y is then given by

$$Ey=\sum^{n} E_{x_i} y=\sum^{n}\frac{(x_i, y)}{(x_i, x_i)}\cdot x_i .$$

It follows that every non-degenerate, finite-dimensional subsapce of X is regular.

PROPERTY 2.13. *If Y is non-degenerate and Y or $Y^{\perp}$ finite-dimensional, then $Y\in\mathscr{B}(X)$.* □

III. *J*-Spaces

This section deals with a very special, and most regular, class of self-dual vector spaces, *J*-spaces. We start directly with the definition.

DEFINITION 3.1. Let X be a self-dual space and let

$$\mathscr{J}(X)=\{J\in\mathscr{L}(X):\ J=J^*=J^{-1} \text{ and } (x, y)_J=(x, Jy) \text{ is a Hilbert-space product on } X\}.$$

If $\mathscr{J}(X)$ consists of at least two different elements, then X is called a *J-space*. If $J\in\mathscr{J}(X)$, then the Hilbert space $\langle X, (\ ,\ )_J\rangle$ is denoted by X_J. The norm in X_J is denoted by $\|\ \|_J$ and the adjoint in $\mathscr{L}(X_J)$ by A^J. We shall refer to *J*-topology, *J*-continuity, etc.

It can be easy to see that the most general *J*-space may be obtained in the following way: let X_1 and X_2 be two Hilbert spaces; then the algebraic direct sum $X=X_1+X_2$ equipped with the scalar product $(x, y)=(x_1, y_1)-(x_2, y_2)$ is a *J*-space.

Remark. If X is a self-dual space and $\mathscr{J}(X)$ is empty, we have nothing to say. If $\mathscr{J}(X)$ consists of exactly one element, then (x, y) is a Hilbert-space scalar product from the very beginning. Conversely, if X is already a Hilbert space (when equipped with the (x, y)

form), then $\mathscr{J}(X)=\{I\}$. If however, there are at least two different elements in $\mathscr{J}(X)$, then there is at least a continuum.

PPROPERTY 3.1. *For every $J\in\mathscr{J}(X)$, $\mathscr{L}(X_J)$ and $\mathscr{L}(X)$ coincide. We have*

$$A^*=JA^JJ \quad \text{and} \quad A^J=JA^*J\,.$$

If $A=A^*\in\mathscr{L}(X)$, then A is J-hermitian if and only if $AJ=JA$. Conversely, if $A=A^J$, then A is hermitian if and only if A commutes with J. If A is unitary, then A is J-unitary if and only if $JAJ=A$. Conversely, a J-unitary operator A is unitary if and only if A and J commute. If $E\in\mathscr{B}(X)$, then E is a J-projection if and only if $EJ=JE$. Conversely, if E is a J-projection, then $E\in\mathscr{B}(X)$ is equivalent to $JE=EJ$. For every $J\in\mathscr{J}(X)$, J is unitary and hermitian on both X and X_J.

Proof: If $A\in\mathscr{L}(X)$, then $(Ax,y)_J=(Ax,Jy)=(x,A^*Jy)=(x,JA^*Jy)_J$. Thus $A\in\mathscr{L}(X_J)$ and $A^J=JA^*J$ (see Property 2.5). The converse follows in much the same way. The rest of the statement is simply a consequence of the relation $JA^J=A^*J$. □

Remark. It follows from the statement above that we have, in fact, a unique concept of continuity for linear operators on X. We shall simply talk about continuous, or bounded operators.

PROPERTY 3.2. *$\mathscr{L}(X)$ equipped with any J-norm is a Banach *-algebra.*

Proof: Since $\mathscr{L}(X)$ and $\mathscr{L}(X_J)$ coincide as algebras, it follows that $\langle\mathscr{L}(X), \|\cdot\|_J\rangle$ is a Banach algebra. But J is unitary on X_J, so

$$\|A^*\|_J=\|JA^JJ\|_J=\|A^J\|_J=\|A\|_J\,. \;\square$$

PROPERTY 3.3. *For each $J\in\mathscr{J}(X)$, the J-topology of X is compatible with the metric.*

Proof: If f is J-continuous, then $f(x)=(y,x)_J=(y,Jx)=(Jy,x)$, or $f=f_{Jy}$. If f is weakly continuous, then $f(x)=(y,x)=(Jy,x)_J$ and so, f is J-continuous. □

Remark. By Propositions 2.4 and 3.3 we may simply talk about *closed* or *dense* linear subspaces of X.

