**Advances in
Applied Clifford Algebras**

Quantum Fractals on n–Spheres.
Clifford Algebra Approach

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Abstract. Using the Clifford algebra formalism we extend the quantum jumps algorithm of the Event Enhanced Quantum Theory (EEQT) to convex state figures other than those stemming from convex hulls of complex projective spaces that form the basis for the standard quantum theory. We study quantum jumps on n-dimensional spheres, jumps that are induced by symmetric configurations of non-commuting state monitoring detectors. The detectors cause quantum jumps via geometrically induced conformal maps (Möbius transformations) and realize iterated function systems (IFS) with fractal attractors located on n-dimensional spheres. We also extend the formalism to mixed states, represented by "density matrices" in the standard formalism, (the n-balls), but such an extension does not lead to new results, as there is a natural mechanism of purification of states. As a numerical illustration we study quantum fractals on the circle (one-dimensional sphere and pentagon), two–sphere (octahedron), and on three-dimensional sphere (hypercube-tesseract, 24 cell, 600 cell, and 120 cell). The attractor, and the invariant measure on the attractor, are approximated by the powers of the Markov operator. In the appendices we calculate the Radon-Nikodym derivative of the $SO(n+1)$ invariant measure on S^n under $SO(1, n+1)$ transformations and discuss the Hamilton's "icossian calculus" as well as its application to quaternionic realization of the binary icosahedral group that is at the basis of the 600 cell and its dual, the 120 cell.

As a by-product of this work we obtain several Clifford algebraic results, such as a characterization of positive elements in a Clifford algebra $\mathcal{C}(n+1)$ as generalized Lorentz "spin–boosts", and their action as Möbius transformation on n-sphere, and a decomposition of any element of $Spin^+(1, n+1)$ into a spin–boost and a spin–rotation, including the explicit formula for the pullback of the $SO(n+1)$ invariant Riemannian metric with respect to the associated Möbius transformation.

Keywords. Quantum fractals, Event Enhanced Quantum Theory, quantum jumps, Clifford algebra, Möbius transformation.

1. Introduction

"The accepted outlook of quantum mechanics (q.m.) is based entirely on its the-
ory of measurement. Quantitative results of observations are regarded as the only
accessible reality, our only aim is to predict them as well as possible from other
observations already made on the same physical system. This pattern is patently
taken over from the positional astronomer, after whose grand analytical tool (an-
alytical mechanics) q.m. itself has been modelled. But the laboratory experiment
hardly ever follows the astronomical pattern. The astronomer can do nothing but
observe his objects, while the physicist can interfere with his in many ways, and
does so elaborately. In astronomy the time–order of *states* is not only of para-
mount practical interest (e.g. for navigation), but it was and is the only method
of discovering the *law* (technically speaking: a hamiltonian); this he rarely, if ever,
attempts by following a single system in the time succession of its states, which
in themselves are of no interest. The accepted foundation of q.m. claims to be
intimately linked with experimental science. But actually it is based on a scheme
of measurement which, because it is entirely antiquated, is hardly fit to describe
any relevant experiment that is actually carried out, but a host of such as are for
ever confined to the imagination of their inventors."

So wrote Ervin Schrödinger fifty years ago [1]. Today the standard scheme
of q.m. is as antiquated as it ever was, and provides no answer to the most funda-
mental questions such as "what is time?", and how to describe *events* that happen
in a single physical system, such as our Universe.[1] The present paper follows the
line of ideas developed in a series of papers that has led to the Event Enhanced
Quantum Theory (EEQT), as summarized in [3], and recently extended in [4], but
we now go beyond that framework. While, following von Neumann, we keep the
algebraic structure as one of the most important for the mathematical formalism
of q.m., and we propose to dispose of the concept of "observables" and of "expec-
tation values" at the fundamental level. We also dispose of the concept of "time",
understood as a "continuous parameter", external to the theory. Our philosophy,
concerning "time" is that of the German social philosopher Ernest Bloch:

> "*Zeit ist nur dadurch, daß etwas geschieht und nur dort wo etwas
> geschiecht.*"

So, time is only *then*, when something happens, and only *there* where something
happens. Therefore the primary concept is that of an *event*, and of the *process*
- that is a sequence of events. Time, as a continuous, global variable, comes in
only in the limit of a large number of events. The primary process is that of
"quantum jumps". It is an irreversible process in an open system, and every system
in which the "future" is only "probable", rather than determined, is necessarily

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[1]Nowadays the defenders of the "antiquated scheme" of q.m. go as far as to assign "crackpot
index" to those who question this scheme. So, for instance, 10 points (on the scale of 1–50), are
assigned for each claim that quantum mechanics is fundamentally misguided, and another 10
points for arguing that while a current well-established theory predicts phenomena correctly, it
doesn't explain "why" they occur, or fails to provide a "mechanism" [2].

an open system. The mathematical formalism of the standard quantum theory is based on complex Hilbert spaces and Jordan algebras of self–adjoint operators. It involves interpretational axioms for expectation values and eigenvalues of self–adjoint operators as "possible results of measurements", yet it does not provide a framework for *defining the measurements* [5, 6]. In view of these considerations, Gell-Mann would certainly score a high crackpot index [2] for this statement [7, p. 165]:

> "Those of us working to construct the modern interpretation of quantum mechanics aim to bring to an end the era in which Niels Bohr's remark applies: 'If someone says that he can think about quantum physics without becoming dizzy, that shows only that he has not understood anything whatever about it'."

The same can be said about the last paragraph of Schrödingers paper [1], where he wrote

> "We are also supposed to admit that the extent of what is, or might be, observed coincides exactly with what quantum mechanics is pleased to call observable. I have endeavored to adumbrate that it does not. And my point is that this is not an irrelevant issue of philosophical taste; it will compel us to recast the conceptual scheme of quantum mechanics."

The need for an open–minded approach is well noted by John A. Wheeler, who ends his book "Geons, Black Holes & Quantum Foam" [8] with the following quote from Niels Bohr's friend Piet Hein:

> I'd like to know
> what this whole show
> is all about
> before it's out.

Alain Connes and Carlo Rovelli [9] proposed to explain the classical time parameter as arising from the modular automorphism group of a KMS state on a von Neumann algebra over the field of complex numbers $\mathbb{C}$.[2] But their philosophy applies, at most, to equilibrium states, while "quantum foams" before the Planck era are certainly far from equilibrium. David Hestenes [10, 11] proposed to understand the role of the complex numbers in quantum theory in terms of the Clifford algebra. This is also our view. L. Nottale, in his theory of "scale relativity" [12] proposed an alternative idea, where the complex structure arises from a stochastic differential equation in a fractal space–time. We think that our approach may serve as a connecting bridge between fractality, the nontrivial topology of dodecahedral models of space–time, as discussed by J–P. Luminet et al. [13] (cf. also [14]), and the late thoughts of A. Einstein [15, p. 92], who wrote:

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[2]Cf. also I. and G. Bogdanov, *Avant le Big Bang : La création du monde*, second, revised and extended edition, (LGF, Paris 2006), where a similar idea, based on a KMS equilibrium state is discussed in a broader, philosophical framework

"To be sure, it has been pointed out that the introduction of
a space-time continuum may be considered as contrary to na-
ture in view of the molecular structure of everything which hap-
pens on a small scale. It is maintained that perhaps the success
of the Heisenberg method points to a purely algebraical method
of description of nature, that is to the elimination of continuous
functions from physics. Then, however, we must also give up, by
principle, the space-time continuum. It is not unimaginable that
human ingenuity will some day find methods which will make it
possible to proceed along such a path. At the present time, how-
ever, such a program looks like an attempt to breathe in empty
space."

