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ON A THEOREM OF MACKEY, STONE AND VON NEUMANN FOR PROJECTIVE IMPRIMITIVITY SYSTEMS *

by

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Summary

Let X be a standard, transitive Borel G-space for an lcsc group G. A multiplier for (G,X) is a Borel function ω from $G\times G\times X$ into the circle group satisfying a cocycle-like relation $\omega(g,h;x)\,\omega(gh,k;x)=\omega(h,k;g^{-1}x)\,\omega(g,hk;x)$ for almost all $x\in X$, and normalization condition $\omega(g,e;x)=\omega(e,g;x)=1$ (x-a.e.) A projective imprimitivity system /with a multiplier ω/ in a separable Hilbert space $\mathcal{H}$ is a pair (E,U), where E is a spectral measure on X, and $g\to U_g$ is a Borel map from G into the unitary group of $\mathcal{H}$ such that: $U_e=1$, $U_gE(S)U_g^*=E(gS)$ and $U_gU_h=\int\omega(g,h;x)\,dE(x)\,U_{gh}$. If $(g,h)\to\omega(g,h;\cdot)$ is continuous in the w*-topology of $\mathcal{L}^\infty(X)$, we require $g\to U_g$ to be strongly continuous. We prove that if ω is a continuous multiplier for (G,G) then there exists a unique irreducible ω-imprimitivity system. In the course of the proof we construct a kind of a regular ω-imprimitivity system for a general (G,X)-multiplier. The theorem generalizes theorems of Stone, von Neumann and Mackey on uniqueness of a solution of canonical commutation relations of quantum mechanics to cases in which translational symmetry is broken by external electromagnetic fields.

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1. Introduction

Heisenberg's commutation relations for kinematical variables $\{p_i, q_i\}_{i=1,2,3}$ of a quantum particle read:

$$[q_i, q_j] = 0 , \qquad /1.1/$$

$$[q_i, p_j] = i\,\delta_{ij} , \qquad /1.2/$$

$$[p_i, p_j] = 0 . \qquad /1.3/$$

Weyl /8/ put them into an exponential form /CCR/:

$$U_{\underline{a}} V_{\underline{b}} = \exp(i\underline{ab}) V_{\underline{b}} U_{\underline{a}} ,$$

$$V_{\underline{b}} = \exp(i\underline{qb}),\ U_{\underline{a}} = \exp(i\underline{pa}),$$

and von Neumann /7/ proved that every irreducible representation of CCR is unitarily equivalent to the Schrödinger one:

$$(p_i f)(x) = -i(\partial f/\partial x_i)(x),$$

$$(q_i f)(x) = x_i f(x).$$

Mackey /3,4/ replaced $V_{\underline{b}}$ by its spectral measure E such that

$$V_{\underline{b}} = \int \exp(i\underline{xb})\, dE(\underline{x}).$$

Then (E,U) is an imprimitivity system for R^3 based on R^3 and the uniqueness theorem follows from the Imprimitivity Theorem /4,5/.

According to these results there is a unique method of quantizing free particles. If, however, external fields are present then there are no reasons to insist on existence of a representation U of the translation group /since translations are no longer symmetries of the system/ and so, the theorems of Stone, von Neumann and Mackey are no longer applicable.

In spite of this, physicists use a definite prescription: if a magnetic field $\underline{B}(\underline{x})$ is present, they replace canonical p_i-s by "gauge invariant" variables $\pi_i = p_i - A_i$, where $\underline{A}$ is a vector-potential of $\underline{B}$: $\underline{B} = \text{curl}\,\underline{A}$. The observables q_i and π_i satisfy now generalized Heisenberg's commutation relations

$$[q_i, q_j] = 0 , \qquad /1.4/$$

$$[q_i, \pi_j] = i\,\delta_{ij}, \qquad /1.5/$$

$$[\pi_i, \pi_j] = i\,\varepsilon_{ijk} B_k(q). \qquad /1.6/$$

In this framework it is important to know, whether every irreducible representation of the commutation relations /1.4-6/ is unitarily equivalent to the conventional one: $\underline{\pi} = \underline{p} - \underline{A}(q)$, where p and q are the Schrödinger operators.