DEFINITION 3.2. In a J-space X we define

$$\mathscr{N}(X)=\{N\in\mathscr{L}(X):\, N=N^*\,,\; N^{-1}\in\mathscr{L}(X) \text{ and } (x,Nx)\geqslant 0\;\; \forall x\in X\}\,.$$

PROPERTY 3.4. *If $N\in\mathscr{N}(X)$, then $(x,y)_N=(x,\,Ny)$ is a Hilbert-space scalar product on X. With $N\in\mathscr{N}(X)$ and $J\in\mathscr{J}(X)$, the two scalar products $(x,y)_N$ and $(x,y)_J$ are topologically equivalent.*

Proof: With $N\in\mathscr{N}$ and $J\in\mathscr{J}$, we have $N\in\mathscr{L}(X_J)$ and so

$$\|x\|_N^2=(x,Nx)=(x,JNx)_J\leqslant\|x\|_J^2\cdot\|JN\|_J=\|N\|_J\cdot\|x\|_J^2\,.$$

On the other hand, $N^{-1}\in\mathscr{L}(X_J)$ and JN is J-positive. Thus

$$\|x\|_N^2=(x,Nx)=(x,JNx)_J\geqslant\|N^{-1}\|_J^{-1}\cdot\|x\|_J^2\,. \;\square$$

Property 3.5. *$\mathcal{N}(X)$ is a convex subset of $\mathcal{L}(X)$. We have* conv$(\mathcal{J}(X))\subset\mathcal{N}(X)$. *$\mathcal{J}(X)$ is closed and $\mathcal{N}(X)$ is open in a uniform topology of $\mathcal{L}_H(X)$. ($\mathcal{L}_H(X)$ is the set of all hermitian elements of $\mathcal{L}(X)$). There exists a mapping $N\to J_N$ from $\mathcal{N}(X)$ to $\mathcal{J}(X)$ with the following property: for every $A\in\mathcal{L}(X)$, if A commutes with N, then $AJ_N=J_NA$.*

Proof: If $N_1, N_2\in\mathcal{N}(X)$ then $(x, N_ix)\geqslant m_i\cdot||x||_J^2$ with $m_i>0$. It follows that with $N=tN_1+(1-t)N_2$ we have $(x, Nx)\geqslant m||x||_J^2$ with $m>0$. Thus N is bicontinuous. Clearly, conv$(\mathcal{J}(X))\subset\mathcal{N}(X)$ and also $\mathcal{N}(X)$ is open in $\mathcal{L}_H(X)$. If $J_n\in\mathcal{J}(X)$ and $J_n\to J$ uniformly, then $J^{-1}=J^*=J$ and $(x, Jx)\geqslant 0$. Thus $J\in\mathcal{J}(X)$. Finally, let $N\in\mathcal{N}(X)$ and $N=\int\lambda dE(\lambda)$ be a spectral decomposition of N on X_N. (It follows directly that N is hermitian on X_N.) Now every hermitian operator on X_N that commutes with N is hermitian on X. It follows that $E(\Delta)$ are in $\mathcal{B}(X)$. Clearly $O\notin\mathrm{Sp}(N)$. We define $E_+=E(0,\infty)$ and $E_-=(-\infty,0)$. By $E_++E_-=1$ and $E_+E_-=0$ it follows that $J_N=E_+-E_-$ satisfies $J_N=J_N^*=J_N^{-1}$. On the other hand, $(x, J_Nx)\geqslant 0$. It follows that $J_N\in\mathcal{J}(X)$. Also J_N commutes with every operator commuting with N. □

