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On Quantum Iterated Function Systems

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Abstract: A Quantum Iterated Function System on a complex projective space is defined through a family of linear operators on a complex Hilbert space. The operators define both the maps and their probabilities by one algebraic formula. Examples with conformal maps (relativistic boosts) on the Bloch sphere are discussed.

Keywords: ifs, qifs, quantum iterated function systems, quantum fractals, quantum measurement
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1 Introduction

Iterated Function Systems [1] generate fractal sets due to non-commutativity of maps. In quantum theory, position and momentum operators do not commute (which leads to Heisenberg's uncertainty relations), and different components of spin also do not commute. This suggests that fractal patterns and chaos may arise as a result of certain quantum measurement processes, for instance in a continuous monitoring of several non-commuting observables. A typical continuous monitoring takes place, for instance, in a cloud chamber. Different regions of the chamber are active in parallel, and they are activated sequentially, each at a different time, by a charged quantum particle that leaves the track. Moreover, in Heisenberg's picture, position operators at different times do not commute. Parallel arrangement of an experimental setup is realized by the addition of operators, while serial arrangements lead to multiplication of the corresponding operators [2]. While sums of operators are commutative, their products, in general, depend on the order of the factors. Sums of terms usually appear in time evolution generators. Products

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appear in transitions from one quantum state to another, as the result of wave function collapse resulting from measurement events. Repeated application, with non-commuting operators, leads to iterated function systems, chaos and fractal attractors on the manifold of quantum states. In the present paper we describe the present status of this new research avenue, and we point out some open questions.

2 The two–sphere S^2 as the canvas

The flagship example of a classical "Iterated Function System" (IFS) is the Sierpinski fractal. It is generated by an application of 3×3 matrices $A[i], i = 1, 2, 3$, in a random

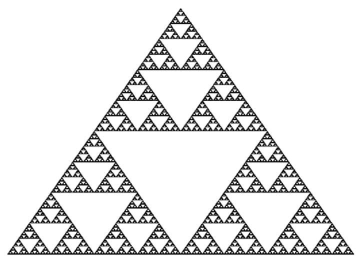

Fig. 1 The classical fractal: Sierpinski Triangle generated by an Iterated Function System.

order, to the vector:

$$v_0 = \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} \tag{1}$$

where $A[i]$ is given by

$$A[i] = \begin{pmatrix} 0.5 & 0 & ax_i \\ 0 & 0.5 & ay_i \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

and $ax_1 = 0.0, ay_1 = 0.0, ax_2 = 0.5, ay_2 = 0.0, ax_3 = 0.25, ay_3 = 0.5$. (Our 3×3 matrices encode affine transformations (of a two–plane)—usually separated into a 2×2 matrix and a translation vector.) At each step one of the three transformations $A[i], i = 1, 2, 3$ is selected with probability $p[i] = 1/3$. After each transformation the transformed vector is plotted on the (x, y) plane.

The important property of the maps $A[i]$ is that they are contractions. The Sierpinski triangle, as well as another well known example, the fern [1], live on a 2-dimensional plane. Quantum iterated function systems (QIFS) live on complex projective spaces, the simplest one being $CP(1)$—a 2-dimensional sphere S^2, also known as the Bloch sphere.

In fact, there are at least two ways in which S^2 is important in physics. First, as the Bloch sphere, it represents pure states of the simplest quantum system—spin 1/2. Second, it represents directions in our three-dimensional space. The last statement is not relativistically invariant. But there is another, relativistically invariant interpretation of S^2, namely as the space of directions of light rays. We will start with this second interpretation.