On a formal, algebraic level, the answer is "yes". In fact, it follows from /1.5/ that $[\pi_i, f(q)] = -i\,\partial_i f(q)$, and the Jacobi identity, when applied to π_i, π_j, π_k , gives $\partial_i B_i = 0$. It "follows" that there exists $\underline{A}$ such that $\underline{B} = \text{curl}\,\underline{A}$, and with $\underline{p} := \underline{\pi} + \underline{A}$ one finds that p and q satisfy /1.1-3/. Moreover, (q,p) is irreducible if and only if (q,π) is such. It is then natural to ask whether these formal manipulations can be put into a rigorous form. One way to do so is to replace q_i-s by a spectral measure E and π_i-s by unitary operators $U_{\underline{a}} = \exp(-i\underline{\pi}\,\underline{a})$ analogously as Mackey did. Formal series expansion leads then to

$$U_{\underline{a}} E(S) U_{\underline{a}}^{*} = E(S+\underline{a}), \qquad /1.7/$$

and

$$U_{\underline{a}} U_{\underline{b}} = \Omega(\underline{a},\underline{b}) U_{\underline{a}+\underline{b}} \qquad /1.8/$$

where

$$\Omega(\underline{a},\underline{b}) = \int \omega(\underline{a},\underline{b};\underline{x})\, dE(\underline{x}) \qquad /1.9/$$

$$\omega(\underline{a},\underline{b};\underline{x}) = \exp\{i\Phi(\underline{a},\underline{b};\underline{x})\},$$

and $\Phi(\underline{a},\underline{b};\underline{x})$ is a flux of $\underline{B}$ through a triangular surface spanned by vectors $-\underline{a}$ and $-\underline{b}$ at $\underline{x}$. It is interesting that associativity requirement for $U_{\underline{a}}$ /which is an integral analogue of the Jacobi identity/ is equivalent to the condition that the flux of $\underline{B}$ through "almost all" tetrahedrons is an integral multiple of 2π /2/. Thus fields satisfying div $\underline{B} = \sum g_i \delta(\underline{x}-\underline{x}^{(i)})$ are admissible /Dirac's magnetic poles/ and one can hardly expect that $\underline{B}$ is of the form curl $\underline{A}$.

In the present paper we generalize relations /1.7-9/ to an arbitrary G-space X (ω-imprimitivity system/ and prove, under appropriate continuity conditions, a uniqueness of an irreducible ω-imprimitivity system in the case of G acting on itself.

We wish to draw the reader attention to a possible connection between ω-imprimitivity systems and connexions in principal bundles /in an analogy with the electromagnetic case one can expect that ω is connected to a curvature form/. Only in the case of a constant curvature, U reduces to an ordinary projective representation, and in the case of curvature zero projective imprimitivity system reduces to an ordinary one.

2. Projective Imprimitivity Systems

Let X be a set and let $\mathcal{B}$ be a σ-algebra of subsets of X. Then X is said to be a Borel space with a Borel structure $\mathcal{B}$ ($\equiv \mathcal{B}(X)$). A Borel space X is standard if its Borel structure is is isomorphic to the natural Borel structure of a Borel subset of a complete, separable metric space.

Let G be a locally compact group satisfying the second axiom of countability and let X be a standard Borel space. Assume that there is given a homomorphism $t: g \to t_g$ of G into the group of Borel automorphisms of X such that

1°/ X is a transitive G-space i.e. for each pair $x, y \in X$ there is $g \in G$ such that $y = t_g(x)$,

2°/ the mapping $(g,x) \to t_g(x)$ from $G \times X$ into X is Borel.

Then X is said to be a <u>transitive standard Borel G-space</u> /we shall write gx instead of $t_g(x)$/. There are quasi-invariant σ-finite Borel measures on X, every two of them being mutually absolutely continuous /6, p.25/. If α is such a measure, and if $\mathcal{C}_\alpha$ is the class of all Borel sets of α-measure zero, then $\mathcal{C}_\alpha$ is α-independent and so, there is a unique invariant measure class $\mathcal{C}$ on X /6, Ch. VIII/. We shall write "$f(x) = g(x)$ (x-a.e.)" if and only if the set $\{x : f(x) \neq g(x)\}$ is in $\mathcal{C}$.

Let us fix G, X and α throughout this section, X being a transitive standard Borel G-space and α -being a quasi-invariant σ-finite measure on X.