Lemma 3.1. *With $J, J'\in\mathcal{J}(X)$ the following statements hold*

(i) *JJ' is positive on both X_J and $X_{J'}$.*

(ii) *$t(J, J')=(JJ')^{1/2}$ is positive on X_J, $X_{J'}$ and unitary on X; we have*

$$t(J, J')^*=t(J, J')^{-1}=t(J', J).$$

(iii) *$t(J, J')$ commutes with every $A\in\mathcal{L}(X)$ that commutes with J and J'; we have*

$$t(J, J')J't(J, J')^*=J.$$

(iv) *$\lambda\in\mathrm{Sp}\,[t(J, J')]$ if and only if $\lambda^{-1}\in\mathrm{Sp}\,[t(J, J')]$.*

(v) *We have* $||t(J, J')||_J^2=||t(J, J')||_{J'}^2=||t(J', J)||_{J'}^2=||t(J', J)||_J^2=$

$$=\sup\{||x||_J^2/||x||_{J'}^2:\ x\neq 0\}=\sup\{||x||_{J'}^2/||x||_J^2:\ x\neq 0\}.$$

(vi) *If $\{x_\alpha\}$ is an orthonormal basis for $X_{J'}$, then $\{t(J, J')x_\alpha\}$ is an orthonormal basis for X_J.*

Proof: We have $(x, JJ'x)_J=(x, J'x)=||x||_{J'}^2\geqslant 0$ and $(x, JJ'x)_{J'}=(J'x, J'x)_J=||J'x||_J^2\geqslant 0$. Now, $t(J, J')=(JJ')^{\frac{1}{2}}$ is defined by a series, positive on X_J and on $X_{J'}$. Let us observe that $t(J, J')^*=Jt(J, J')J=J't(J, J')J'$ is also J- and J'-positive and satisfies

$$t(J, J')^*\,t(J, J')^*=[t(J, J')^2]^*=J'J=t(J, J')^{-2}.$$

Therefore, $t(J, J')^*=t(J, J')^{-1}=t(J', J)$. Now, $t(J, J')J'=t(J, J')J'JJ=t(J', J)J$, so (iii) holds. (iv) is an immediate consequence of $t(J, J')^{-1}=Jt(J, J')J$. To prove (v) we use the formula for a norm of a positive operator on a Hilbert space:

$$||t(J, J')||_J^2=||t(J, J')^2||_J=||JJ'||_J=\sup\{(x, JJ'x)_J/||x||_J^2\}=\sup\{(x, J'x)/||x||_J^2\}$$
$$=\sup\{||x||_{J'}^2/||x||_J^2\}.$$

On the other hand, since $t(J, J')^2$ is positive on both X_J and $X_{J'}$ the J- and J'-norms coincide and equal to the spectral radius of JJ' (=spectral radius of $J'J$). Finally, let $\{x_\alpha\}$ be an orthonormal basis for $X_{J'}$. Since $t(J, J')$ is bicontinuous, if follows that $\{t(J, J')\, x_\alpha\}$ is total. On the other hand

$$(t(J, J')x_\alpha, t(J, J')x_{\alpha'})_J = (t(J, J')x_\alpha, Jt(J, J')x_{\alpha'}) = (x_\alpha, t(J', J), Jt(J', J)^* x_{\alpha'})$$
$$= (x_\alpha, J'x_{\alpha'}) = (x_\alpha, x_{\alpha'})_{J'} = \delta_{\alpha\alpha'}. \square$$

Definition 3.3. *The common value in* (v) *which coincides with the spectral radius of JJ' is denoted by $\nu^2(J, J')$. We also define*

$$d(J, J') = \log \nu(J, J').$$

Property 3.6. *For all J, $J' \in \mathcal{J}(X)$, $x \in X$ and $A \in \mathcal{L}(X)$, the following inequalities hold*:

(i) $\|x\|_J \leqslant \nu(J, J') \cdot \|x\|_{J'}$, $\|x\|_{J'} \leqslant \nu(J, J') \cdot \|x\|_J$;

(ii) $\|A\|_J \leqslant \nu^2(J, J') \cdot \|A\|_{J'}$, $\|A\|_{J'} \leqslant \nu^2(J, J') \cdot \|A\|_J$;

(iii) $\nu(J, J') \geqslant 1$ *and* $\nu(J, J') = 1$ *if and only if* $J = J'$.

