

## PIECEWISE DETERMINISTIC QUANTUM DYNAMICS AND QUANTUM FRACTALS ON THE POINCARÉ DISK

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It is shown that piecewise deterministic dissipative quantum dynamics in a vector space with indefinite metric can lead to well defined, positive probabilities. The case of quantum jumps on the Poincaré disk is studied in detail including results of numerical simulations of quantum fractals.

**Keywords:** iterated function systems, dynamical semigroups, EEQT, indefinite metric, quantum jumps, quantum measurement, non-commuting observables, fractals, Poincaré disk.

### 1. Introduction

One of the most important characteristic features of a quantum theory, as opposite to a pure classical theory, is that the set of pure states of a quantum system is not a simplex. Consider a quantum system described in a complex Hilbert space  $\mathcal{H}$ . Its pure states are described by rays  $\{e^{i\phi}\psi : \phi \in [0, 2\pi]\} \subset \mathcal{H}$ ,  $\|\psi\| = 1$  or, equivalently, by one-dimensional orthogonal projections  $P = |\psi\rangle\langle\psi|$ . Let  $\mathcal{S}$  be the set of all pure states. If  $\mu$  is a probabilistic measure on  $\mathcal{S}$  then, according to the standard, linear, quantum mechanics, only the density matrix  $\rho = \int_{\mathcal{S}} P d\mu(P)$  is observable, not the measure  $\mu$  itself. And yet, as demonstrated in [1], while searching for the dynamical mechanism of particle track creation and event generation, the measurement process introduces a mild non-linearity and leads to a unique random process on  $\mathcal{S}$ —provided the quantum system is coupled, in an appropriate way, to a classical system.<sup>1</sup> This mild non-linearity is related to the “time-of-events” observable studied in [4], and calls for a careful re-examination of the fundamental axioms of quantum mechanism. In particular, if the Event Enhanced Quantum Theory (EEQT) developed by Blanchard and the present author will prove to be correct, then precise measurements of timing of events may give us more information about the actual state of the quantum system than it is usually assumed.

The present paper, although stemming from the above ideas, deals with a pure formal aspect of EEQT—namely with a piecewise deterministic process (PDP) of

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<sup>1</sup>A similar approach has recently been discussed by A. Peres [2, 3].

jumps (sometimes also called *state reductions*) on the space of pure states in a space with indefinite metric. Spaces with indefinite metric are usually considered as too pathological for applications in quantum theory [5]. And yet within the formalism of EEQT, where none of the standard probabilistic axioms of quantum theory are required, and that because all the interpretation comes from a dynamical coupling between a classical and a quantum systems, it is possible to have a well-defined dynamics with all positive probabilities. The main goal of the present paper is to show, using the simplest possible model, that a continuous monitoring of several noncommuting observables leads to a piecewise deterministic random process, with positive transition probabilities, and with a fractal attractor set, also in the case of indefinite metric, when the state space is noncompact.

In the present paper we will follow some of the ideas and notation developed in [6], though we will make an effort at presenting this paper as self-contained.

We will start with a linear Linblad's type master equation for observables in a Krein space, with a particular kind of coupling between a quantum system and a finite-state classical system—cf. Eq. (6). We skip the possible unitary Hamiltonian part of the evolution, and we concentrate on the dissipative part alone. By Theorem 2 in Section 3 the coupling generates a unique, piecewise deterministic process on the space of pure states of the total system. Next, we farther specify the coupling in such a way that we can take the trace over the classical system and get the effective piecewise deterministic process on the hyperboloid of pure states (of positive norm squared) of the quantum system alone—cf. PDP Algorithm in Section 3. We then examine in detail the particular case of Krein's space  $V_{1,1}$  when the space of pure states is isomorphic to the Poincaré disk.

We specify the coupling operators to be fuzzy projections as defined in Eq. (25), on a family of symmetrically distributed states on the disk (a selection of 33 points from the hyperbolic tiling with Schlafli symbol (3,8)—cf. Fig. 1). The plot of the sequence of 100 mln of quantum jumps generated by the PDP process suggests that there is a fractal attractor set.

It should be mentioned that, according to the linear paradigm of quantum mechanics, mentioned at the beginning of this section, the patterns displayed in the figures, although mathematically unique, are physically unobservable. According to this paradigm, the only observable is the statistical matrix resulting from the integration of projections with respect to the measure. In our case, due to symmetry of the pattern, it is clear that this statistical state is a mixture of the maximal entropy state  $\frac{1}{2}I$ , and the orthogonal projection on the eigenstate of  $\sigma_3$  corresponding to the eigenvalue 1. It is worthwhile to notice that Lozinski et al. [7] recently studied iterated function systems on the space of statistical operators of a given quantum system. Their approach, even though different in spirit, has several areas of overlap with piecewise deterministic processes generated by EEQT.