The present paper is a technical one. It fills the empty space with discrete
structures, and it deals with the discrete random aspects of quantum jumps gen-
erated by the algebraic structure of real Clifford algebras of Euclidean spaces, and
of their conformal extensions. The jumps are generated by Möbius transforma-
tions and lead to iterated function systems with place dependent probabilities,
thus to fractal patterns on n–spheres. Our ideas are close to those of W. E. Baylis,
who also noticed [16] the similarities between the Clifford algebra scheme and the
formal algebraic structure of q.m. Our results concern the case of the signature
$(1, n + 1)$. With some adaptation, the methods and the ideas developed in the
present paper should be also applicable to the "hyperbolic quantum formalism",
such as developed in recent papers by A. Khrennikov [17].

In Sec. 2 we introduce our notation, which is kind of a mixture of that used
by Deheuvels [18] on one hand, and of Gilbert and Murray [19] on the other. In
Proposition 1 we recall the vector space isomorphism between the Clifford algebra
and the exterior algebra, and in Proposition 2 we define the trace functional, and
list its properties that are important for applications to quantum probabilities.
In Sec. 3 we review the necessary concepts and results from the monograph by
Gilbert and Murray [19], and discuss in details the algebra isomorphism between
$\mathcal{C}^+(1, n + 1)$ and the algebra $\mathbb{R}(2, \mathcal{C}(n + 1))$. The main results of this section are
given in Theorem 1.

In Sec. 4 we use the Clifford algebra approach to discus Möbius transfor-
mations of the spheres S^n, as well as their natural extensions to their interiors
B^{n+1}. The key concept here is that of "positivity". In Propositions 3 and 4 we
characterize the positive elements of the $Spin^+$ group, and in Corollary 1 we prove
the polar decomposition of any element of the $Spin^+$ group into a product of a
positive spin–boost and of a unitary spin–rotation. In Theorem 3 we describe the
action of $Spin^+(1, n + 1)$ as the two–fold covering group of the group of Möbius
transformations of S^n, and give the explicit formula for the action of spin–boosts
on S^n (cf. Eq. (4.37)). We also calculate the Radon-Nikodym derivatives of the
transformed surface area and the volume (cf. Eqs. (4.42), (4.43)). In subsection
4.2 we discuss the stereographic projection, and we use the exponential form of

the spin–boosts in order to describe their (singular) action on $\mathbb{R}^n$ rather than on S^n.

In Sec. 5 we discuss iterated function systems (IFS) of conformal maps and introduce the important concept of the Markov operator, which is later being used in our numerical simulations (cf. Sec. 6). Proposition 5 of this section is important in applications to quantum theory. One of the most important features of the standard, linear, quantum mechanics is the fact that "observables" are restricted to bilinear functions on pure states. Therefore different mixtures of pure states leading to the same "density matrix" are claimed to be experimentally indistinguishable. In our Proposition 5, and in Corollary 2, we show that if the probabilities of the iterated function systems of Möbius transformations are given by geometrical factors derived from the maps themselves (cf. Eqs. (4.37),(5.59)), and also satisfy the additional balancing condition (5.58), then the Markov operator restricts to the space of functions on S^n given by the trace on the Clifford algebra, thus leading to a linear Markov semi-group. Corollary 3 gives the explicit form of the Markov operator for the case when the IFS of Möbius transformations is endowed with geometrical probabilities given by Eq. (5.59).

Sec. 6 contains the results of the numerical simulations of IFS of Möbius transformations that lead to "quantum fractals". We study quantum fractals on the circle (one-dimensional sphere and pentagon), two–sphere (octahedron), and on three-dimensional sphere (hypercube-tesseract, 24 cell, 600 cell, and 120 cell). The last section contains the summary and conclusions and also points out some open problems.

In the Appendix 1, which is of independent interest, we discuss the Möbius transformation in terms of the group $SO^+(1, n+1)$ and derive the Radon–Nikodym derivative formula for a general $SO^+(1, n+1)$ transformation. Appendix 2 reproduces the original Hamilton's paper of 1856 introducing the "icossian calculus", while in Appendix 3 we discuss its application to quaternionic realization of the binary icosahedral group that is at the basis of 600 cell and its dual, the 120 cell.

2. Notation

We will denote by $\mathbb{R}$ the field of real numbers, and by $\mathbb{R}^*$ the multiplicative group $\mathbb{R} \setminus \{0\}$. Let V be an n–dimensional real vector space endowed with a non–degenerate quadratic form Q of signature (r, s), $r + s = n$. That is V admits an orthonormal basis e_i, with $Q(e_1) = \ldots = Q(e_r) = 1$, $Q(e_{r+1}) = \ldots = Q(e_n) = -1$. Let $\mathcal{C} = \mathcal{C}(V, Q)$ the Clifford algebra of (V, Q). The even and the odd parts of $\mathcal{C}$ are denoted as $\mathcal{C}^+$ and $\mathcal{C}^-$ respectively. We shall consider $\mathbb{R}$ and V as vector subspaces of $\mathcal{C}$, so that $v^2 = Q(v) \in \mathcal{C}$, $v \in V$.

The principal automorphism of $\mathcal{C}$ is denoted by π and is determined by $\pi(v) = -v$, $v \in V$, while the principal anti–automorphism τ, denoted also as $\tau(a) = a^\tau$, is determined by $v^\tau = v$, $v \in V$. Their composition ν is the unique anti–automorphism satisfying $\nu(v) = -v$ for all $v \in V$. We will denote by Δ the norm

function $\Delta : \mathcal{C} \longrightarrow \mathcal{C}$, defined by

$$\Delta(a) = a^\nu a. \tag{2.1}$$

We recall that, cf. [19, 5.14–5.16], if $\Delta(a), \Delta(b) \in \mathbb{R}$, then $\Delta(ab) = \Delta(a)\Delta(b)$, $\Delta(\pi(a)) = \Delta(\tau(a)) = \Delta(a^\nu) = \Delta(a)$ and, for all $\lambda \in \mathbb{R}$, $\Delta(\lambda a) = \lambda^2 \Delta(a)$. Moreover, if $\Delta(a) \in \mathbb{R}^*$, then a is invertible, and $a^{-1} = (1/\Delta(a))a^\nu$. In particular, if $\Delta(a) \in \mathbb{R}$, then also $aa^\nu = \Delta(a)$.

We denote by $Spin^+(V, Q)$ the group:

$$Spin^+(V, Q) = \{g \in \mathcal{C}^+(V, Q) : \Delta(g) = 1, \ gVg^{-1} = V\}. \tag{2.2}$$

Every element $g \in Spin^+(V, Q)$ is a product of an even number of positive unit vectors (i.e. vectors $v \in v$ such that $Q(u) = +1$) and an even number of negative unit vectors (i.e. $v \in V$ such that $Q(u) = -1$) – cf. [18, Definition IX.4.C]. The map $\sigma : Spin^+(V, Q) \to SO^+(V, Q), \quad \sigma(g) : v \mapsto gvg^{-1}$ is a two–fold covering homomorphism from $Spin^+(V, Q)$ onto $SO^+(V, Q)$, the connected group of "proper rotations", that is orthogonal transformations of (V, Q) of determinant one, which preserve the orientation of maximal negative subspaces of V.[3]

We denote by $\mathbb{R}(n)$ (resp. $\mathbb{R}(n, \mathcal{C}) = \mathcal{C} \otimes \mathbb{R}(2)$) the algebra of $n \times n$ matrices with entries from $\mathbb{R}$ (resp. from $\mathcal{C}$).