2.1 S^2 as the projective light cone

Affine transformations form a natural group of transformations acting on the plane. What is the natural group of transformations acting on the two-sphere? One would think it is the rotation group $O(3)$. But rotations are volume-preserving and would not mimic contractions. The next candidate in line is the Lorentz group $O(3,1)$. It is not so well known that the Lorentz group acts on a manifold that is diffeomorphic to the sphere S^2 in a natural way. One way to see that this is the case is to notice that the Lorentz group is the group preserving the space-time metric $s^2 = -x_0^2 + x_1^2 + x_2^2 + x_3^2$ and thus the light cone $C = \{x = (x^0, x^1, x^2, x^3) : -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 0\}$. Therefore, because it acts by linear transformations, it also preserves the projective light cone $\mathbb{P}C$, that is the set of equivalence classes $\hat{x} : x \in C$, with respect to the equivalence relation $R \subset (C \setminus \{0\}) \times (C \setminus \{0\})$ where xRy iff $x = \lambda y, \lambda \neq 0$. Each equivalence class has a unique representative with $x^0 = 1$, so $\mathbb{P}C$ can be identified with the sphere $S^2 = \{\mathbf{n} \in \mathbb{R}^3 : \mathbf{n}^2 = 1\}$. The Lorentz group $O(3,1)$ consists of 4×4 real matrices $\Lambda = (\Lambda^\mu{}_\nu)$ satisfying $\Lambda^T \eta \Lambda = \eta$, where $\eta = (\eta_{\mu\nu}) = diag(-1,1,1,1)$ is the diagonal metric matrix. The action $S^2 \ni \mathbf{n} \mapsto \Lambda(\mathbf{n})$ of $O(3,1)$ on S^2 is given explicitly by the formula:

$$\Lambda(\mathbf{n})^i = \frac{\Lambda^i{}_0 + \Lambda^i{}_j n^j}{\Lambda^0{}_0 + \Lambda^0{}_j n^j}, \tag{3}$$

where we use Einstein's summation convention over repeated indices. The group $O(3,1)$ has four connected components. We will need only the connected component of the identity $SO_+(3,1)$ consisting of those matrices Λ in $O(3,1)$ for which $\det(\Lambda) = 1$ and $\Lambda^0{}_0 > 0$. The group $SO_+(3,1)$, though connected, is not simply connected. Its simply connected double covering group is the group $SL(2,\mathbb{C})$ of 2×2 complex matrices of determinant 1. By polar decomposition, every matrix $A \in SL(2,\mathbb{C})$ can be uniquely decomposed as $A = PU$, into a positive part P and a unitary part $U \in SU(2)$.[†] The group $SU(2)$ is the double covering of the rotation group $SO(3)$. Nontrivial positive matrices in $SL(2,\mathbb{C})$ have two eigenvalues $\lambda_1 < 1$ and $\lambda_2 = 1/\lambda_1 > 1$. It is the positive matrices in $SL(2,\mathbb{C})$ that will generate our iterated function systems.

To describe the $2:1$ group homomorphism $A \mapsto \Lambda(A)$ from $SL(2,\mathbb{C})$ to $SO_+(3,1)$, and also to describe algebraically the action of $SL(2,\mathbb{C})$ on S^2, it is convenient to use the Pauli spin matrices σ_0, σ_1, σ_2, σ_3 defined by

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[†] In Relativity the positive matrices represent "Lorentz boosts."

The homomorphism $SL(2,\mathbb{C}) \to SO_+(3,1)$ is then given by the formula:

$$\Lambda(A)^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu A \sigma_\nu A^\star), \tag{4}$$

where $A^\star$ denotes the Hermitian conjugate of A. Every Hermitian 2×2 matrix X can be uniquely represented as $X = x^\mu \sigma_\mu$, with x^μ real. For every $\epsilon \in [0,1]$ and every unit length vector $\mathbf{n} \in S^2$, let

$$P(\mathbf{n},\epsilon) = \frac{1}{2}(I + \epsilon\sigma(\mathbf{n})), \tag{5}$$