## 2. Indefinite metric space

Let  $V$  be a complex vector space of finite dimension  $n$  with a nondegenerate sesquilinear form  $\langle \psi, \phi \rangle$ .<sup>2</sup> Let  $L(V)$  be the algebra of all linear operators on  $V$  endowed with the star operation defined by  $\langle A\psi, \phi \rangle = \langle \psi, A^*v \rangle$ .  $A \in L(V)$  is said to be *symmetric* if  $A = A^*$ .  $U \in L(V)$  is said to be *unitary* if  $U^* = U^{-1}$ . Let  $\mathcal{H}(V)$  and  $\mathcal{U}(V)$  denote the set of all symmetric and all unitary operators resp. Let  $\mathcal{J}(V)$  be the set of all  $J \in \mathcal{H}(V) \cap \mathcal{U}(V)$  such that the sesquilinear form  $(\psi, \phi)_J \doteq \langle \psi, J\phi \rangle$  is positive definite. We will assume that  $\mathcal{J}(V)$  has at least two elements. This is equivalent to assuming that the scalar product  $\langle \psi, \phi \rangle$  is indeed indefinite, that is that there exists a basis  $\{e_1, \dots, e_k, e_{k+1}, \dots, e_{k+l}\}$  such that  $\langle e_\alpha, e_\beta \rangle = g_{\alpha\beta}$  where

$$\{g_{\alpha\beta}\} = \text{diag}(+1, \dots, +1, -1, \dots, -1) \tag{1}$$

with  $k, l \geq 1$ . Whenever we will need to specify the signature of  $V$ , we will use the notation  $V_{(k,l)}$  for  $V$ . There is one-to-one correspondence between elements of  $\mathcal{J}(V)$  and  $k$ -dimensional subspaces of  $V$  on which  $\langle \psi, \phi \rangle$  is positive definite. The group  $\mathcal{U}(V)$  is isomorphic to  $U(k, l)$  and acts transitively on  $\mathcal{J}(V)$  with the fixpoint group isomorphic to  $U(k) \times U(l)$ , thus

$$\mathcal{J}(V) \cong \frac{U(k, l)}{U(k) \times U(l)} = \frac{SU(k, l)}{S(U(k) \times U(l))}. \tag{2}$$

In particular, for  $k = l = 2$  we get the Cartan symmetric domain  $A(2) \equiv D_4 \equiv SO(4, 2)/S(O(4) \times O(2))$ , with Shilov boundary isomorphic to the compactified Minkowski space, while for  $k = l = 1$  we get the Poincaré disk  $A(1) \equiv D_1 \equiv SO(1, 2)/SO(2)$  whose Shilov boundary is a circle, with a physical interpretation of it being a compactified, circular, time.

Given  $J \in \mathcal{J}(V)$  we have positive-definite scalar product  $(\psi, \phi)_J \doteq \langle \psi, J\phi \rangle$ . Let us denote by  $A^J$  the Hermitian conjugation with respect to this scalar product. We call  $A \in L(V)$   $J$ -Hermitian if  $A = A^J$ . It is then easy to see that  $A \mapsto JA$  gives a one-to-one correspondence between symmetric and  $J$ -Hermitian operators.

DEFINITION 1. We will use the following notation:

$$V_+ = \{\psi \in V : \langle \psi, \psi \rangle \geq 0\}, \tag{3}$$

$$\mathcal{L}_+(V) = \{A \in L(V) : AV_+ \subset V_+\}. \tag{4}$$

For  $\psi \in V_+$  we will write  $\|\psi\|$  for  $\sqrt{\langle \psi, \psi \rangle}$ .

Notice that  $A \in \mathcal{L}_+(V)$  if and only if  $\|A\psi\|^2 \geq 0$  for all  $\psi \in V_+$ . Each  $J \in \mathcal{J}(V)$  is a difference of two orthogonal, complementary projections:

$$J = E - F = 2E - I, \tag{5}$$

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<sup>2</sup>It should be noted that the term "Hilbert space" is usually reserved for a space with a positive definite scalar product. The term "Krein space" though less known, is more appropriate here.

where  $E = E^* = E^2$  projects on a  $k$ -dimensional subspace on which  $\langle \psi, \phi \rangle$  is positive definite, and  $F = F^* = I - E$  projects on the complementary  $l$ -dimensional subspace, where  $\langle \psi, \phi \rangle$  is negative definite.