2.1. Vector space isomorphism between the Clifford and the Grassmann algebra

Let us recall that, as a vector space, Clifford algebra is naturally graded and isomorphic to the exterior algebra. In particular we have the following result :

Proposition 1. *Let e_i, $i = 1, 2, \ldots, n$ be an orthonormal basis for V, and let e_I : $I = (i_1, i_2, \ldots, i_p), 1 \le i_1 < i_2 < \ldots < i_p \le n$ be defined as the Clifford products $e_I = e_{i_1} e_{i_1} \ldots e_{i_p}$, with $e_I = 1$ for $I = \emptyset$. Then the set $\{e_I\}$ of 2^n vectors in $\mathcal{C}$ is a linear basis of $\mathcal{C}$, the subspaces $\mathcal{C}_p$ generated by e_I, $I = (i_1, \ldots i_p)$ are independent of the choice of the orthonormal basis e_i, and $\mathcal{C}$ is the direct sum of vector subspaces $\mathcal{C}_p$:*

$$\mathcal{C} = \bigoplus_{k=0}^{n} \mathcal{C}_p \tag{2.3}$$

Moreover, for each $p = 0, \ldots, n$ the skew–symmetric map α_p from $V \times V \times \ldots \times V$ (p times) to $\mathcal{C}$ given by:

$$\alpha_p(x_1, x_2, \ldots, x_p) = \frac{1}{p!} \sum_\sigma (-1)^\sigma x_{\sigma 1} x_{\sigma 2} \ldots x_{\sigma p},$$

determines an isomorphism of the vector subspace $\Lambda^p V$ of the exterior algebra ΛV onto $\mathcal{C}_p$ that sends $e_{i_1} \wedge \ldots \wedge e_{i_p} \in \Lambda^p V$ to $e_{i_1} \ldots e_{i_p} \in \mathcal{C}_p \subset \mathcal{C}$.

Proof. cf. [18, Theoreme VIII.10]   □

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[3]The group $Spin^+$ is denoted simply as $Spin$ in Refs [18],[20, 2.4.2], and as $Spin_0$ in [19]. The case of $(r, s) = (1, 1)$ is special, as in this case the group $Spin^+$ has two disconnected components – cf. Ref [18, p. 369].

Of particular interest for us will be the subspace $\mathcal{C}_0 \oplus \mathcal{C}_1 \subset \mathcal{C}$ of *paravectors*. We will denote this subspace by V^1 and endow it with the quadratic form Q^1 defined by

$$Q^1(x^0, v) = (x^0)^2 - v^2, \, x^0 \in \mathbb{R}, v \in V. \tag{2.4}$$

If Q is of signature (r, s), then Q^1 has signature $(s + 1, r)$.

2.2. The trace

We denote by Φ the linear functional on $\mathcal{C}$ assigning to each element $a \in \mathcal{C}$ its scalar part $\Phi(a) = a_0 \in C_0$ in the decomposition (2.3). Then the following proposition holds:

Proposition 2. *The functional Φ has the following properties:*

  (i) $\Phi(1) = 1$,

 (ii) $\Phi(a^\tau) = \Phi(a)$, $\forall a \in \mathcal{C}$,

(iii) $\Phi(ab) = \Phi(ba)$, $\forall a, b \in \mathcal{C}$,

 (iv) $(a, b) \overset{df}{=} \Phi(a^\tau b)$ *is a nondegenerate, symmetric, bilinear form on $\mathcal{C}$, that is positive definite if the original quadratic form on V is positive definite. We have $\Phi(a) = (1, a) = (a, 1)$, $\forall a \in \mathcal{C}$.*

  (v) $(ab, c) = (b, a^\tau c) = (a, cb^\tau)$, $\forall a, b, c \in \mathcal{C}$.

Proof. (i) and (ii) follow immediately from the definition. In order to prove (iii) notice that if $\{e_i\}$, $i = 1, \ldots, n$ is an orthonormal basis in V, $\{e_I\}$, $I = \{i_1 < \ldots < i_p\}$ is the corresponding basis in $\mathcal{C}$, and $a = \sum_I a_I e^I$, $b = \sum_I b_I e_I$ are the decompositions of a and b in the basis e_I, then $\Phi(ab) = \sum_I a_I b_I \Phi(e_I e_I) = \Phi(ba)$. From the very definition of the scalar product (a, b) it follows that $(a, b) = \Phi(a^\tau b) = \Phi((a^\tau b)^\tau) = \Phi(b^\tau a) = (b, a)$. Moreover, we have $(e_I, e_J) = 0$ if $I \neq J$, and also $(e_I, e_I) = e_{i_1}{}^2 \ldots e_{i_p}{}^2 = (-1)^{s(I)}$, where $s(I)$ is the number of negative norm square vectors in I. In particular e_I is orthonormal with respect to the scalar product in $\mathcal{C}$, and so (iv) holds. We have $(ab, c) = \Phi((ab)^\tau c) = \Phi(b^\tau a^\tau c) = \Phi(a^\tau cb^\tau) = (a, cb^\tau)$, which establishes (v). $\square$

Remark 1. *It is easy to see that $\Phi(a) = (1/2^n)\mathrm{tr}(L(a))$, where $L(a)$ is the left multiplication by a acting on $\mathcal{C} : L(a)b = ab$, and the trace is taken over $\mathcal{C}$, see e.g. [21, p. 601] for a general discussion. Because of this property Φ will be called a trace.*

We will call an element $a \in \mathcal{C}$ *positive*, which we will write $a \geq 0$, if $a = a^\tau$ and $(v, av) \geq 0$ for all $v \in \mathcal{C}$. Equivalently, $a \geq 0$ if and only if a is of the form $a = b^\tau b$, for some $b \in \mathcal{C}$ (cf. e.g. [22, 7.27]). If a is positive and $a \neq 0$, we will write $a > 0$. If $a \geq 0$, then, in particular, $\Phi(a) = (1, a1) \geq 0$ and, if $a > 0$ then $\Phi(a) > 0$.

3. Algebra Isomorphism Between $\mathcal{C}^+(V^1, Q^1)$ and $\mathbb{R}(2, \mathcal{C}(V,Q))$

It is a well known fact (see e.g. Ref. [19, 6.13]) that the algebras $\mathcal{C}^+(V^1, Q^1)$ and $\mathbb{R}(2, \mathcal{C})$ are isomorphic. For the purpose of the present paper it is useful to have a description of this isomorphism in some details.

**Notation**: In what follows we will use the notation $\mathcal{C} = \mathcal{C}(V,Q)$, and $\mathcal{C}^1 = \mathcal{C}(V^1, Q^1)$.