Proof: (i) follows directly from Lemma 3.1 (v), and (ii) is an immediate consequence of (i). (iii) follows from Lemma 3.1 (iv). □

Property 3.7. *$d(J, J')$ is a metric on $\mathcal{J}(X)$. We have*

$$d(J, J') = \tfrac{1}{2} \log(1 + \|J' - J\|_J),$$
$$\|J' - J\|_J = \exp\{2d(J, J')\} - 1.$$

Proof: By Lemma 3.1 and Property 3.6 we have $d(J, J') = d(J', J) \geqslant 0$ and $= 0$ iff $J = J'$. We also have

$$\nu^2(J, J') = \sup\{(x, Jx)/(x, J'x)\} = \sup\{[(x, Jx)/(x, J''x)][(x, J''x)/(x, J'x)]\}$$
$$\leqslant \sup\{(xJx)/(x, J''x)\} \cdot \sup\{(x, J''x)/(x, J'x)\} = \nu^2(J, J'') \cdot \nu^2(J'', J').$$

Thus $d(J, J') \leqslant d(J, J'') + d(J'', J')$. Finally, $\|J' - J\|_J = \|JJ' - 1\|_J = \|JJ'\|_J - 1 = \nu^2(J, J') - 1$ what completes the proof. □

Property 3.8. *The uniform and metric topologies on $\mathcal{J}(X)$ coincide. Uniformly bounded and metrically bounded subsets of $\mathcal{J}(X)$ coincide.*

Proof: Straightforward from Property 3.7. □

Property 3.9. *$\mathcal{J}(X)$ is a complete metric space.*

Proof: Follows from Property 3.8 and the fact that $\mathcal{J}(X)$ is uniformly closed in $\mathcal{L}(X)$. □

PROPERTY 3.10. *$\mathscr{J}(X)$ and d are invariant under the action of the unitary group $\mathscr{U}(X)$ of X. $\mathscr{U}(X)$ acts on $\mathscr{J}(X)$ transitively. Moreover, if $J, J' \in \mathscr{J}(X)$, then there exists $U(J, J') \in \mathscr{J}(X)$ such that*

$$U(J, J')\, JU(J, J')^* = J' \quad \text{and} \quad U(J, J')\, J'U(J, J')^* = J\,;$$

$U(J, J')$ is given by

$$U(J, J') = t(J, J')\, J' = t(J', J)\, J = J\, t(J, J') = J'\, t(J', J)$$

and commutes with every $A \in \mathscr{L}(X)$ that commutes with both J and J'.

Proof: We have

$$d(VJV^*, VJ'V^*) = \tfrac{1}{2}\log\left(\sup \frac{(x, VJV^*x)}{(x, VJ'V^*x)}\right) = \tfrac{1}{2}\log\left(\sup \frac{(V^*x, JV^*x)}{(V^*x, J'V^*x)}\right) = d(J, J')\,.$$

To prove that $U(J, J') = t(J, J')\, J'$ satisfies the requirements of the statement, we observe that

$$t(J, J')\, J' = Jt(J, J') = J'J'Jt(J, J') = J't(J', J) = t(J', J)\, J$$

by Lemma 3.1 (iii) and the definition of t. We thus have $U(J, J')^* = U(J, J')$ and also

$$(x, U(J, J')x) = (x, Jt(J, J')x) = (x, t(J, J')x)_J \geq v^{-1}(J, J')\cdot(x, x)_J\,.$$