### 3. Linblad's type semigroups and associated piecewise deterministic processes in $V_{(k,l)}$

It is well known that certain types of quantum dynamical semigroups [8, 9] are in one-to-one correspondence with piecewise deterministic Markov processes on the space of pure states of the algebra of observables of the system—cf. [10, 11] and references therein). By analyzing the proof of the existence and uniqueness theorems it is easy to see that these results extend to  $V_{k,l}$  provided the operators that implement the jumps are in  $\mathcal{L}_+(V)$ .

**THEOREM 2.** *Let  $S$  be a finite set. Let  $V = V_{k,l}$ , and let  $\mathcal{A}$  be the  $\star$ -algebra  $\mathcal{A} = \bigoplus_{\alpha \in S} \mathcal{L}(V)$ . A typical element of  $\mathcal{A}$  is a sequence  $A = (A_\alpha)_{\alpha \in S}$ ,  $A_\alpha \in \mathcal{L}(V)$ . For each pair  $\alpha, \beta \in S$ ,  $\alpha \neq \beta$ , let there be given an operator  $g_{\alpha\beta} \in \mathcal{L}_+(V)$ . Consider semigroup of linear maps  $\phi_t(A)_\alpha \doteq \exp(Lt) : \mathcal{A} \rightarrow \mathcal{A}$ ,  $t \geq 0$ , with infinitesimal generator  $L$  defined by*

$$L(A)_\alpha = \sum_{\beta \neq \alpha} g_{\beta\alpha}^* A_\beta g_{\beta\alpha} - \frac{1}{2} (\Lambda_\alpha A_\alpha + A_\alpha \Lambda_\alpha), \quad (6)$$

where

$$\Lambda_\alpha = \sum_{\beta \neq \alpha} g_{\beta\alpha}^* g_{\beta\alpha}. \quad (7)$$

The formula

$$\text{tr}(P\phi_t(A)_\alpha) = \sum_{\beta} \int \text{tr}(AQ) dp(t; P, \alpha; Q, \beta) \quad (8)$$

gives then one-to-one correspondence between semigroups  $\phi_t$  and piecewise deterministic Markov processes on  $\mathcal{P}_+ \times S$  with transition probabilities  $p(t; \alpha, P; \beta, dQ)$  described by the following algorithm:

Suppose that at time  $t_0$  the system is described by a pair  $(P \in \mathcal{P}_+, \alpha \in S)$ , with  $P$  being an orthogonal projection on a unit vector  $\psi \in V_+$ . Choose a uniform random number  $p \in [0, 1]$ , and proceed with the continuous time evolution by solving the differential equation

$$\dot{\psi}_t = \exp\left(-\frac{1}{2} \Lambda_\alpha\right) \psi_t \quad (9)$$

with the initial vector  $\psi$  until  $t = t_1$ , where  $t_1$  is determined by

$$\|\psi_{t_1}\|^2 = p. \quad (10)$$

Then jump. When jumping, change  $\alpha \rightarrow \beta$  with probability

$$p_{\alpha \rightarrow \beta} = \|g_{\beta\alpha} \psi_{t_1}\|^2 / (\psi_{t_1}, \Lambda_\alpha \psi_{t_1}) \quad (11)$$

and, if  $p_{\alpha \rightarrow \beta} \neq 0$ , change

$$\psi_{t_1} \rightarrow \psi_1 = g_{\beta\alpha} \psi_{t_1} / \|g_{\beta\alpha} \psi_{t_1}\|.$$

Repeat the steps substituting  $t_1, \psi_1, \beta$  for  $t_0, \psi_0, \alpha$ .

*Proof:* Algebraically, the proof goes, step by step, exactly the same way as in [10, 11] (cf. also [12] for the uniqueness), the only thing that we need to check is that positivity conditions required in the Hilbert space case also hold when the scalar product is indefinite. If  $f(t) \doteq \|\psi_t\|^2$ , then the function  $f(t)$  is real analytic, with  $f(0) = 1$  and  $\dot{f}(t) = -\langle \psi_t, \Lambda_\alpha \psi_t \rangle$ . As long as  $f(t) > 0$ , that is as long as  $\psi_t \in V_+$ , the derivative  $\dot{f}(t)$  is nonpositive, because  $\langle \psi_t, \Lambda_\alpha \psi_t \rangle = \sum_{\beta \neq \alpha} \|g_{\alpha\beta} \psi_t\|^2$ , and  $g_{\alpha\beta} \in \mathcal{L}_+(V)$ . Therefore  $f$  is monotonically decreasing from  $f(0) = 1$  and either it never reaches the given value  $p \in (0, 1)$ , so the jump never happens, or, if it reaches the value  $p$  at a finite time  $t_1$ , then  $\psi_{t_1} \in V_+$  and therefore, because  $g_{\alpha\beta} \in \mathcal{L}_+(V)$ , the jump probabilities  $p_{\alpha \rightarrow \beta}$  are non-negative. Finally, notice that whenever the denominator in Eq. (11) vanishes, this can happen only for discrete values of  $t_1$ , thus on a set of measure zero.  $\square$