where $\sigma(\mathbf{n}) \doteq n^1\sigma_1 + n^2\sigma_2 + n^3\sigma^3$. It is easy to see that a Hermitian matrix $X \neq I$ is idempotent if and only if it is of the form $X = P(\mathbf{n},1)$ for some $\mathbf{n} \in S^2$. We will write $P(\mathbf{n}) \doteq P(\mathbf{n},1)$. It is also easy to check that a matrix P is positive if and only if it is of the form $P = c\ P(\mathbf{n},\epsilon)$ for some $c > 0,\ \epsilon \in [0,1],\ \mathbf{n} \in S^2$. Notice that $\det(P) = 1$ if and only if $\epsilon < 1$ and $c = 2(1-\epsilon^2)^{-1/2}$. We will use the matrices $P(\mathbf{n},\epsilon)$, with the same ϵ but different vectors $\mathbf{n}$ to generate IFSs on S^2. The formula (3) describing the action of the Lorentz group on S^2 is not the most convenient one for our needs. Another way of describing the same action is by noticing that, for $\mathbf{r} \in S^2$, the following identity holds[‡]

$$P(\mathbf{n},\epsilon)P(\mathbf{r})P(\mathbf{n},\epsilon) = \lambda(\epsilon,\mathbf{n},\mathbf{r})P(\mathbf{r}'), \tag{6}$$

where $\lambda(\epsilon,\mathbf{n},\mathbf{r}) \geq 0$ is given by:

$$\lambda(\epsilon,\mathbf{n},\mathbf{r}) = \frac{1+\epsilon^2+2\epsilon(\mathbf{n}\cdot\mathbf{r})}{4}, \tag{7}$$

while

$$S^2 \ni \mathbf{r}' = \frac{(1-\epsilon^2)\mathbf{r} + 2\epsilon(1+\epsilon(\mathbf{n}\cdot\mathbf{r}))\mathbf{n}}{1+\epsilon^2+2\epsilon(\mathbf{n}\cdot\mathbf{r})} \tag{8}$$

where $(\mathbf{n}\cdot\mathbf{r})$ denotes the scalar product

$$\mathbf{n}\cdot\mathbf{r} = n_1r_1 + n_2r_2 + n_3r_3. \tag{9}$$

[§] The map $\mathbf{r} \mapsto \mathbf{r}'$ is the same as the one described in Eq. (3), with $\Lambda = \Lambda\left(2P(\mathbf{n},\epsilon)/\sqrt{1-\epsilon^2}\right)$. Notice that the dilation coefficient $2/\sqrt{1-\epsilon^2}$ is not important here because it would cancel out anyway in Eq.(3). The transformation $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ implemented by Λ can be explicitly described by the formula known from texts on special relativity:

$$\begin{aligned} x^{0\prime} &= \cosh(\alpha)x^0 + \sinh(\alpha)(\mathbf{x}\cdot\mathbf{n}), \\ \mathbf{x}' &= \mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\mathbf{n} + [\sinh(\alpha)x^0 + \cosh(\alpha)(\mathbf{x}\cdot\mathbf{n})]\mathbf{n}, \end{aligned} \tag{10}$$

where the "velocity" $\beta = \tanh(\alpha) = 2\epsilon/(1+\epsilon^2)$.[¶] What is important for us is the fact that the coefficient $\lambda(\epsilon,\mathbf{n},\mathbf{r})$ in Eq.(6) is positive, and thus can be interpreted as a

[‡] A more general formula is discussed in Sec. 5.4, Eq. (23).

[§] Because of the property $\lambda(\epsilon,\mathbf{n},\mathbf{r}) \geq 0$, λ will later be interpreted as the probability of a jump— cf. Eq. (12) and Sec. 5.4. This interpretation is natural within quantum measurement theory, but its physical meaning within the framework of light directions and Lorentz boosts is not clear.

[¶] Notice that for $\epsilon \to 1$, $\beta \to 1$ – thevelocityoflight. In this limit, the maps $\mathbf{r} \mapsto \mathbf{r}'$ degenerate to $\mathbf{r} \mapsto \mathbf{n}$ and become non-invertible.

(relative) probability associated with the transformation $\mathbf{r} \mapsto \mathbf{r}'$. In other words, relative probabilities associated with maps implemented by $P(\mathbf{n},\epsilon)$ are naturally associated with the maps. It should be noticed that positivity of λ is guaranteed by the algebraic properties of the operators involved. Indeed, because $P(\mathbf{n},\epsilon) = P(\mathbf{n},\epsilon)^\star$, and because $P(\mathbf{r}) = P(\mathbf{r})^\star = P(\mathbf{r})^2$ is an orthogonal projection, the right hand side in Eq.(6) can be represented as $A^\star A$, with $A = P(\mathbf{r})P(\mathbf{n},\epsilon)$, and is, therefore, automatically positive.