It follows that $U(J, J') \in \mathscr{J}(X)$. □

PROPERTY 3.11. *If $J, J' \in \mathscr{J}(X)$, then J and J' commute if and only if $J = J'$.*

Proof: If $JJ' = J'J$, then $t(J, J') = t(J, J')^{-1}$ and by Lemma 3.1 (iv) it follows that $t(J, J') = 1$ or, $JJ' = 1$. □

PROPERTY 3.12. *The classes: $\mathscr{F}_{\mathscr{C}}$ — of compact operators, $\mathscr{F}_{\mathscr{S}}$ — of Hilbert–Schmidt operators, and $\mathscr{F}_{\mathscr{T}}$ — of trace operators, coincide for different J in $\mathscr{J}(X)$. If $|\cdot|_J$ is a $\mathscr{H}$-$\mathscr{S}$ norm for X_J, then*

$$|A|_J \leq v^2(J, J')\, |A|_{J'}\,.$$

With $A, B \in \mathscr{F}_{\mathscr{S}}$ we have

$$(A, B)_J = (t(J', J)\, At(J, J'), t(J', J)\, Bt(J, J'))_{J'}\,.$$

In particular, if A and B commute with JJ', then $(A, B)_J = (A, B)_{J'}$. If $A \in \mathscr{F}_{\mathscr{T}}$, then $\mathrm{Tr}(A)_J = \mathrm{Tr}(A)_{J'}$ for all $J, J' \in \mathscr{J}(X)$.

Proof: Follows immediately from the properties of t. □

IV. Spectral decomposition for hermitian operators and related topics

We have seen in the last section that a unitary operator on X may have a purely real spectrum. Similarly, a hermitian operator may have a purely imaginary spectrum. It is clear that for such operators we cannot expect any similarity to hermitian operators on a Hilbert space. On the other hand, even if the spectrum of a hermitian operator is real, this does not suffice for the existence of a spectral decomposition. We distinguish, in this section, a class of hermitian operators for which a satisfactory theory of spectral decomposition can be developed.

DEFINITION 4.1. Let $\mathscr{A}$ be any subset of $\mathscr{L}(X)$; then $\mathscr{A}'$ stands for the *commutant* of $\mathscr{A}$ and $\mathscr{J}_{\mathscr{A}}=\mathscr{J}\cap\mathscr{A}'$. In particular, $\mathscr{J}_A$ is the set of all J in $\mathscr{J}(X)$ that commute with A. A hermitian operator A is said to be *elliptic* provided $\mathscr{J}_A\neq\varnothing$.

LEMMA 4.1. *Let $\mathscr{A}$ be a family of hermitian* (resp. *unitary*) *operators in $\mathscr{L}(X)$. Then $\mathscr{J}_{\mathscr{A}}\neq\varnothing$ if and only if there exists a bicontinuous operator $W\in\mathscr{L}(X)$ and $J\in\mathscr{J}(X)$ such that all WAW^{-1}, $A\in\mathscr{A}$ are J-hermitian* (resp. *J-unitary*).

Proof: If $\mathscr{J}_{\mathscr{A}}\neq\varnothing$ and $J\in\mathscr{J}_{\mathscr{A}}$, then all A in $\mathscr{A}$ are hermitian (resp. unitary) on X_J. Conversely, assume WAW^{-1} is hermitian on X_J. By the polar decomposition theorem we may assume that W is J-positive. Then $N=JW^2$ is in $\mathscr{N}(X)$. It is easy to see that A commutes with N and so, by Property 3.5, $J_N\in\mathscr{J}_{\mathscr{A}}$. □

DEFINITION 4.2. A *Boolean algebra of projections* on X is a subset $\mathscr{B}$ of $\mathscr{B}(X)$ containing 0 and 1, which is a Boolean algebra under the operations $E\vee F=E+F-EF$ and $E\wedge F=EF$ (in particular, $\mathscr{B}$ is commutative in $\mathscr{L}(X)$). $\mathscr{B}$ is *complete* (*σ-complete*) if for every subset (sequence) $\{E_\alpha\}\subset\mathscr{B}$, the projections $\vee E_\alpha$ and $\wedge E_\alpha$ are in $\mathscr{B}$. A Boolean algebra $\mathscr{B}$ is *bounded* if $\|E\|_J\leqslant M_J$ for all $E\in\mathscr{B}$ and some (equivalently: all) J in $\mathscr{J}(X)$.