Here, as in [14] we will be interested in a special case of the above process that leads to simple iterated function systems with place-dependent probabilities, as discussed in [13]. Let  $N$  be a natural number, and let  $\mathcal{S} = 2^N$  be the set of all binary sequences of length  $N$ . Assume  $\mathcal{L}_+(V) \ni g_{\alpha\beta} = g_i \neq 0$  when  $\alpha$  differs from  $\beta$  only at one, the  $i$ -th bit, otherwise  $g_{\alpha\beta} = 0$ . Let

$$\Lambda = \sum_{i=1}^N g_i^* g_i. \tag{12}$$

If we take a trace over the index  $\alpha$ , we end up with a piecewise deterministic process on  $\mathcal{P}_+$  that can be described as follows.

**PDP ALGORITHM 1.** *Start, at time  $t_0$ , with a unit vector  $\psi \in V_+$ . Choose a uniform random number  $p \in [0, 1]$ , and proceed with the continuous time evolution by solving the differential equation*

$$\dot{\psi}_t = \exp(-\frac{1}{2}\Lambda)\psi_t \tag{13}$$

with the initial vector  $\psi$  until  $t = t_1$ , where  $t_1$  is determined by

$$\|\psi_{t_1}\|^2 = p. \tag{14}$$

Then jump. When jumping, change

$$\psi_{t_1} \rightarrow \psi_1 = g_i \psi_{t_1} / \|g_i \psi_{t_1}\|$$

with probability

$$p_i(\psi) = \|g_i \psi_{t_1}\|^2 / \sum_{j=1}^N \|g_j \psi_{t_1}\|^2. \tag{15}$$

Repeat the steps replacing  $t_0, \psi_0$  with  $t_1, \psi_1$ .

#### 4. Example: $V_{1,1}$ —the Poincaré disk

As a simple example consider the case of  $k = l = 1$ . Let us choose  $J_0 \in \mathcal{J}(V)$  and let  $e_1, e_2$  be an orthonormal basis in  $V$  diagonalizing  $J_0$ . Thus  $\langle e_1, e_2 \rangle = -\langle e_2, e_2 \rangle = +1$ ,  $V$  can be identified with  $\mathbb{C}^2$ —the set of all column vectors

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad (16)$$

where  $a_1$  and  $a_2$  are complex numbers,  $J_0$  has now the matrix form

$$J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (17)$$

while the positive definite scalar product  $(\psi, \phi)_{J_0}$  is nothing but the standard

$$(a, b)_{J_0} = \bar{a}_1 b_1 + \bar{a}_2 b_2, \quad (18)$$

where the bar in  $\bar{c}$  stands for the complex conjugation of the complex number  $c$ . Let us denote by  $A^\dagger$  the Hermitian conjugate matrix of  $A$ . Using our previous notation  $A^\dagger = A^{J_0}$ , the most general  $J_0$ -Hermitian matrix,  $A = A^\dagger$ , is of the form

$$A = x^0 I + \sigma(\mathbf{x}), \quad (19)$$

where

$$\sigma(\mathbf{x}) = x^1 \sigma_1 + x^2 \sigma_2 + x^3 \sigma_3 \quad (20)$$

and  $\sigma_i$  are the Pauli matrices:

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (21)$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (22)$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (23)$$

and  $x^\mu$ ,  $\mu = 0, 1, 2, 3$ , are real numbers. Note that now  $J_0$  is represented by  $\sigma_3$ . It follows that the most general symmetric (with respect to  $\langle \ , \ \rangle$ ) operator on  $V$  is of the form

$$\sigma_3(x^0 + \sigma(\mathbf{x})) = x^3 - ix^2 \sigma_1 + ix^1 \sigma_2 + x^0 \sigma_3, \quad (24)$$

where  $x^\mu$  are real. In other words, moving between symmetric and  $J$ -Hermitian operators is accomplished by the "Wick rotation"  $x^1 \mapsto -ix^2, x^2 \mapsto ix^1$  in the parameters  $x^1, x^2$ .