Let the map $A : \mathcal{C} \times \mathcal{C} \to \mathbb{R}(2, \mathcal{C})$ be defined by

$$A(a,b) = \left\{ \begin{pmatrix} a & b \\ \pi(b) & \pi(a) \end{pmatrix} : a, b \in \mathcal{C} \right\}, \tag{3.5}$$

and let $\gamma : V^1 \to \mathbb{R}(2, \mathcal{C})$ be the linear map given by

$$\gamma(x^0, v) = \begin{pmatrix} 0 & x^0 + v \\ x^0 - v & 0 \end{pmatrix} = \begin{pmatrix} 0 & x^0 + v \\ \pi(x^0 + v) & 0 \end{pmatrix}. \tag{3.6}$$

Then γ is evidently the Clifford map, $\gamma(x^0, v)^2 = Q^1(x^0, v)I$, and therefore it extends to a unique algebra homomorphism, which we will denote by the same symbol γ, from $\mathcal{C}^1$ to $\mathbb{R}(2, \mathcal{C})$. We will define now the following maps, and study their properties:
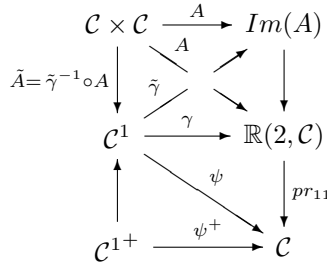
The map $pr_{11} : \mathbb{R}(2, \mathcal{C}) \to \mathcal{C}$ assigns to each matrix in $\mathbb{R}(2, \mathcal{C})$ its top–left entry. For instance $pr_{11}(A(a,b)) = a$. $Im(A)$ is the set of all matrices of the form (3.5). We will not distinguish between the maps $A : \mathcal{C} \times \mathcal{C} \to \mathbb{R}(2, R)$ and $A : \mathcal{C} \times \mathcal{C} \to Im(A)$, which differ only by the canonical inclusion $Im(A) \to \mathbb{R}(2, \mathcal{C})$. But we will distinguish between $\gamma : \mathcal{C}^1 \to \mathbb{R}(2, \mathcal{C})$ and $\tilde{\gamma} : \mathcal{C}^1 \to Im(A)$. The latter map is an algebra isomorphism, therefore $\tilde{\gamma}^{-1} : Im(A) \to \mathcal{C}^1$ is well defined. The map ψ is defined as $\psi = pr_{11} \circ \tilde{\gamma}$, and is an algebra isomorphism, and ψ^+ is its restriction to $\mathcal{C}^{1+}$. We will use the notation $\tilde{A}(a,b)$ for $\tilde{\gamma}^{-1}(A(a,b))$.

Theorem 1.   (i) *Let us realize the Clifford algebra $\mathcal{C}(1, -1)$ as the matrix algebra $\mathbb{R}(2)$ using the following basis*

$$f_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, f_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, f_{01} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.7}$$

so that we have

$$f_0^{2k} = 1_2, f_0^{2k+1} = f_0, f_1^{2k} = (-1^k)1_2, f_1^{2k+1} = (-1)^k f_1. \tag{3.8}$$

Let $\{e_0 \in \mathbb{R}, e_i \in V, i = 1, \ldots, n+1\}$, be an orthonormal basis of V^1. Then, in terms of this basis the map $\gamma : \mathcal{C}^1 \longrightarrow Mat(2, \mathcal{C})$ reads:

$$
\begin{aligned}
\gamma(1) &= 1_2 \otimes 1_{\mathcal{C}} \\
\gamma(e_0 e_{i_1} \ldots e_{i_{2k}}) &= (-1)^k f_0 \otimes e_{i_1} \ldots e_{i_{2k}}, \\
\gamma(e_0 e_{i_1} \ldots e_{i_{2k+1}}) &= (-1)^k f_{01} \otimes e_{i_1} \ldots e_{i_{2k+1}}, \\
\gamma(e_{i_1} \ldots e_{i_{2k}}) &= (-1)^k 1_2 \otimes e_{i_1} \ldots e_{i_{2k}}, \\
\gamma(e_{i_1} \ldots e_{i_{2k+1}}) &= (-1)^k f_1 \otimes e_{i_1} \ldots e_{i_{2k+1}}.
\end{aligned}
$$

(ii) $ker(\psi) = \mathcal{C}^{1^-}$, and ψ restricts to the algebra isomorphisms ψ^+ from $\mathcal{C}^{1^+}$ onto $\mathcal{C}$. In terms of the basis we have:

$$
\left.
\begin{aligned}
\psi^+(1_{\mathcal{C}^1}) &= 1_{\mathcal{C}}, \\
\psi^+(e_{i_1} \ldots e_{i_{2k}}) &= (-1)^k e_{i_1} \ldots e_{i_{2k}}, \\
\psi^+(e_0 e_{i_1} \ldots e_{i_{2k+1}}) &= (-1)^k e_{i_1} \ldots e_{i_{2k+1}}.
\end{aligned}
\right\}
\tag{3.9}
$$

(iii) With the notation as above, we have

$$
A(a,b)A(a',b') = A(a'',b''), \quad \text{where} \quad a'' = aa' + b\pi(b'), \; b'' = ab' + b\pi(a'). \tag{3.10}
$$

The principal involution π and the principal anti–involution τ of $\mathcal{C}^1$ can be expressed through their corresponding operations in $\mathcal{C}$ as

$$
\pi(\tilde{A}(a,b)) = \tilde{A}(a, -b), \tag{3.11}
$$

$$
\tau(\tilde{A}(a,b)) = \tilde{A}(\nu(a), \tau(b)). \tag{3.12}
$$

The even subalgebra $\mathcal{C}^{1^+}$ of $\mathcal{C}^1$ can then be identified with the set of all $A(a,b)$, with $b = 0$, that is, using the map pr_{11}, with $\mathcal{C}$.

(iv) Denoting by Φ^1 (resp. Δ^1), and Φ (resp. Δ) the trace (resp. norm function) of $\mathcal{C}^1$ and $\mathcal{C}$ respectively, we have

$$
\Phi^1 = \Phi \circ \psi, \tag{3.13}
$$

$$
\Delta^1(\tilde{a}) = \Delta(\psi^+(\tilde{a})), \quad \forall \tilde{a} \in \mathcal{C}^1, \tag{3.14}
$$

$$
(\tilde{a}, \tilde{b}) = (\pi(\psi^+(\tilde{a})), \psi^+(\tilde{b})), \quad \forall \tilde{a}, \tilde{b} \in \mathcal{C}^1. \tag{3.15}
$$

(v) $\tilde{g} \in Spin(V^1, Q^1)$ if and only if $g = \psi^+(\tilde{g})$ satisfies
   a) $\Delta(g) = 1$, and
   b) $gV^1 g^\tau = V^1$.

Proof. (i) and (ii) follow by a straightforward calculation.
    (iii) By a straightforward matrix multiplication we get from (3.5) that

$$
A(a,b)A(a',b') = A(a'',b''), \text{ where } a'' = aa' + b\pi(b'), \; b'' = ab' + b\pi(a'). \tag{3.16}
$$

It follows that the range (image) of the map A is an algebra and, because it has the right dimension $2 \times dim(V)$, the Clifford map γ extends to the isomorphism of $\mathcal{C}^1$ onto $Im(A)$. It is also clear that the even subalgebra of $\mathcal{C}^1$ is represented by the matrices $A(a,0)$, while the odd subspace is represented by matrices $A(0,b)$.