<u>Definition 2.1.</u> A <u>multiplier for (G,X)</u> is a Borel function ω

$$\omega : (g,h,x) \to \omega(g,h;x)$$

from $G \times G \times X$ into the circle group, such that

/i/ for all $g,h,k \in G$

$$\omega(g,h;x)\omega(gh,k;x) = \omega(h,k;g^{-1}x)\omega(g,hk;x) \qquad \text{/x-a.e./},$$

/ii/ for all $g \in G$

$$\omega(g,e;x) = \omega(e,g;x) = 1 \qquad \text{/x-a.e./}.$$

We say that ω is <u>continuous</u> if for every Borel function $f \in \mathcal{L}^1(X,\alpha)$ the mapping $g,h \to \int \omega(g,h;x)f(x)d\alpha(x)$ is continuous on $G \times G$.

Let us observe that if $\omega(g,h;x)$ is x-independent, then the above definition reduces to a usual definition of a multiplier for G.

A class of multipliers for (G,X) can be obtained in the fol- -lowing way: let w be a Borel map $w:(g,x)\to w(g;x)$ from $G\times X$ into the circle group, such that $w(e;x)=1$ /x-a.e./. Let

$$\omega_w(g,h;x) := w(g;x)w(h;g^{-1}x)\,w(gh;x)^{-1} .$$

It is easy to see that ω_w is a multiplier for (G,X). Moreover, if $g\to w(g;\cdot)$ is continuous in the w*-topology of $\mathcal{L}^\infty(X)$, then ω_w is continuous. It will be shown in Sec.3 that every continuous multiplier for (G,G) is of the form ω_w.

Let $\mathcal{H}$ be a separable Hilbert space. We denote by $\mathcal{U}(\mathcal{H})$ the unitary group of $\mathcal{H}$ equipped with a strong /equivalently: weak/ operator topology. $\mathcal{P}(\mathcal{H})$ will stand for an orthocomplemented lattice of orthogonal projections in $\mathcal{H}$.

<u>Definition 2.2.</u> Let ω be a multiplier /resp. continuous multiplier/for (G,X), and let $\mathcal{H}$ be a separable Hilbert space. A pair (U,E), where U is a map $U:g\to U_g$ from G to $\mathcal{U}(\mathcal{H})$ and $E:S\to E(S)$ is a projection-valued measure from $\mathcal{B}(X)$ into $\mathcal{P}(\mathcal{H})$, is said to be an <u>ω-imprimitivity system</u> /<u>ω-IS</u>/ if

/i/ $g\to U_g$ is Borel /resp. continuous/,

/ii/ $U_e=I$,

/iii/ $U_gE(S)U_g^*= E(gS)$,

/iv/ $U_gU_h = \Omega(g,h)U_{gh}$,

where

/v/ $\Omega(g,h) = \int\omega(g,h;x)dE(x)$.

A pair (U,E) is said to be a /<u>continuous</u>/ <u>projective</u> <u>impri-</u>

-mitivity system if there exists a /continuous multiplier for (G,X), such that (U,E) is an ω-imprimitivity system.

The following result can be easily extracted from a standard version of the Imprimitivity Theorem /see e.g. 6, p. 72-76/ :

Lemma 2.1. Let U and E satisfy the condition /iii/ of the Def. 2.2. There is a Hilbert space $\mathcal{K}$ and an isometry T from $\mathcal{H}$ onto $\mathcal{L}^2(X, \mathcal{K}, \alpha)$ such that $TET^{-1}=E^c$, where E^c is a canonical spectral measure in $\mathcal{L}^2(X,\mathcal{K},\alpha)$, i.e. for every $f \in \mathcal{L}^2(X,\mathcal{K},\alpha)$:

$$(E^c(S)f)(x) = \mathcal{X}_S(x)f(x) \qquad /x\text{-a.e.}/.$$

Lemma 2.2. Let ω be a multiplier /resp. continuous multiplier/ for (G,X), and let $(\mathcal{H},U,E)$ be an ω-IS. There exist: a unitary representation $V:g\to V_g$ of G in $\mathcal{H}$, and a Borel /resp. continuous/ map $W:g\to W_g$ from G into $\mathcal{U}(\mathcal{H})$ such that

/i/ $W_e = I$,

/ii/ W_g commutes with E(S) for all $g\in G$, $S\in \mathcal{B}(X)$,

/iii/ $\Omega(g,h)= W_g V_g W_h V_g^* W_{gh}^*$ for all $g,h\in G$,

/iv/ $U_g = W_g V_g$ for all $g\in G$,

/v/ $(\mathcal{H},V,E)$ is a 1-IS.