#### 4.0.1. Fuzzy projections

In our toy model the state of the system is represented by a unit vector  $\phi \in V_+$ . Proportional vectors describe the same state, therefore it is better to represent states by one-dimensional orthogonal projections  $P$  on positive subspaces. The set of all states will be denoted by  $\mathcal{P}_+$ . Quantum jumps will be implemented by operators in  $\mathcal{L}_+(V)$ . In this paper, as in [14], we will be interested in fuzzy projections of the form

$$Q(J, \epsilon) = \frac{1}{2}(I + \epsilon J), \quad (25)$$

which have the above property. Notice that for  $\epsilon = 1$  we have  $Q(J, 1) = \frac{1}{2}(I + (2E_J - I)) = E_J$ , and thus  $Q(J, 1)$  is a sharp projector. Notice that in  $V_{1,1}$  we have  $Q(J, 1) \in \mathcal{P}_+$  because in this case maximal positive subspaces are one-dimensional. We want to know for which values of  $\epsilon$ ,  $Q(J, \epsilon)$  is in  $\mathcal{L}_+(V)$ . For this it is enough to calculate the trace of  $Q(J, 1)Q(J', \epsilon)Q(J, 1)$ , and to find the conditions on  $\epsilon$  which guarantee its positivity for all  $J \in \mathcal{J}(V)$ . Let us use the representation using  $J_0$  and the Pauli matrices as above.

**DEFINITION 3.** For all  $\mathbf{m} = (m_0, m_1, m_2), \mathbf{n} = (n_0, n_1, n_2) \in \mathbb{R}^3$  set  $\langle \mathbf{m} \cdot \mathbf{n} \rangle = m_0n_0 - m_1n_1 - m_2n_2$ . Let  $\mathcal{H} = \{\mathbf{m} = (m_0, m_1, m_2) : \langle \mathbf{m} \cdot \mathbf{m} \rangle = 1, m_0 > 0\}$ .

**LEMMA 4.** Each  $J \in \mathcal{J}(V)$  is uniquely representable in the form

$$J = J(\mathbf{m}) = \hat{\sigma}(\mathbf{m}) = m_0\sigma_3 - im_2\sigma_1 + im_1\sigma_2, \quad (26)$$

where  $\mathbf{m} \in \mathcal{H}$ .

*Proof:*  $J$  must be symmetric, therefore  $J = x^3 - ix^2\sigma_1 + ix^1\sigma_2 + x^0\sigma_3$ , with  $x^\mu$  real.  $J^2 = I$  implies then that  $x^3 = 0$  and  $(x^0)^2 - (x^1)^2 - (x^2)^2 = 1$ . Now,  $\langle \psi, J\psi \rangle = \langle \psi, J_0J\psi \rangle_{J_0}$ , therefore, for  $\langle \psi, J\psi \rangle$  to be non-negative for all  $\psi$ ,  $J_0J = \sigma_3J = x^0 + x^1\sigma_1 + x^2\sigma_2$  must be positive in the standard sense. That means  $x^0 > 0$ .  $\square$

**DEFINITION 5.** We will write  $Q(\mathbf{m}, \epsilon)$  for  $Q(J(\mathbf{m}), \epsilon)$ , i.e.

$$Q(\mathbf{m}, \epsilon) = \frac{1}{2}(I + \epsilon\hat{\sigma}(\mathbf{m})), \quad (27)$$

or, explicitly, in a matrix form

$$Q(\mathbf{m}, \epsilon) = \frac{1}{2} \begin{pmatrix} 1 + \epsilon m_0 & \epsilon(m_1 - im_2) \\ \epsilon(-m_1 + im_2) & 1 - \epsilon m_0 \end{pmatrix}. \quad (28)$$

Note that  $\text{Tr}(Q(\mathbf{m}, \epsilon)) = 1$ .

**THEOREM 6.**

(i) For each  $\mathbf{m}, \mathbf{r} \in \mathcal{H}$ ,  $\epsilon \in \mathbb{R}$  we have

$$Q(\mathbf{m}, \epsilon)Q(\mathbf{r})Q(\mathbf{m}, \epsilon) = \lambda(\epsilon, \mathbf{m}, \mathbf{r})Q(\mathbf{r}'), \quad (29)$$

where

$$\lambda(\epsilon, \mathbf{m}, \mathbf{r}) = \frac{1 + \epsilon^2 + 2\epsilon\langle \mathbf{m} \cdot \mathbf{r} \rangle}{4} \quad (30)$$

and

$$\mathbf{r}' = \frac{(1 - \epsilon^2)\mathbf{r} + 2\epsilon(1 + \epsilon\langle \mathbf{m} \cdot \mathbf{r} \rangle)\mathbf{m}}{1 + \epsilon^2 + 2\epsilon\langle \mathbf{m} \cdot \mathbf{r} \rangle} \in \mathcal{H}. \quad (31)$$

(ii) The fuzzy projections  $Q(\mathbf{m}, \epsilon)$  map  $V_+$  into  $V_+$  if and only if  $\epsilon \geq 0$ .