3 Quantum Iterated Function Systems on S^2

Given a sequence $\mathbf{n}_i$, $i = 1, \ldots, N$ of vectors in S^2, we associate with this sequence an iterated function system $\{w_i, p_i\}$ on S^2 with place-dependent probabilities, defined as follows:

$$w_i(\mathbf{r}) = \mathbf{r}' = \frac{(1-\epsilon^2)\mathbf{r} + 2\epsilon(1+\epsilon(\mathbf{n}_i\cdot\mathbf{r}))\mathbf{n}_i}{1+\epsilon^2+2\epsilon(\mathbf{n}_i\cdot\mathbf{r})}, \tag{11}$$

$$p_i(\mathbf{r}) = \frac{\lambda(\epsilon,\mathbf{n}_i,\mathbf{r})}{\sum_{j=1}^N \lambda(\epsilon,\mathbf{n}_j,\mathbf{r})}. \tag{12}$$

The system $\{w_i, p_i\}$ defined by these formulae will be called a *Quantum Iterated Function System* or *QIFS* . ‖ The formula for the probabilities simplifies whenever

$$\sum_{i=1}^N \mathbf{n}_i = 0. \tag{13}$$

In the following, we will always assume that the vectors $\mathbf{n}_i$ defining the transformations w_i add to zero. In this case, the p_i are given by:

$$p_i(\mathbf{r}) = \frac{1+\epsilon^2+2\epsilon(\mathbf{n}_i\cdot\mathbf{r})}{N(1+\epsilon^2)}. \tag{14}$$

In [3] we examined QIFSs corresponding to several most symmetric configurations, where the vectors $\mathbf{n}_i$ were placed at the vertices of regular polyhedra: the tetrahedron (4), octahedron (6), cube (8), icosahedron (12), dodecahedron (20), double tetrahedron (8), and icosidodecahedron (30). In each case, numerical simulation of the Markov process, starting with a random original point, led to fractal-like patterns on the sphere. For ϵ close to 1, the operators $P(\mathbf{n},\epsilon)$ are close to projections; therefore the attraction centers are very distinctive. For ϵ close to 0, the operators $P(\mathbf{n},\epsilon)$ induce maps close to the identity map—the patterns are fuzzy. Typical patterns are shown in Fig. 2.

It seems that the fractal dimension depends on the value of ϵ. The Hausdorff dimension of the limit set for the tetrahedral case has been numerically estimated in Ref. [4] and shown to decrease from 1.44 to 0.49 while ϵ increases from 0.75 to 0.95.

‖ A short justification for the term "quantum" will be given in the closing section of this paper.

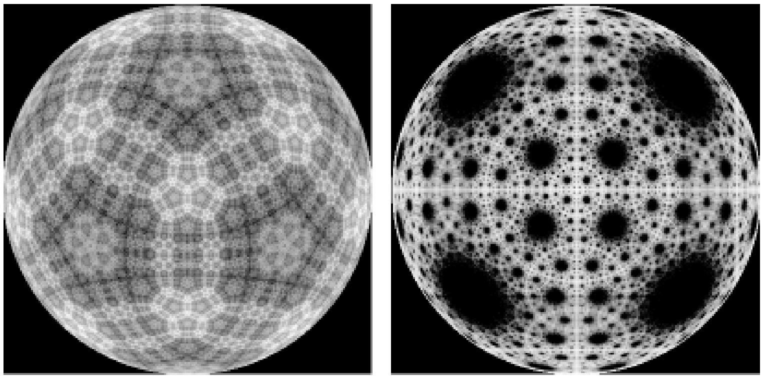

Fig. 2 Quantum Dodecahedron ($\epsilon = 0.78$) and Quantum Octahedron ($\epsilon = 0.58$) The darker the place, the smaller the probability of it being visited.