It follows from the very definition that π and τ defined by (3.11) and (3.12) are involutions, and that $\pi(\psi(w)) = \psi(-w)$, $\tau(\gamma(w)) = \gamma(w)$ for $w \in V^1$. Therefore we need to show that π, defined by (3.11), is an automorphism, and that τ, defined by (3.12), is an anti–automorphism. (Notice that although, by abuse of the notation, we denote by the same symbol π the main automorphisms of $\mathcal{C}$ and $\mathcal{C}^1$, the meaning is always clear from the context.)

Let C be the matrix[4]: $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then

$$CA(a,b)C^{-1} = A(a,-b), \tag{3.17}$$

therefore the formula (3.11) defines an involutive automorphism of $\mathcal{C}^1$, and, since it reverses the signs of vectors, it defines the principal involution of $\mathcal{C}^1$.

Proving that τ is an anti–automorphism of $\mathcal{C}^1$ follows by a straightforward calculation using (3.5) and the properties of π and τ on $\mathcal{C}$.

(iv) follows from (i)–(ii). Finally, (v) follows from (ii) and (iv) – (cf. also Ref. [19, Theorem 6.12]). $\qquad\square$

From now on we will assume that (V, Q) is an $(n+1)$– dimensional Euclidean space, that is that Q has the signature $(n + 1, 0)$.

4. Möbius Transformations of S^n and their Extensions to $\bar{B}^{n+1}$

Remark 2. *In chapter 2 of reference [20], Pierre Anglès gives an explicit construction of covering groups of the conformal group of a standard regular pseudo-euclidean space endowed with a quadratic form of signature (p,q), together with a geometrical construction of this conformal group and shows explicitly that this group is isomorphic to $PO(p + 1, q + 1)$, by using a wider Clifford algebra associated with a pseudo-euclidean regular standard space endowed with a quadratic form of signature $(p + 1, q + 1)$ -cf., for example, [20, 2.4.2.5.2]. By using his table given in [20, 2.4.2.4], one can characterize the elements of the conformal group of the sphere S^n stereographically projected onto E^n. We follow another algebraic method, considering the particular case of signature $(n, 0)$.*

Let (V, Q) be an $(n + 1)$– dimensional Euclidean space, $n > 0$. Vectors in V will be denoted by bold symbols: $\mathbf{x}, \mathbf{y}$, etc. The bold symbol $\mathbf{n}$ will be reserved for unit vectors. We will denote by B^{n+1} the open unit ball $B^{n+1} = \{\mathbf{x} \in V : \mathbf{x}^2 < 1\}$, by $\bar{B}^{n+1}$ its closure $\bar{B}^{n+1} = \{\mathbf{x} \in V : \mathbf{x}^2 \leq 1\}$, and by S^n its boundary, the unit sphere $S^n = \{\mathbf{n} \in V : \mathbf{n}^2 = 1\}$. We will denote by $\mathcal{C}$ the Clifford algebra $\mathcal{C}(V, Q)$, by $Spin^+$ the group $Spin^+(V, Q)$, and by $Spin^{1^+}$ the group $\psi^+(Spin^+(V^1, Q^1))$, described by the conditions a) and b) in Theorem 1, (v). Following Ref. [19] we define the Clifford group $\Gamma(V)$ as

$$\Gamma(V) = \{w_1 \ldots w_k : w_j \in V^1, \quad \Delta(w_j) \neq 0\}.$$

---

[4]Cf. also [18, Xh. VIII.6, p. 310], where the matrix C is used to define an anti–involution of the algebra $\mathbb{C}(2)$.

It is evident that this group is closed under π, τ, ν.

We will describe the action of $Spin^{1^+}$ on the unit sphere S^n, and on its interior B^{n+1}. As the main tool we will use the special class of elements of $\mathcal{C}$, that are called *transformers* .

4.1. Transformers

Following Gilbert and Murray [19, 5.21] we define a transformer to be any element a of $\mathcal{C}$ with the property that for every element $w \in V^1$ there exists another $w' \in V^1$ such that

$$aw = w'\pi(a). \tag{4.18}$$

The set $\mathcal{T}$ of all transformers is a multiplicative semigroup. Moreover we have the following important result proven in [19, 5.24–5.29]:

Theorem 2. *The set of all transformers $\mathcal{T}$ is closed under the principal automorphism π. Moreover, for every $a \in \mathcal{T}$, $\Delta(a) \in \mathbb{R}$, and if $\Delta(a) \neq 0$, then also $a^\tau \in \mathcal{T}$. The set of all invertible transformers coincides with Clifford group $\Gamma(V)$.* $\qquad\square$

Lemma 1. *If a is an invertible transformer, then for every $w \in V^1$ we have*

$$\sigma_a(w) \overset{df}{=} awa^\tau \in V^1. \tag{4.19}$$

Proof. We first notice that $\tau(w) = w, \forall w \in V^1$. Applying τ to both sides of the defining equation (4.18) we get $wa^\tau = a^\nu w'$. Multiplying by a from the left, we get $awa^\tau = aa^\nu w'$. But since $\Delta(a) = \Delta(\pi(a)) = a^\nu a = aa^\nu \in \mathbb{R}$, we get $awa^\tau = \Delta(a)w' \in V^1$. $\qquad\square$

Motivated by the above lemma we define the subsets $\mathcal{M}, \mathcal{M}_+ \in \mathcal{C}$ as follows:

$$\mathcal{M} \;=\; \{a \in \mathcal{C} : aV^1 a^\tau \subset V^1\}. \tag{4.20}$$
$$\mathcal{M}_+ \;=\; \{a \in \mathcal{M} : a > 0,\ \Phi(a) = 1\} \tag{4.21}$$

Definition 1. *We define the following important subsets of $\mathcal{T}$ and of $\mathcal{M}_+$:*

$$\mathcal{G} \;=\; \{a \in \mathcal{T} : \Delta(a) = 1\}, \tag{4.22}$$
$$\mathcal{G}_R \;=\; \{a \in \mathcal{G} : aa^\tau = 1\}, \tag{4.23}$$
$$\mathcal{G}_+ \;=\; \{a \in \mathcal{G} : a \geq 0\}, \tag{4.24}$$

$$\mathcal{M}_{1+} \;=\; \{a \in \mathcal{M}_+ : \Delta(a) > 0\}, \tag{4.25}$$
$$\bar{\mathcal{M}}_{1+} \;=\; \{a \in \mathcal{M}_+ : \Delta(a) \geq 0\}, \tag{4.26}$$
$$\mathcal{M}_{0+} \;=\; \{a \in \mathcal{M}_+ : \Delta(a) = 0\}. \tag{4.27}$$

Notice that, by the Theorem 2, $\mathcal{G}$ is invariant under both π and τ. It is sometimes denoted as $Spin_0(V)$, and the map σ (cf. 4.19) is a two–fold covering homomorphism: $\mathcal{G} \mapsto SO^+(V^1, Q^1)$ – cf. [19, 6.12]. Thus $\mathcal{G}$ is nothing but $Spin^{1^+}$. $\mathcal{G}_R$ leaves the subspace $V \subset V^1$ invariant and σ, when restricted to $\mathcal{G}_R$, is a two–fold covering homomorphism of $SO(V, Q)$. The elements of $\mathcal{G}_+$, that will be studied in our paper

in some details, will be called *spin–boosts*. $\mathcal{M}$ is a multiplicative semigroup, and $\mathcal{G} \subset \mathcal{M}$. We will show that $\mathcal{M}_{0+}$ is naturally isomorphic to the unit sphere S^n, while $\mathcal{M}_{1+}$ (resp. $\bar{\mathcal{M}}_{1+}$) corresponds to the open unit ball B^{n+1} (resp its closure $\bar{B}^{n+1}$).