*Proof:* (i) Follows by inspection, or by applying the ‘‘Wick rotation’’ to the formulae in [14].

(ii) Let  $A = A^*$  be a linear operator on  $V$  which maps  $V_+$  into  $V_+$ , then we must have  $\langle A\psi, A\psi \rangle = \langle \psi, A^2\psi \rangle \geq 0$  for all  $\psi \in V$  with  $\langle \psi, \psi \rangle = 1$ . Observe that  $\langle \psi, A^2\psi \rangle = \text{Tr}(EA^2E) = \text{Tr}(AEA)$ , where  $E$  is the orthogonal projection on  $\psi$ . Since

$$\lambda(\epsilon, \mathbf{m}, \mathbf{r}) = \text{Tr}(Q(\mathbf{m}, \epsilon)Q(\mathbf{r})Q(\mathbf{m}, \epsilon)), \quad (32)$$

it is enough to analyze the behaviour of the function  $\lambda$ . The unitary group of  $V$  acts on  $\mathcal{J}(V)$  transitively by its natural action:  $J \mapsto UJU^*$ . As the phase of the unitary operator cancels out, we can restrict ourselves to the special unitary group, in our case  $SU(1, 1)$ . We have 2 : 1 homomorphism  $U \mapsto \Lambda(U)$  of  $SU(1, 1)$  onto  $SO(2, 1)$  given by

$$UJ(\mathbf{m})U^* = J(\Lambda\mathbf{m}). \quad (33)$$

The function  $\lambda$  in Eq. (30) is  $SO(2, 1)$ -invariant. Therefore to analyze its behaviour we can always make an  $SO(2, 1)$  transformation so that  $\mathbf{r} = (1, 0, 0)$ , then, as a function of  $\mathbf{m} \in \mathcal{H}$ , we get  $4\lambda(\mathbf{m}, \epsilon) = 1 + \epsilon^2 + 2\epsilon m_0$ , which is bounded from below only when  $\epsilon \geq 0$ . If  $\epsilon > 0$ , then  $\lambda(\mathbf{m}, \epsilon)$ , with  $\mathbf{m} \in \mathcal{H}$ , attains minimum equal to  $\frac{(1+\epsilon)^2}{4}$  at  $\mathbf{m} = (1, 0, 0)$ .  $\square$

#### 4.1. Poincaré disk model

The hyperboloid  $\mathcal{J}(V) \approx \mathcal{P}_+$  is isomorphic to the Poincaré disk  $D$

$$D = \{z \in \mathbb{C} : |z| < 1\} \quad (34)$$

by the isomorphism:

$$J(\mathbf{m}) = \hat{\sigma}(\mathbf{m}) = m_0\sigma_3 + im_1\sigma_2 - im_2\sigma_1 \Leftrightarrow z, \quad (35)$$

where  $m_1 + im_2 = 2z/(1 - |z|^2)$ .

The action of the unitary group  $SU(V)$  on  $\mathcal{J}(V)$  is isomorphic to the action of  $SU(1, 1)$  on the Poincaré disk  $D$  via fractional transformations:  $A : z \mapsto (a_{11}z + a_{12})/(a_{21}z + a_{22})$ , the isomorphism being given by  $A = \sigma_2 U \sigma_2$ .

#### 5. Piecewise deterministic process on $\mathcal{P}_+$

As an example consider the piecewise deterministic process described by Eqs. (9)–(15), with  $g_i = \sqrt{\frac{4}{N}}Q(\mathbf{m}[i], \epsilon)$ ,  $i = 1, 2, \dots, N$ . In [14], where we were



studying quantum jumps on the complex projective space  $PC(1) \approx S^2$ , we could neglect completely the continuous part of evolution of the state vector, as given by Eq. (9), and that because we could choose symmetrically distributed configuration of fuzzy projections  $Q(\mathbf{n}[i], \epsilon)$  in such a way that  $\sum_i Q(\mathbf{m}[i], \epsilon)^2 = \text{const} \cdot I$  due to  $\sum_i \mathbf{m}[i] = 0$ . Such a choice is not possible in our case, with sphere replaced by the positive hyperboloid, because vectors on the positive hyperboloid cannot balance to the zero vector. Given a sequence  $\mathbf{m}[1], \mathbf{m}[2], \dots, \mathbf{m}[N]$ ,  $\langle \mathbf{m}[i], \mathbf{m}[i] \rangle = 1$ , we have