Conversely, if $(\mathcal{H},V,E)$ is a 1-IS, $g\to W_g$ is a Borel /resp. continuous/ map from G into $\mathcal{U}(\mathcal{H})$ satisfying the conditions /i-iii/, with Ω as in Def. 2.2., and if U is given by /iv/, then $(\mathcal{H},U,E)$ is an ω-IS.

Proof. With T and E^c as in Lemma 2.1., let V^c be a unitary representation of G in $\mathcal{L}^2(X,\mathcal{K},\alpha)$, such that $/V^c,E^c/$ is a 1-IS /for an existence see e.g. 6, p. 60/. Let $V := TV^cT^{-1}$ and $W:=UV$, then /i-v/ are automatically satisfied. The converse is also easily verified.

If ω is of the form ω_w, then one can easily classify all irreducible ω-IS-s by reducing the problem to 1-IS-s.

<u>Theorem 2.1.</u> Let ω be a multiplier for (G,X) of the form ω_w. Let π be an irreducible representation of the stability subgroup G_o of an $x_o \in X$, and let $(\mathcal{H}^\pi, V^\pi, E^\pi)$ be a unique irreducible 1-IS associated with π /see 6, p. 78/. Then, with $\tilde{W}_g = \int w(g;x)dE^\pi(x)$ and $U^\pi = \tilde{W}V^\pi$, (E^π, U^π) is an irreducible ω-IS. Every irreducible ω-IS is of the above form. Moreover, (E^π, U^π) is unitarily equivalent to $(E^{\pi'}, U^{\pi'})$ if and only if π is equivalent to π'. In particular, if G_o is trivial, then there exists a unique ω-IS.

<u>Proof.</u> The first part of the theorem is an immediate consequence of Lemma 2.2. To prove the converse, let $(\mathcal{H},U,E)$ be an irreducible ω-IS, let $\tilde{W}_g = \int w(g;x)dE(x)$, and let V and W be as in Lemma 2.2. Now, $\{\tilde{W}_g\} \in E''$ /the von Neumann algebra generated by E/, and with $Q_g := \tilde{W}_g^* W_g$ it follows that $Q_g V_g Q_h V_g^* = Q_{gh}$. Therefore $g \to Q_g V_g$ is a unitary representation of G, and (E,QV) is a 1-IS. Moreover, (E,QV) is irreducible /since $QV = \tilde{W}^* U$/ and so, there exists π such that $(E,QV) = (E^\pi, V^\pi)$. It follows that $U = \tilde{W}V^\pi$. Clearly, $(E^\pi, U^\pi) \simeq (E^{\pi'}, U^{\pi'})$ if and only if $(E^\pi, V^\pi) \simeq (E^{\pi'}, V^{\pi'})$.

The next theorem assures that for every multiplier ω for (G,X) an ω-IS exists. In order to prove this, we construct a kind of a regular projective IS.

<u>Theorem 2.2.</u> Let ω be a multiplier for (G,X) and let $= \mathcal{L}^2(G \times X, \mu \times \alpha)$, where μ is a left Haar measure on G. Let us define

$$(V_g f)(k,x) = f(g^{-1}k,x) ,$$

$$(W_g f)(k,x) = \omega(k^{-1},g;x) f(k,x) ,$$

$$(E(S)f)(k,x) = \chi_S(kx) f(k,x) .$$

Then $(\mathcal{H},E,V)$ is a 1-IS, and W satisfies the conditions /i-iii/ of Lemma 2.2. In particular, with $U = WV$, (E,U) is an ω-IS. Moreover, if ω is continuous, then U is such.