PROPERTY 4.1. *A Boolean algebra $\mathscr{B}$ of projections is bounded if and only if $\mathscr{J}_{\mathscr{B}}\neq\varnothing$.*

Proof: If $J\in\mathscr{J}_{\mathscr{B}}$, then $\|E\|_J=1$ and so $\mathscr{B}$ is bounded. Conversely, every bounded Boolean algebra of idempotents on a Hilbert space $\mathscr{H}$ is similar to a Boolean algebra of $\mathscr{H}$-hermitian projections (see [4], p. 58). The statement follows thus by Lemma 4.1. □

PROPERTY 4.2. *Every bounded Boolean algebra of projections is contained in a complete Boolean algebra of projections.*

Proof: If $\mathscr{B}$ is bounded, then $\mathscr{B}$ is a Boolean algebra of J-projections for $J\in\mathscr{J}_{\mathscr{B}}$. Now the set $\tilde{\mathscr{B}}$ of all projections in the weak closure of $\mathscr{B}$ is a complete Boolean algebra of projections on X_J. On the other hand, by the property of a weak closure, we have $EJ=JE$ for all $E\in\tilde{\mathscr{B}}$ and so $\tilde{\mathscr{B}}\subset\mathscr{B}(X)$. □

PROPERTY 4.3. *Let $\mathscr{B}$ be a σ-complete Boolean algebra of projections. Then $\mathscr{J}_{\mathscr{B}}\neq\varnothing$ and $\mathscr{B}$ is bounded. For every sequence $\{E_\alpha\}\subset\mathscr{B}$ (subset, if $\mathscr{B}$ is complete) $E=\vee E_\alpha$ is a projection onto the closed linear span of $\{E_\alpha X\}$ and $F=\wedge E_\alpha$ is a projection onto $\bigcap(E_\alpha X)$. Observe that in general $\bigcap X_\alpha\neq\wedge X_\alpha$!)*

Proof: By a result of Bade every σ-complete Boolean algebra of projection on a Banach space is bounded ([1], Theorem 2.2). Given $J\in\mathscr{J}(X)$, X_J is a Banach space, so $\mathscr{B}$ is bounded. By Property 4.1, $\mathscr{J}_{\mathscr{B}}\neq\varnothing$ and given $J\in\mathscr{J}_{\mathscr{B}}$, $\mathscr{B}$ is a Boolean algebra of projections on a Hilbert space X_J. □

Property 4.4. *If $\mathscr{B}_1$ and $\mathscr{B}_2$ are bounded and $\mathscr{B}_1 \subset \mathscr{B}_2'$, then $\mathscr{J}_{\mathscr{B}_1} \cap \mathscr{J}_{\mathscr{B}_2} \neq \emptyset$. There exists a complete Boolean algebra of projections $\mathscr{B}$ that contains both $\mathscr{B}_1$ and $\mathscr{B}_2$.*

Proof: Follows by a result of Wermer on Boolean algebras of idempotents (see [10], Theorem 1). □

Definition 4.3. A (real) *spectral measure* is a σ-homomorphism $E : \Delta \to E(\Delta)$ from a Boolean algebra of all Borel subsets of the real line into $\mathscr{B}(X)$. A spectral measure $\boldsymbol{E}$ is *compact* if $E(\Delta)$ vanishes outside some compact Δ_0.