$$\sum_{i=1}^N \mathbf{m}[i] = Nc \mathbf{m}_c, \quad (36)$$

where  $c$  is a constant  $c > 0$  and  $\mathbf{m}_c$  is the centroid vector of unit length square  $\langle \mathbf{m}_c, \mathbf{m}_c \rangle = 1$ . The EEQT algorithm, as described, for instance in [10], tells us that when the quantum jumps are implemented by operators  $g_i$  then time evolution between jumps is implemented by

$$\psi(t) = R(t)\psi(0) = e^{-\frac{2t}{N} \sum_{i=1}^N Q(\mathbf{m}[i], \epsilon)^2} \psi(0). \quad (37)$$

Due to the fact that  $\hat{\sigma}(\mathbf{m}_c)^2 = I$ , the exponential is easily calculated,

$$R(t) = e^{-\frac{t}{2}(1+\epsilon^2)} [\cosh(\epsilon ct) - \sinh(\epsilon ct) \hat{\sigma}(\mathbf{m}_c)]. \quad (38)$$

We will choose our configuration so that  $\mathbf{m}_c = (1, 0, 0)$ , thus  $\hat{\sigma}(\mathbf{m}_c) = \sigma_3$ , or, in the matrix form

$$R(t) = e^{-\frac{t}{2}(1+\epsilon^2)} \begin{pmatrix} e^{-\epsilon ct} & 0 \\ 0 & e^{\epsilon ct} \end{pmatrix}. \quad (39)$$

It is easy to see that when  $Q(\mathbf{r})$  is the projection on  $\psi(0)$ ,  $Q(\mathbf{r}) = |\psi(0)\rangle\langle\psi(0)|$ , then

$$\|\psi(t)\|^2 = \text{Tr}(R(2t)Q(\mathbf{r})), \quad (40)$$

which can be easily calculated from the explicit matrix formulae above,

$$p(t) \doteq \|\psi(t)\|^2 = \exp(-t(1 + \epsilon^2)) (\cosh(2\epsilon ct) - m_0 \sinh(2\epsilon ct)). \quad (41)$$

For  $\epsilon > 0$  and  $m_0 \geq 1$  the function  $p(t)$  is monotonically decreasing from  $p(t_0) = 1$  to  $p(t_1) = 0$ , where  $t_0 = 0$  and  $t_1 = (\log((m_0 + 1)/(m_0 - 1)))/4\epsilon c$ . In numerical simulations of the PDP process the time of jump is calculated by selecting a uniformly distributed random number  $p \in (0, 1)$ , and then numerically solving the equation  $p(t) = p$ . One convenient method of doing it is by introducing the variables  $y = \exp(4\epsilon ct)$  and  $s = (1 + \epsilon^2 + 2\epsilon c)/4\epsilon c$ , and solving numerically the equation  $2py^s + y(m_0 - 1) - (m_0 + 1) = 0$  for  $y$ . To illustrate the resulting iterated function system we took  $N = 33$  points  $\mathbf{m}[i]$  from a hyperbolic tiling with Schläfli symbol  $(8, 3)$  and plot the results of 100 mln jumps for two  $\epsilon = 0.75$  and  $\epsilon = 1.75$ .

Figures 2 and 3 show the results of the iteration process for two different values of the fuzziness parameter  $\epsilon = 0.75$ , and  $\epsilon = 1.75$  (for  $\epsilon = 1.0$  the fuzzy projections are sharp). Due to the fact that our process is a hybrid one—it has

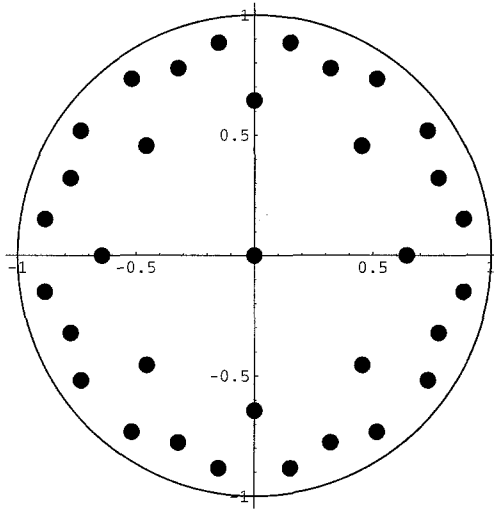


Fig. 1. 33 points from the hyperbolic tiling of the Poincaré disk with Schläfli symbol  $(8, 3)$ .

jumps with place-dependent probabilities, as in iterated function systems, but it also has dissipative parts of a continuous evolution with a random time length, and also due to the fact that the Poincaré disk is non-compact, there are no ready to use theorems that we could apply in order to ascertain the existence and uniqueness of the invariant measure, as it was done in the case of positive definite metric [14]. Our numerical simulations suggest a conjecture that the probabilities of getting outside a bounded region decrease fast enough with the dimension of this region, so that the attractor set and the invariant measure supported by this set are well defined.