4 Transfer (Markov) Operator and Invariant Measure

Stenflo [5] gives a brief review of the problem of the existence and uniqueness of the invariant measures which is quite useful in our case. We will follow the notation and the terminology of [5]. The transfer operator T for the system is defined by the formula:

$$(Tf)(\mathbf{r}) = \sum_{i=1}^{N} p_i(\mathbf{r}) f(w_i(\mathbf{r})), \tag{15}$$

where $f \in C(S^2)$—the space of all continuous functions on S^2. By the Riesz Representation Theorem, T induces the dual operator $T^\star : \mu \mapsto T^\star\mu$ on the space $M(S^2)$ of Borel probability measures on S^2 via the formula:

$$\int_{S^2} (Tf)(\mathbf{r})\, d\mu(\mathbf{r}) = \int_{S^2} f(\mathbf{r})\, d(T^\star\mu)(\mathbf{r}).$$

Since S^2 is compact, there always exists an invariant probability measure μ that is invariant, i.e. $T^\star\mu = \mu$. Numerical simulations of QIFSs seem to indicate that such a measure is also unique, and that it is concentrated on a unique attractor set, though different for different $\epsilon \in (0, 1)$. As each of the normalized operators $2P(\mathbf{n}_i, \epsilon)/\sqrt{1-\epsilon^2} \in SL(2, \mathbb{C})$ has two eigenvalues, $(1+\epsilon)/(1-\epsilon)$ and $(1-\epsilon)/(1+\epsilon)$, the standard contraction arguments do not apply. Nevertheless we have the following theorem:

Theorem 4.1. For the Quantum Octahedron, the invariant measure is unique in the whole parameter range $0 < \epsilon < 1$.

Proof. In Ref. [5], Stenflo states the following classical results, attributed to Barnsley, et al. [6]: *Let* $\{(X, d), p_i(x), w_i(x), i \in S = \{1, 2, \ldots, N\}\}$ *be an IFS with place-dependent probabilities, with all* w_i *being Lipshitz continuous, and all* p_i *being Dini-continuous, and bounded away from* 0. *Suppose*

$$\sup_{x \neq y} \sum_{i=1}^{N} p_i(x) \log\left(\frac{d\,(w_i(x), w_i(y))}{d(x, y)}\right) < 0. \tag{16}$$

Then the generated Markov chain has a unique invariant probability measure.

The log-average contraction condition (16) is somewhat more general than the average contraction condition

$$Max(w) \doteq \sup_{x \neq y} \sum_{i=1}^{N} p_i(x) \frac{d\left(w_i(x), w_i(y)\right)}{d(x,y)} < 1. \tag{17}$$

In our case, $w_i(x)$ and $p_i(x)$ are analytic, with $p_i(\mathbf{r}) \geq \frac{1-\epsilon^2}{N(1+\epsilon^2)}$ We made a numerical estimation of the LHS of the inequality (17), with d being the natural, rotation-invariant, *arc* distance on S^2, for the Quantum Octahedron, with $0 < \epsilon < 1$, and obtained the epsilon dependence shown in Fig. 3, thus assuring the uniqueness of the invariant measure

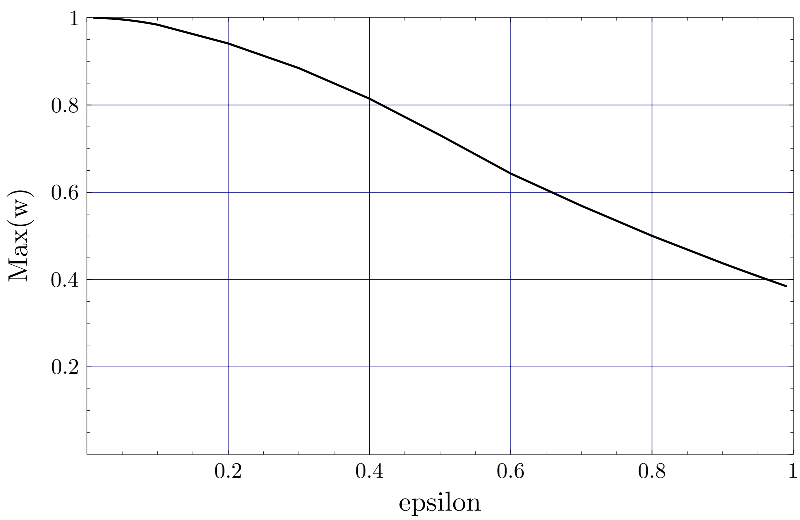

Fig. 3 ϵ–dependence of the average contraction parameter $Max(w)$ for the quantum Octahedron.