<u>Proof</u>. One immediately verifies that (E,V) is a 1-IS, and that W is Borel and satisfies the conditions /i-ii/ of Lemma 2.2. Moreover, if ω is continuous, then a routine application of the Fubini theorem and Lebesgue Dominated Convergence theorem shows that $g \to W_g$ is continuous. It is therefore enough to show that /iii/ of Lemma 2.2. holds. First of all, it follows from the definitions that

$$(W_g V_g W_h V_g^* f)(k,x) = \omega(k^{-1};g;x)\,\omega(k^{-1}g,h;x)\,f(k,x) \ .$$

Then, for fixed $g,h,k \in G$, there is an $N_{g,h}(k) \in \mathcal{C}$ such that

$$\omega(k^{-1},g;x)\,\omega(k^{-1}g,h;x) = \omega(g,h;kx)\,\omega(k^{-1},gh;x)$$

for all $x \notin N_{g,h}(k)$ /see Def. 2.1.,/i//. It follows that for all $x \notin N_{g,h}(k)$

$$(W_g V_g W_h V_g^* f)(k,x) = \omega(g,h;x)\,\omega(k^{-1},gh;x)\,f(k,x) =$$
$$= \omega(g,h;kx)(W_{gh} f)(k,x).$$

Thus, for almost all (k,x) we get

$$(W_g V_g W_h V_g^* f)(k,x) = (\Omega(g,h) W_{gh} f)(k,x),$$

since it follows from the very definition of E that

$$(\Omega(g,h) f)(k,x) = \omega(g,h;kx)\,f(k,x) \quad /k,x\text{-a.e.}/.$$

3. <u>The Case of G Acting on Itself</u>.

Let us assume that G acts on X not only transitively but also freely i.e. that X is an affine G-space. Without lost of generality we may then assume that $X=G$ and that $\alpha = \mu$ - a left Haar measure on G.

<u>Theorem 3.1.</u> Every continuous multiplier for (G,G) is of the form ω_w, $g \to w(g;\cdot)$ being continuous in the w^*-topology of $\mathcal{L}^\infty(G)$.

<u>Proof.</u> Let $(\mathcal{H},\hat{E},\hat{U})$ be the regular ω-IS. For every $f\in\mathcal{H}$ let $(Tf)(x,y)=f(yx,y^{-1})$. It is easy to see that T is a unitary operator in $\mathcal{H}$ with $T^2=I$ that commutes with V of the Theorem 2.2. Let $(E,U):(T\hat{E}T,T\hat{U}T)$, then (E,U) is an ω-IS and

$$(E(S)f)(x,y)=\chi_S(x)f(x,y),$$
$$(U_gf)(x,y)=\omega(y^{-1}x^{-1},g;y^{-1})f(g^{-1}x,y).$$

We now identify $\mathcal{H}$ with $\mathcal{L}^2(G,\mathcal{K},\mu)$, $\mathcal{K}=\mathcal{L}^2(G,\mu)$, and define for every $g,k\in G$

$$(w(g;x)\phi)(y):=\omega(y^{-1}x^{-1},g;y^{-1})\,\phi(y)\quad,(\phi\in\mathcal{K}).$$

Then $w(g;x)$ is a unitary operator in $\mathcal{K}$ and with

$$(W_gf)(x)=w(g;x)f(x)$$

we get $U_g=W_gV_g$ /observe that $(g,x)\to w(g;x)$ is a Borel map/. Now, since $W_gV_gW_hV_g^*=\Omega(g,h)W_{gh}$, it follows that

$$w(g;x)w(h;g^{-1}x)=\omega(g,h;x)w(gh;x)\qquad /x\text{-a.e.}/$$

and so, there exists $x_0\in G$ such that

$$w(g;x_0)w(h;g^{-1}x_0)=\omega(g,h;x_0)w(gh;x_0)\quad /g,h\text{-a.e.}/\qquad /3.1/$$

Let p be a projection in $\mathcal{L}^2(G,\mu)$, and let

$$p(x):=w(x_0x^{-1},x_0)^{-1}p\;w(x_0x^{-1},x_0).\qquad /3.2/$$

The function $x\to p(x)$ is a Borel function from G to $\mathcal{P}(\mathcal{K})$. It follows that $(Pf)(x):=p(x)f(x)$ defines a projection in $\mathcal{H}$ and it follows from the very definition that P reduces E. Let us show that P reduces U also. First, we observe that for almost all g,x