Property 4.5. *Every spectral measure E is bounded and $\mathscr{J}_E \neq \emptyset$. $\Delta \to E(\Delta)x$ is strongly continuous for all x in X.*

Proof: It is sufficient to notice that the range of $\boldsymbol{E}$ is a σ-complete Boolean algebra of projections. □

Property 4.6. *Let $\Delta \to E(\Delta)$ be a spectral measure and let f be a continuous complex function on R. If $\boldsymbol{E}$ is compact, then the integral $A(f) = \int f(\lambda)\, dE(\lambda)$ exists in the uniform topology of $\mathscr{L}(X)$ and*

$$\left\| \int f(\lambda)\, dE(\lambda) \right\|_J \leqslant \sup_{\lambda \in \Delta_0} |f(\lambda)| \cdot M_{E,J} .$$

If f is real, then $A(f)$ is elliptic. There exists $J \in \mathscr{J}(X)$ commuting with all $A(f)$.

Proof: Follows by Property 4.5 and a corresponding Hilbert-space theorem. □

Property 4.7. *If A is elliptic, then there exists a unique spectral measure E such that $A = \int \lambda\, dE(\lambda)$.*

Proof: If $J \in \mathscr{J}_A$, then A is J-hermitian. Its spectral resolution on X_J commutes with J and so, is a spectral measure according to Definition 4.3. □

We have thus established a one-to-one correspondence between spectral measures and elliptic operators in $\mathscr{L}(X)$. It is possible to generalize it for unbounded operators. We shall not deal with these problems. (The reader is referred to the paper of Jalava [6a] where a generalization of Property 4.7 is proved for the unbounded case.) Let us observe that a simplest hermitian operator with $\mathscr{J}_A$ empty is of the following form

$$A_x y = (x, y)x , \quad (x, x) = 0 .$$

A_x is a nilpotent, its spectrum consists of one point only, $\lambda = 0$.

Definition 4.4. A semigroup $\{V\}$ of unitary operators on X is *bounded* if $\|V\|_J$ are bounded or, equivalently, if the orbits of $\{V\}$ on $\mathscr{J}(X)$ are bounded.

Property 4.8. *With $A = A^* \in \mathscr{L}(X)$, the following conditions are equivalent:*

(i) *A is elliptic,*

(ii) *$\{e^{iAt}\}$ is bounded,*

(iii) *for some (equivalently every) $J \in \mathscr{J}(X)$, $\sup_{n,\lambda>0} \lambda^n \|(A - \lambda I)^n\|_J < \infty$.*

Proof: (i) implies (ii) and (ii) is equivalent to (iii) by [8], Theorem 12.3.1. If (ii) holds, then $V=e^{iA}$ satisfies $\|V^n\|_J<M_J$, $n=\pm 1, \pm 2, \dots$ It follows, by a theorem due to Nagy [9] that V is similar to a J-unitary operator. The statement follows now by Lemma 4.1. □

PROPERTY 4.8. *If $A_1, \dots, A_n$ are elliptic and commute, then $\bigcap \mathscr{J}_{A_i}\neq\emptyset$.*

Proof: Follows immediately by Property 4.4. □

The rest of this section we devote to some remarks connected with unitary group representations on X. We assume that there is given a locally compact group G and a unitary representation $g\to V_g$ of G on X. Now, if $g\to V_g$ is weakly continuous, then it is weakly continuous on each X_J. It follows that $g\to V_g$ is strongly continuous (see [4], p. 57). We assume that $g\to V_g$ is continuous.

PROPERTY 4.9. *If G is amenable and $\{V_g\}$ is bounded, then $\mathscr{J}_G\neq\emptyset$.*

Proof: By [3], Theorem 3.4.1, $\{V_g\}$ is similar to a group of unitary operators on X_J. The statement follows from Lemma 4.1. □

As a corollary we obtain Phillips result [8], Theorem 6.1

PROPERTY 4.10. *If $\mathscr{A}$ is an abelian *-subalgebra of $\mathscr{L}(X)$ and* $\sup\{\|A\|_J\colon A\in\mathscr{A}\}<\infty$, *then $\mathscr{J}_{\mathscr{A}}\neq\emptyset$.*