Lemma 2. *Let* $a \in V^1$, $a \neq 0, 1$, $\Phi(a) = 1$, $\Delta(a) \geq 0$. *Then* $a > 0$, *and* a *is of the form* $a = 1 + \alpha \mathbf{n}$, $0 < \alpha \leq 1$, $\mathbf{n} \in V$, $\mathbf{n}^2 = 1$. *If* $\Delta(a) > 0$, *then*

$$\sqrt{a} = \frac{1}{\sqrt{1+\epsilon^2}}(1 + \epsilon \mathbf{n}), \quad \text{where} \quad \epsilon = \frac{1 - \sqrt{1-\alpha^2}}{\alpha}. \tag{4.28}$$

If $\Delta(a) = 0$, *then* $\alpha = 1$, $a = 1 + \mathbf{n}$, *and* $\sqrt{a} = \frac{1}{\sqrt{2}}a$.

Proof. Since $a \in V^1$, $a = x^0 + \mathbf{x}$, $x^0 \in \mathbb{R}$, $\mathbf{x} \in V$. Since $\Phi(a) = x^0$, and $\Delta(a) = (x^0)^2 - \mathbf{x}^2$, it follows that $x^0 = 1$, $\mathbf{x}^2 \leq 1$. Let us write a as $a = 1 + \alpha \mathbf{n}$, $0 < \alpha \leq 1$, $\mathbf{n}^2 = 1$. Consider first the case of $\Delta(a) = 1 - \alpha^2 > 0$, i.e. $\alpha < 1$. Let $b \overset{df}{=} \frac{1}{\sqrt{1+\epsilon^2}}(1 + \epsilon \mathbf{n})$, where $\epsilon = \frac{1-\sqrt{1-\alpha^2}}{\alpha}$. Then $b^\tau = b$, and, by simple algebra, we get $0 < \epsilon < 1$, $b^\tau b = b^2 = a$. Thus $a > 0$. But now b has the same form as a (up to a positive multiplicative factor), therefore also $b > 0$. Then $b = \sqrt{a}$ follows from the uniqueness of a positive square root of a positive element. If $\Delta(a) = 0$, i.e. $a = 1 + \mathbf{n}$, then $a^2 = (1 + \mathbf{n})^2 = 1 + 2\mathbf{n} + \mathbf{n}^2 = 2a$, and therefore $a > 0$, and $\sqrt{a} = a/\sqrt{2}$. $\square$

The following proposition characterizes explicitly the sets $\mathcal{M}_{0+}, \mathcal{M}_{1+}, \bar{\mathcal{M}}_{1+}$.

Proposition 3. *Let* $P : V \supset \bar{B}^{n+1} \to V^1 \subset \mathcal{C}$ *be the map*

$$P(\mathbf{x}) = 1 + \mathbf{x}, \quad \mathbf{x} \in \bar{B}^{n+1}. \tag{4.29}$$

Then P *is a bijection* $P : \bar{B}^{n+1} \to \bar{\mathcal{M}}_1^+$, $P(S^n) = \mathcal{M}_{0+}$, *and* $P(B^{n+1}) = \mathcal{M}_1^+$.

Proof. If $\mathbf{x} = 0$, then $P(\mathbf{x}) = 1$, which is evidently in $\bar{\mathcal{M}}_1^+$. Let us therefore assume $0 < \mathbf{x}^2 \leq 1$. With $a = P(\mathbf{x})$, we have $a = a^\tau$, $\Phi(a) = 1$, $\Delta(a) = 1 - \mathbf{x}^2 \geq 0$, therefore, by Lemma 2, $a > 0$. Moreover, by a simple calculation, we find that if $w = y^0 + \mathbf{y} \in V^1$, then

$$awa = y^0(1 + \mathbf{x}^2) + 2(\mathbf{x} \cdot \mathbf{y}) + (1 - \mathbf{x}^2)\mathbf{y} + 2(y^0 + (\mathbf{x} \cdot \mathbf{y}))\mathbf{x} \in V^1. \tag{4.30}$$

Therefore $a \in \bar{\mathcal{M}}_1^+$. To show that P is a surjection onto $\bar{\mathcal{M}}_{1+}$, let a be an arbitrary element in $\bar{\mathcal{M}}_{1+}$. Then $a^2 = a(1 + \mathbf{0})a$ must be in V^1. Let us therefore write $a^2 = y^0 + \mathbf{y}$. Now $y^0 = \Phi(a^2) > 0$, and $\Delta(a^2) = \Delta(a)^2 \geq 0$, Therefore, we can write $a^2 = y^0(1 + \alpha \mathbf{n})$, $\alpha \leq 1$. Then it follows from Lemma 2 that a^2 has a square root in V^1 and, because of the uniqueness of the square root, a itself must be in V^1. But, since $\Phi(a) = 1$, and $\Delta(a) \geq 0$, it follows that $a = 1 + \mathbf{x}$, $\mathbf{x}^2 \leq 1$. This shows that P is a bijection. The remaining statements follow from $\Delta(P(\mathbf{x})) = \Delta(1 + \mathbf{x}) = 1 - \mathbf{x}^2$. $\square$

The following proposition and its corollary describe the set of spin–boosts $\mathcal{G}_+$, and the Iwasawa–type decomposition of $\mathcal{G}$.

Proposition 4. $m \in \mathcal{G}_+$ *if and only if m is of the form*

$$m = \frac{1 + \alpha\mathbf{n}}{\sqrt{1 - \alpha^2}}, \quad \mathbf{n} \in S^n, \quad 0 \leq \alpha < 1. \tag{4.31}$$

An equivalent form is that of

$$m = \exp\left(\frac{\eta}{2}\mathbf{n}\right), \quad \alpha = \tanh(\eta/2), \eta > 0. \tag{4.32}$$

Proof. The sufficient condition: With m, $\mathbf{n}$, and α as in (4.31), it follows from the Proposition 3 that $1 + \alpha\mathbf{n} > 0$. On the other hand $\Delta(m) = 1$, thus $m \in \mathcal{G}_+$. On the other hand, since $\mathbf{n}^2 = 1$, is easy to calculate the exponential in (4.32), the result being:

$$\exp\left(\frac{\eta}{2}\mathbf{n}\right) = \cosh(\eta/2) + \sinh(\eta/2)\mathbf{n}. \tag{4.33}$$

It is then easy to see that by setting $\alpha = \tanh(\eta/2)$, $\eta > 0$, we recover (4.31).
The necessary condition. We can assume that $m \neq 1$. Suppose $m \in \mathcal{G}_+$, then $\Phi(m) > 0$, and thus $m/\Phi(m) \in \mathcal{M}_1^+$. It follows from the Lemma 5 that m is proportional to $1 + \alpha\mathbf{n}$, $0 < \alpha < 1$, $\mathbf{n}^2 = 1$. Then, from $\Delta(m) = 1$ it follows that the proportionality coefficient is $1/\sqrt{1 - \alpha^2}$. $\square$