in this particular case.□

Let μ_0 be the natural, rotation-invariant, normalized measure on S^2. Then, for any finite n, the measure $T^{\star n}\mu_0$ is continuous with respect to μ_0 and therefore can be written as

$$T^{\star n}\mu_0(\mathbf{r}) = f_n(\mathbf{r})\mu_0(\mathbf{r}).$$

The sequence of functions $f_n(\mathbf{r})$ gives a convenient graphic representation of the limiting invariant measure. In our case, the functions f_n can be computed explicitly via the following recurrence formula:

$$f_{n+1}(\mathbf{r}) = \sum_{i=1}^{n} p_i\left(w_i^{-1}(\mathbf{r})\right) \frac{d\mu_0\left(w_i^{-1}(\mathbf{r})\right)}{d\mu_0(\mathbf{r})} f_n\left(w_i^{-1}(\mathbf{r})\right) \tag{18}$$

or, explicitly:

$$f_{n+1}(\mathbf{r}) = \frac{(1-\epsilon^2)^4}{N(1+\epsilon^2)} \sum_{i=1}^{N} \frac{f_n\left(w_i^{-1}(\mathbf{r})\right)}{(1+\epsilon^2 - 2\epsilon\mathbf{n}_i \cdot \mathbf{r})^3} \tag{19}$$

where

$$w_i^{-1}(\mathbf{r}) = \frac{(1-\epsilon^2)\mathbf{r} - 2\epsilon(1-\epsilon\mathbf{n}_i \cdot \mathbf{r})\mathbf{n}_i}{1+\epsilon^2 - 2\epsilon\mathbf{n}_i \cdot \mathbf{r}}. \tag{20}$$

Fig. 4 shows a plot of $\log(f_5(\mathbf{r})+1)$ for the Quantum Octahedron, $\epsilon = 0.58$, using the stereographic projection $\mathbf{n} \mapsto z = \frac{n^1 - in^2}{1-n^3}$ from S^2 to the complex plane. It should be noted that via the stereographic projection, the maps $\mathbf{r} \mapsto w_i(\mathbf{r})$ become fractional and thus conformal transformations of the complex plane: $z \mapsto w_i(z) = \frac{az+b}{cz+d}$, with $a = 1 + \epsilon n_i^3$, $b = \epsilon(n_i^1 - \mathrm{i}n_i^2)$, $c = \epsilon(n_i^1 + \mathrm{i}n_i^2)$, $d = 1 - \epsilon n_i^3$.

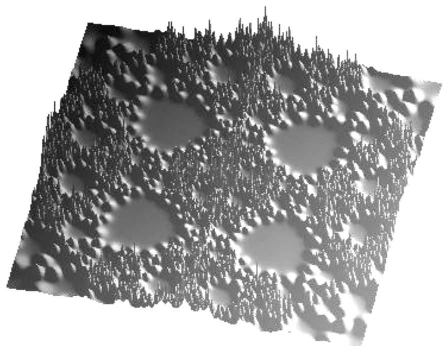

Fig. 4 An approximation of the invariant measure: Plot of $f_5(\mathbf{r})$ for Quantum Octahedron ($\epsilon = 0.58$).

5 Concluding Remarks

In this section we will place QIFSs within a larger field of piecewise-deterministic Markov processes and their connection to dissipative dynamics of mixed quanto-classical dynamical systems.