## 6. Concluding remarks

In the present paper a model nonunitary quantum system in a Krein's space  $V_{1,1}$  is analyzed. The model is described via a quantum dynamical semigroup and, from a computational point of view, as a generalization of classical iterated function system. It has been shown that with a careful treatment, within the general framework of EEQT, a continuous monitoring of several noncommuting observables can be modelled by a piecewise deterministic random process (with positive jump-time distribution and positive transition probabilities) also in the case of a quantum system whose pure states are represented by vectors in indefinite metric space. The formalism described in Sections 2 and 3 applies to an arbitrary Krein space, in particular, when appropriate care is taken, also to infinite-dimensional spaces arising naturally in explicitly covariant theories of spinor and vector relativistic wave equations. In the present paper we have chosen the simplest case of the two-dimensional space  $V_{1,1}$  mainly because, in this case, it is easy to visualize the resulting fractal structure on the Poincaré disk.

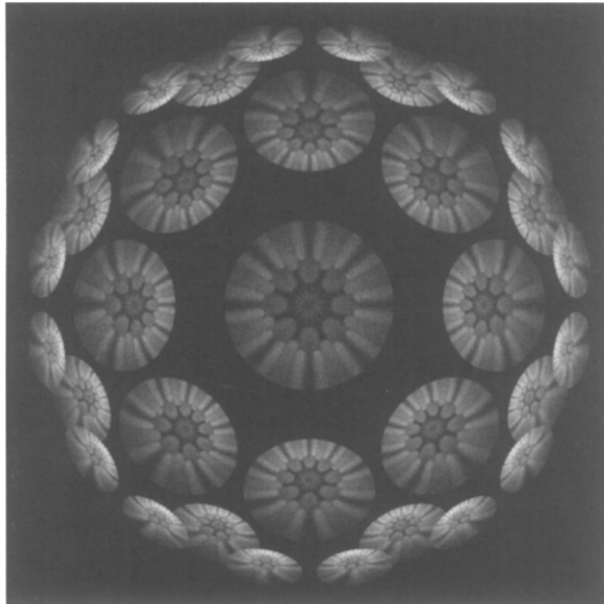


Fig. 2. Hyperbolic quantum fractal  $\epsilon = 0.75$ . Gray scale of a given pixel  $i$  is proportional to  $\log((1+n_i)/n_{\max})$ , where  $n_i$  is the number of jumps to the area covered by the pixel. In particular the black color area is never visited (or is visited with a negligible probability).

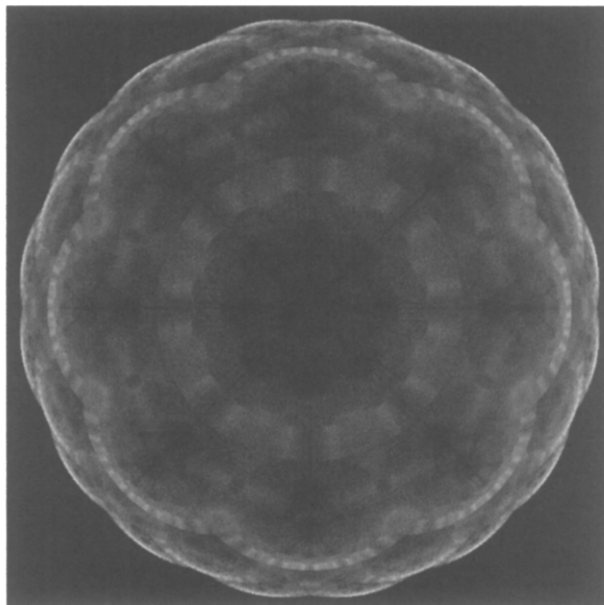


Fig. 3. Hyperbolic quantum fractal  $\epsilon = 1.75$ .

The main difference between the case of  $CP^1 = S^2$ , studied in [14], and the indefinite metric case studied in the present paper, comes from the fact that, in the latter case, the space of pure states  $V_+$  is noncompact and has no centroid. This implies a nontrivial continuous, nonunitary drag between jumps. This drag is represented by straight line segments towards the detectors in Figures 2 and 3. In realistic models such drag would be superimposed on the continuous, unitary Hamiltonian evolution.

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