Corollary 1. $\mathcal{G} = \mathcal{G}_+\mathcal{G}_R$. *Every element $g \in \mathcal{G}$ has a unique decomposition into the product*

$$g = mu, \quad m \in \mathcal{G}_+, u \in \mathcal{G}_R. \tag{4.34}$$

Proof. Let $g \in \mathcal{G}$. If $g = 1$, then there is nothing to prove, as we take $m = 1, u = 1$. Let us therefore assume $g \neq 1$. Using the Polar Decomposition Theorem (cf. e.g. [22, p. 153]), g can be written, in a unique way, as $g = mu$, where $m^2 = gg^\tau > 0$, and $uu^\tau = u^\tau u = 1$. We need to show that $m \in \mathcal{G}_+$ and $u \in \mathcal{G}_R$. Now, since $\mathcal{G}$ is invariant under τ, it follows that $m^2 = gg^\tau \in \mathcal{G}_+$. Therefore, by (i), m^2 can be written as $m^2 = \exp(\eta\mathbf{n}/2)$ and, from the uniqueness of the square root, $m = \exp(\eta\mathbf{n}/4)$. Therefore $m \in \mathcal{G}_+$. It follows that $u = m^{-1}g \in \mathcal{G}$, and so $u \in \mathcal{G}_R$. $\square$

Remark 3. *The decomposition given in (4.34) corresponds to the well known decomposition of Lorentz transformations into "boosts" and "space rotations." The special case of $n = 2$, and $SO^+(1,3)$, though not at the Clifford algebra level, is treated in details in Ref. [23]*

Let us now describe the action of the group $\mathcal{G}$ on $\bar{B}^{n+1}$. We will need the following lemma, which is the result of a simple, though somewhat lengthy, calculation in the Clifford algebra $\mathcal{C}$.

Lemma 3. *If $m = P(\alpha\mathbf{n})/\sqrt{1 - \alpha^2} \in \mathcal{G}_+$, then for all $\mathbf{x} \in \bar{B}^{n+1}$ we have*

$$P(\alpha\mathbf{n})(1 + \mathbf{x})P(\alpha\mathbf{n}) = (1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x}))(1 + \mathbf{x}'), \tag{4.35}$$

$$m(1 + \mathbf{x})m = \frac{1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x})}{1 - \alpha^2}(1 + \mathbf{x}'), \tag{4.36}$$

where

$$\mathbf{x}' = \frac{(1 - \alpha^2)\mathbf{x} + 2\alpha(1 + \alpha\,(\mathbf{n} \cdot \mathbf{x}))\mathbf{n}}{1 + \alpha^2 + 2\alpha\,(\mathbf{n} \cdot \mathbf{x})}. \tag{4.37}$$

$\square$

Before stating the next theorem let us notice that if $\mathbf{x}^2 \leq 1$, then $P(\mathbf{x}) = 1 + \mathbf{x} > 0$. If $g \in \mathcal{G}$, then also $gP(\mathbf{x})g^\tau > 0$ and, therefore, $\Phi(gP(\mathbf{x})g^\tau) > 0$. Since $\mathcal{G} \subset \mathcal{M}$, and since $\mathcal{M}$ is a multiplicative semigroup, it follows that $gP(\mathbf{x})g^\tau \in \mathcal{M}$, and therefore $gP(\mathbf{x})g^\tau/\Phi(gP(\mathbf{x})g^\tau) \in \bar{\mathcal{M}}_{1+}$.

We also recall the definition of a conformal transformation (see e.g. Ref. [24, Ch. 3.7]

Definition 2. *A diffeomorphism ϕ of a Riemannian manifold (M, G) is called a conformal transformation if there is a function $\rho > 0$ on M such that*

$$(\phi^*G)_{\alpha\beta} = \rho^2 G_{\alpha\beta}.$$

If $n = \dim(M) \geq 3$ then the group of conformal transformations of M is a Lie group of $\dim \leq \frac{(n+1)(n+2)}{2}$, and for the spheres S^n, that are of particular interest in our paper, the upper limit is reached – cf. e.g. [25, Note 11, p. 309] and also references in [20, Ch. 2].

Remark 4. *The case of $n = 2$ is exceptional, as in this case every complex analytic transformation of the complex plane generates a conformal transformation on the Riemann sphere. In this case it is better to deal with the subgroup of all conformal transformations of S^2, called "Möbius transformations." These are the transformations of S^n that preserve cross–ratios*

$$\frac{d(u, x)d(v, y)}{d(u, v)d(x, y)}, \qquad u, v, x, y \in S^n,$$

where d is the natural distance on S^n. More information about various, equivalent definitions and properties of Möbius transformations of S^n and of B^{n+1} can be found, for example, in Refs. [26, Ch. 4] and [20, Ch. 2].

Theorem 3. (i) *Let for each $g \in \mathcal{G}$, let $\phi_g : \bar{B}^{n+1} \to \bar{B}^{n+1}$ be defined by*

$$\phi_g(\mathbf{x}) = P^{-1}\left(\frac{\sigma_g(P(\mathbf{x}))}{\Phi(\sigma_g(P(\mathbf{x})))}\right). \tag{4.38}$$

Then $g \mapsto \phi_g$ is a homomorphism from $\mathcal{G}$ onto a group of transformations of $\bar{B}^{n+1}$.

(ii) *If $m \in \mathcal{G}_+$ is written as in (4.31): $m = (1 + \alpha\mathbf{n})/\sqrt{1 - \alpha^2}$, then the Möbius transformation ϕ_m is explicitly given by the formula:*

$$\phi_m(\mathbf{x}) = \frac{(1 - \alpha^2)\mathbf{x} + 2\alpha(1 + \alpha\,(\mathbf{n} \cdot \mathbf{x}))\mathbf{n}}{1 + \alpha^2 + 2\alpha\,(\mathbf{n} \cdot \mathbf{x})}, \quad \mathbf{x} \in \bar{B}^{n+1}. \tag{4.39}$$

(iii) *When restricted to the unit sphere S^n, ϕ is a two–fold covering homomorphism from $\mathcal{G}$ onto the group of Möbius transformations of S^n.*

(iv) *For* $m = (1 + \alpha\mathbf{n})/\sqrt{(1 - \alpha^2)} \in \mathcal{G}_+$, *the map* $\phi_m : S^n \ni \mathbf{x} \mapsto \mathbf{x}' \in S^n$, *given by* (4.37), *is conformal with the conformal factor*

$$\rho = \frac{(1 - \alpha^2)}{(1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x}))}. \tag{4.40}$$

That is, if $G = (G_{\alpha\beta})$ *is the natural Riemannian metric on the unit sphere then*

$$(\phi_m^* G)_{\alpha\beta} = \frac{(1 - \alpha^2)^2}{(1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x}))^2} \, G_{\alpha\beta}. \tag{4.41}$$

Thus ϕ_m *does not, in general, preserve the canonical,* $SO(V)$*–invariant, volume form* dS *of* S^n. *Denoting by* dS' *the pullback[5]* $\phi_m^*(dS)$ *of* dS *by* ϕ_m, *for every* $\mathbf{x} \in S^n$ *we have:*

$$\frac{dS'}{dS}(\mathbf{x}) = \left(\frac{1 - \alpha^2}{1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x})} \right)^n. \tag{4.42}$$

If the map (4.37) *is applied to the ball* $B^{(n+1)}$ *(rather than to its boundary* S^n*), and if* dV *denotes the standard Euclidean volume form of* V^1, *then*

$$\frac{dV'}{dV} = \left(\frac{1 - \alpha^2}{1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x})} \right)^{n+2}. \tag{4.43}$$

Remark 5. *It is easy to see that our definition of conformal (Möbius) transformations of* S^n *is equivalent to one given by Pierre Anglès in Ref.* [20, 2.4.1,2.4.2.1]. *In particular* $\mathcal{M}_{0+}$ *can be identified with* $P(Q^1 - \{0\})$ *in the notation of Ref.* [20]. *But we do not need the stereographic projection that distinguishes the vector* $e_{n+1} \in V$.