Proof: Clearly, the uniform closure $\bar{\mathscr{A}}$ of $\mathscr{A}$ also satisfies the assumptions. The unitary group of $\mathscr{A}$ is amenable and bounded. The statement follows by Property 4.9. □

LEMMA 4.2. *Assume that G is compact and $g\to V_g$ continuous. Then $\{V_g\}$ is bounded.*

Proof: Suppose $\|V_{g_\alpha}\|_J\to\infty$. Since G is compact, we may assume that $V_{g_\alpha}\to I$ strongly and $\|V_{g_\alpha}\|_J\to\infty$. Clearly, $V^*_{g_\alpha}\to I$ strongly and $\|V^*_{g_\alpha}\|_J=\|V_{g_\alpha}\|_J\to\infty$. Let us choose a sequence $x_\alpha\in X$ such that $\|x_\alpha\|_J=1$ and $\|V_{g_\alpha}x_\alpha\|_J\to\infty$. By the weak compactness of the unit ball of X_J, we may assume that $x_\alpha\to x_0$ weakly. It follows that $\|x_\alpha-x_0\|<M$. Let $y\in X$ be arbitrary. We have $(V_{g_\alpha}x_\alpha, y)_J=(V_{g_\alpha}x_\alpha, Jy)=(x_\alpha, V^*_{g_\alpha}Jy)$. Now, $V^*_{g_\alpha}Jy\to Jy$ strongly and $x_\alpha\to x_0$ weakly. Thus $(x_\alpha, V^*_{g_\alpha}Jy)\to(x_0, Jy)=(x_0, y)_J$. It follows that $V_{g_\alpha}x_\alpha\to x_0$ weakly. This is a contradiction with $\|V_{g_\alpha}x_\alpha\|_J\to\infty$. □

PROPERTY 4.11. *If G is compact, then $\mathscr{J}_G\neq\emptyset$.*

DEFINITION 4.5. We say that $g\to V_g$ is *irreducible* if the only projections (or regular subspaces) invariant under G are trivial ones : 0 or I.

PROPERTY 4.12. *Every continuous unitary representation of a compact group G on X is reducible.*

Proof: By Property 4.11, $JV_g=V_gJ$ for some $J\in\mathscr{J}(X)$. But $E=\frac{1}{2}(J+1)$ is a projection. □

V. Some unsolved problems

(A) What is the structure of the commutant of $\mathscr{J}(X)$? Is it trivial? Or, more generally, given a subset $J_0\subset\mathscr{J}(X)$, how does J_0' look? In particular, what can be said about hermitian operators commuting with two different J's? (It is easily seen that $JJ'+J'J$ is hermitian and commutes with J and with J'.)

(B) What are the characteristic properties that fix the form of $t(J, J')$ and $U(J, J')$? Consider a triple $J, J', J'' \in \mathscr{J}(X)$. Does $t(J, J')\,t(J', J'')=t(J, J'')$ hold? If not, then what can be said about $R_{J,J',J''}=t(J, J')\,t(J', J'')\,t(J'', J)$? It is easily seen that $R_{J,J',J''}$ commutes with J. What is the behaviour of $R_{J,J',J''}$ if $J, J', J'' \to J_0$.

(C) What are the Banach *-algebras, which have a faithful *-representation on a J-space? It follows directly that a Banach *-algebra with an algebraic, norm-preserving automorphism α satisfying $\alpha^2=1$ and $\|\alpha(A^*)A\|=\|A\|^2$ is such an algebra. Is it possible to characterize all uniformly closed *-subalgebras of $\mathscr{L}(X)$? It can be shown that every positive linear form f on $\mathscr{L}(X)$ vanishes on all finite-dimensional projections. If f is strongly continuous, then f vanishes on all projections. Does it follow that f vanishes on $\mathscr{L}(X)$?

(D) Classify irreducible continuous representations of the Poincaré group on X. Does every reducible representation of this group decompose into irreducible ones? Do there exist irreducible representations with non-elliptic generators of translations? Does every irreducible representation of the Poincaré group have a maximal, positive, invariant subspace?

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