5.1 Classical dynamics

Usually, classical dynamics is described by a 1-parameter group ϕ_t of diffeomorphisms of a smooth manifold X. In classical mechanics, X is a symplectic manifold, the "phase space" of the system, and the flow ϕ_t is generated by a Hamiltonian vector field on X. States of the system are simply points of X, and statistical states are probabilistic measures on X. The set of all statistical states is convex—its extremal elements are called pure states. These are Dirac measures—concentrated at points of X. The flow ϕ_t on X gives rise to a flow on the space of "observables," that is functions on M, and to a flow on the space of "statistical states," that is on the space $M(X)$ of probabilistic measures on X. If X is discrete, then we cannot have a continuous flow on X, but we can still have a continuous family of transformations acting on observables and on statistical states.

5.2 Quantum dynamics

Quantum theory is usually formulated in terms of linear operators acting on a separable complex Hilbert space $\mathfrak{H}$. Observables are represented by Hermitian elements of the algebra $\mathfrak{A} = L(\mathfrak{H})$ of all bounded linear operators on $\mathfrak{H}$. Statistical states are positive, normalized, ultra-weakly continuous functionals on $\mathfrak{A}$. They are represented by positive trace-class operators ρ, $\mathrm{Tr}\,(\rho) = 1$, with $\rho(A) \doteq \mathrm{Tr}\,(\rho A)$, $A \in \mathfrak{A}$. Pure states are represented by ρ of the form $\rho = P$, where P is an orthogonal projection onto a 1-dimensional subspace of $\mathfrak{H}$. The space of pure states can thus be identified with the space of 1-dimensional subspaces of $\mathfrak{H}$. If $\mathfrak{H}$ is finite-dimensional, $\mathfrak{H} \approx \mathbb{C}^n$, then the space of pure states is the complex projective plane CP^{n-1}. Quantum dynamics is usually described in terms of a 1-parameter group of unitary operators $U(t) : t \in \mathbb{R}$. It acts on observables via the automorphism $\alpha_t : A \mapsto U(t)^{-1} A U(t)$.

5.3 Mixed quanto-classical dynamics

We will consider the simple case, where the classical system is finite $X = \{1, \ldots, N\}$. For each $\alpha \in X$, consider the Hilbert space $\mathfrak{H}_\alpha = \mathbb{C}^{n_\alpha}$ and let $M(n_\alpha)$ be the algebra of $n_\alpha \times n_\alpha$ complex matrices.** The observables of the coupled system are now functions $\alpha \mapsto A_\alpha \in M(n_\alpha)$ on X with values in $M(n_\alpha)$. A pure state of the system is a pair (α, P), where $\alpha \in \{1, \ldots, N\}$ and P is a Hermitian projection matrix onto a one-dimensional subspace in $\mathbb{C}^{n_\alpha}$. It is not possible to couple the classical and the quantum degrees of freedom via reversible, unitary dynamics. A 1-parameter semi-group of completely positive maps of the algebra $\mathfrak{A} = \oplus_{\alpha=1}^{N} M(n_\alpha)$ is used instead. We are interested in semigroups with generators of Lindblad's type (also known as "dynamical semigroups"), in particular with generators of the form:

$$L(A)_\alpha = \mathrm{i}[H_\alpha, A_\alpha] + \sum_{\beta \neq \alpha} g^\star_{\beta\alpha} A_\beta \, g_{\beta\alpha} - \frac{1}{2}(\Lambda_\alpha A_\alpha + A_\alpha \Lambda_\alpha), \tag{21}$$

where $g_{\beta\alpha} \in L(\mathfrak{H}_\alpha, \mathfrak{H}_\beta)$ and

$$\Lambda_\alpha = \sum_{\beta \neq \alpha} g^\star_{\beta\alpha} g_{\beta\alpha} \in L(\mathfrak{H}_\alpha). \tag{22}$$

We always assume that the diagonal terms vanish: $g_{\alpha\alpha} = 0$. It has been shown in [7] that there is a one-to-one correspondence between semigroups with generators of the above type and piecewise-deterministic Markov processes on the space of pure states of the system.