$$p(g^{-1}x)=w(g;x)^{-1}p(x)w(g;x)\ .\qquad /3.3/$$

In fact, it follows from /3.1/ and /3.2/ that

$$w(g;x)^{-1}p(x)w(g;x)=w(g;(x_0x^{-1})^{-1}x_0)^{-1}w(x_0x^{-1};x_0)^{-1}pw(x_0x^{-1};x_0)\times$$
$$\times\ w(g;(x_0x^{-1})^{-1}x_0)=$$
$$=w(x_0x^{-1}g;x_0)^{-1}\,\omega(x_0x^{-1},g;x_0)^{-1}p\,\omega(x_0x^{-1},g;x_0)w(x_0x^{-1}g;x_0)=$$
$$=w(x_0x^{-1}g;x_0)^{-1}pw(x_0x^{-1}g;x_0)=p(g^{-1}x),$$

for almost all g,x. This shows that for almost all $g\in G$ $(U_gPf)(x)=(PU_gf)(x)$ /x-a.e./, and so, $U_gP=PU_g$ for almost all $g\in G$. Therefore P commutes with U by the continuity of U.

Now, let ϕ be a unit vector in $\mathcal{K}$, and let p_ϕ be an orthogonal projection onto $\{\mathbb{C}\phi\}$. The subspace $\mathcal{H}_\phi = P_\phi\mathcal{H}$ is now invariant under E and U and so $U^\phi = U\restriction\mathcal{H}_\phi$ and $E^\phi = E\restriction\mathcal{H}_\phi$ form up an ω-IS in $\mathcal{H}_\phi$. For every $\xi\in\mathcal{L}^2(G,\mu)$ let

$$(R\xi)(x)=\xi(x)w(x_ox^{-1},x_o)^{-1}\phi .$$

Then $x\to(R\xi)(x)$ is a Borel function from G into $\mathcal{K}$, and $\|R\xi\| = \|\xi\|$. Moreover, it is easy to see that $P^\phi R\xi = R\xi$ and so, R maps $\mathcal{L}^2(G,\mu)$ into $\mathcal{H}_\phi$ isometrically. In fact, the range of R is equal to $\mathcal{H}_\phi$, since for every $f\in\mathcal{H}_\phi$/i.e. such that $P_\phi f = f$/ R maps the function $\xi(x):=(\phi,w(x_ox^{-1},x_o)f(x))$ into f. Now

$$(R^{-1}E(S)R\xi)(x)= \chi_S(x)\xi(x),$$

and

$$(R^{-1}U_gR\xi)(x)=(\tilde{W}_g\tilde{V}_g\xi)(x),$$

where

$$(\tilde{V}_g\xi)(x)= \xi(g^{-1}x),$$

and

$$(\tilde{W}_g\xi)(x)= \tilde{w}(g;x)\xi(x)=$$

$$= (\phi,w(x_ox^{-1};x_o)w(g;x)w(x_ox^{-1}g;x_o)^{-1}\phi)\xi(x).$$

Since $(R^{-1}ER,R^{-1}UR)$ is an ω-IS, we conclude that $\tilde{w}$ has the required properties.

<u>Corolary 3.1.</u> Let ω be a continuous multiplier for (G,G). There is a unique irreducible ω-IS (E,U), and E is unitarily equivalent to the canonical spectral measure in $\mathcal{L}^2(G,\mu)$.

<u>Proof.</u> Follows immediately from the Theorems 3.1 and 3.2.

4. <u>Final Remarks</u>.

It would be desirable to know, whether the Corollary 3.1 can be obtained under a weaker condition on ω. From a physical point of view one would like to know what happens if U_g is assumed to be locally continuous only. In fact, what one needs is an existence of selfadjoint generators /in case of a Lie group/. On the other hand, one should take into account unobservability of phase facters. It is then natural to consider equivalence classes of multipliers. Bargmann /1/ has shown that on a connected simply connected Lie group G, every multiplier is equivalent to an analytic one. An analogous problem for a general (G,X) case is not solved.

Finally, we would like to draw the reader attention to the fact that our formula for a regular ω-imprimitivity system is a kind of generalization of the Poincaré Lemma. The last deals with differential forms. We deal with their exponents and geodesic simplexes. When written in an infinitezimal form and applied to the electromagnetic case, our formula for a regular representation leads to a bilocal electromagnetic potential

$$A_i(x',x)= \int_0^1 F_{jk}(z)\frac{\partial z^j}{\partial t} \frac{\partial z^k}{\partial x^i} dt$$

where $z = x+t(x'-x)$ and $i,j,k = 1,2,3,0$. This should be compared with B. de Witt's proposal /9/.

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