** In all examples studied so far, the dimensions n_α were the same for all α. But such a restriction is not necessary.

5.4 From dynamical semigroups to QIFSs

Here we are not concerned with the continuous time evolution between jumps, so let us extract from Ref. [7], and also slightly reformulate, the jump process alone. It is determined by the operators $g_{\alpha\beta}$ only, and it is an iterated function system, with place-dependent probabilities that are also determined by the $g_{\alpha\beta}$. Let (α, P) be a pure state, with P an orthogonal projection on a unit vector $\psi \in \mathfrak{H}_\alpha$. Observe that for each $\beta \neq \alpha$ we have:

$$g_{\beta\alpha} P g^\star_{\beta\alpha} = \lambda(\alpha, \beta; P) Q \tag{23}$$

where

$$\lambda(\alpha, \beta; P) = \|g_{\beta\alpha}\psi\|^2 \geq 0 \tag{24}$$

and, if $\lambda(\alpha, \beta; P) > 0$, then Q is a projection operator on the vector $g_{\beta\alpha}\psi / \|g_{\beta\alpha}\psi\|$ in $\mathfrak{H}_\beta$. The probabilities $p(\alpha, \beta; P)$ are defined as

$$p(\alpha, \beta; P) = \frac{\lambda(\alpha, \beta; P)}{\sum_{\beta\neq\alpha} \lambda(\alpha, \beta; P)}. \tag{25}$$

Assume now that all Hilbert spaces $\mathfrak{H}_\alpha \equiv \mathfrak{H}$ are identical. Assume that $X = 2^N$ – the set of N bits, and that $g_{\alpha\beta} = g_i \neq 0$ when α differs from β only at the i-th bit, and otherwise $g_{\alpha\beta} = 0$. We will just have a family of operators g_i and a jump process on pure states P—that is one-dimensional orthogonal projections in $\mathfrak{H}$. The maps and their probabilities are determined by:

$$g_i P g_i^\star = \lambda(i; P) Q, \tag{26}$$

with $\lambda(i; P) = \|g_i\psi\|^2$, $p_i(P) = \lambda(i; P) / \sum_j \lambda(j, P)$, and Q the orthogonal projection on the subspace spanned by the vector $g_i\psi$. We have an iterated function system on the complex projective space $CP(n-1)$ (equivalently, on the grassmannian of one-dimensional subspaces of $\mathbb{C}^n$), with place-dependent probabilities. Both maps and probabilities are determined by the set of linear operators $g_i, \quad i \in \{1, 2, \ldots, N\}$.

5.5 A short history of QIFSs

The idea of coupling a classical and a quantum system via a dynamical semigroup was originally described in [8]. In the first model of a QIFS on S^2, with index α also continuous, $\alpha = \mathbf{n}$ with values in S^2 and, using the notation of Sec. 2.1, $g_\mathbf{n}$ was defined as $g_\mathbf{n} = \exp(i\pi\sigma(\mathbf{n}))$. These maps were unitary, thus measure-preserving, and did not give rise to a fractal attractor. The tetrahedral model was first introduced in [10]. It was then examined analytically and modelled numerically in a Ph.D. thesis by G. Jastrzebski [4]. The model was further exploited in [11], where it was described in some detail, and where the Lyapunov exponent of the semigroup generator was computed. The term QIFS was introduced about that time on the sci.physics.research newsgroup on the Internet. Recently, the term QIFS wasadopted in [12] for another class of maps, namely for maps on the space of all statistical states of a quantum system, with arbitrarily assigned probabilities.

5.6 Some open problems

The question of uniqueness of the invariant measure for a general QIFS is an open technical problem. For a particular case of the Quantum Octahedron, we used numerical estimations of the average contraction parameter. We do not have an analytical proof even in this particular case. Then there is a "philosophical" question: can the fractal patterns derived from the QIFS algorithm be "observed"? Or are they purely mathematical constructs that have no relation to the "real world"? The question is not an easy one to answer, because QIFSs live in the projective Hilbert space of pure states of a quantum system, not in "our space". The question of whether they can be observed is related to the question: can the wave function be observed? A preliminary discussion of this problem has been given in Ref. [9]. We hope to return to this question in our future publications.

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