Remark 6. *The transformations* $\phi_g : \bar{B}^{n+1} \to \bar{B}^{n+1}$, *defined in* (4.38) *are also called Poincaré extensions of those restricted to* S^n *– cf.* [26, Ch. 4.4, 4.5].

Proof. (i) That ϕ_g is a group homomorphism follows directly from the defining formula. In order to show that each ϕ_g maps S^n onto S^n, we first notice that from $\Delta(P(\mathbf{x})) = 1 - \mathbf{x}^2$, it follows that $\mathbf{x} \in S^n$ if and only if $\Delta(P(\mathbf{x})) = 0$. If $\Delta(P(\mathbf{x})) = 0$, then, since $\Delta(g) = \Delta(g^\tau) = 1$, also $\Delta(gP(\mathbf{x})g^\tau) = \Delta(g)^2\Delta(P(\mathbf{x})) = \Delta(P(\mathbf{x})) = 0$, thus $\phi_g(S^n) \subseteqq S^n$. In fact, since $g^{-1} = g^\nu \in \mathcal{G}$, we have that $\phi_g(S^n) = S^n$.
(ii) Follows from (4.36).
(iii) Let us show that ϕ so restricted to S^n has kernel $\mathbb{Z}_2$. We first notice that if $g \in \ker\phi$ then $g^\tau \in \ker\phi$. Indeed, from the very definition of ϕ it follows that $g \in \ker\phi$ if and only if $g(1 + \mathbf{n})g^\tau$ is proportional to $1 + \mathbf{n}$ for all $\mathbf{n} \in S^n$:

$$g(1 + \mathbf{n})g^\tau = \lambda(1 + \mathbf{n}).$$

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[5]Let us recall that if $\phi : M \longrightarrow N$ is a C^1 map between differentiable manifolds M and N, and if ω is a k–form on N, then its pullback $\phi^*(\omega)$ is the k–form on M defined by $\phi^*(\omega)(\xi_{1_p}, \ldots, \xi_{k_p}) = \omega(d\phi_p(\xi_{1_p}), \ldots, d\phi_p(\xi_{k_p}))$ for all $\xi_{1_p}, \ldots, \xi_{k_p} \in T_p(M)$, $p \in M$ where $d\phi_p : T_pM \longrightarrow T_{\phi(p)}N$ is the derivative of ϕ at p. For a composition of maps we have $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ – cf. e.g. [27, Ch. XVI.20].

By applying π to both sides of this equation, we get

$$\pi(g)(1 - \mathbf{n})g^\nu = \lambda(1 - \mathbf{n}).$$

Now, multiplying by g^τ from the left, and by g from the right, and taking into account the fact that $\Delta(g) = g^\nu g = \Delta(g^\tau) = g^\tau \pi(g) = 1$, we find $g^\tau(1 - \mathbf{n})g = (1/\lambda)(1 - \mathbf{n})$, and, since $\mathbf{n} \in S^n$ is arbitrary, $g^\tau \in ker\phi$. Now, assuming that $g \in ker\phi$, let $g = mu$ be the decomposition of g into a spin–boost $m \in \mathcal{G}_+$ and a rotation $u \in \mathcal{G}_R$. Then $g^\tau = u^\tau m \in ker\phi$, and, since the kernel of a group homomorphism is a group, we get $m^2 = gg^\tau \in ker\phi$, i.e. $\phi_{m^2}(\mathbf{x}) = \mathbf{x}, \quad \mathbf{x} \in S^n$. Let us write m^2 as $m^2 = (1 + \alpha\mathbf{n})/\sqrt{1 - \alpha^2}$ and, since we have assumed that $dim(V) \geq 2$, we can choose for $\mathbf{x}$ a unit vector in V, orthogonal to $\mathbf{n}$. Then, from (4.37) we get

$$\mathbf{x} = \phi_{m^2}(\mathbf{x}) = \frac{(1 - \alpha)^2\mathbf{x} + 2\alpha\mathbf{n}}{1 + \alpha^2},$$

which is possible only for $\alpha = 0$, i.e. if $m^2 = 1$. But then, from the uniqueness of the square root, $m = 1$, and so $g = u$. Now, $u(1 + \mathbf{x})u^\tau = (1 + \mathbf{x})$ implies $u\mathbf{x}u^\tau = \mathbf{x}$, which extends, by simple scaling to all $\mathbf{x} \in V$. Since $u^\tau \pi(u) = \Delta(u^\tau) = \Delta(u) = 1$, the last equation can be rewritten as $u\mathbf{x} = \mathbf{x}\pi(u)$, and it follows from [19, Lemma 5.25] that $u \in \mathbb{R}$. Then, since $\Delta(u) = 1$, we get $u^2 = 1$, so that $u = \pm 1$. That the homomorphism ϕ is surjective, as its image is a connected Lie group of conformal transformations of dimension equal to that of $Spin^+(1, n + 1)$, that is $(n + 2)(n + 1)/2$ - cf. Definition 2 and Remark 4.

(iv) Let us endow V with an orthonormal basis $e_1, \ldots, e_{n+1}$, and the corresponding coordinates $x^1, \ldots, x^{n+1}$. Let $G = (G_{ij} = \delta_{ij})$ be the natural Riemannian metric in V. From (4.37) it is then easy to compute $G^*_{ij} = (\phi^*_m G)_{ij} = \frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j}\delta_{kl}$. The result is

$$G^*_{ij} = \rho^2\left(\delta_{ij} + \frac{4\alpha^2(\mathbf{x}^2 - 1)}{f^2}\,n_i n_j - \frac{2\alpha}{f}(n_i x_j + n_j x_i)\right). \qquad (4.44)$$

where

$$f = 1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x}), \qquad \rho = \frac{1 - \alpha^2}{f}. \qquad (4.45)$$

If $\mathbf{v} = (v^i)$ and $\mathbf{w} = (w^i)$ are vectors tangent to S^n, so that $(\mathbf{v} \cdot \mathbf{n}) = (\mathbf{w} \cdot \mathbf{n}) = 0$ then, when computing $G^*_{ij}v^i w^j$, the two last terms vanish, and we obtain $G^*_{ij}v^i w^j = \rho^2 G_{ij}v^i w^j$, which proves (4.41). (4.42) follows immediately from (4.41). It is also easy to calculate the determinant of the matrix G^*. It has eigenvalue equal to ρ^2 on the whole $(n - 1)$– dimensional subspace orthogonal to $\mathbf{n}$ and $\mathbf{x}$, while the product of its two eigenvalues in the subspace spanned by $\mathbf{n}$ and $\mathbf{x}$ is equal to ρ^4. So the determinant is $\rho^{2(n+1)}$, and the square root of the determinant is ρ^{n+2}, which proves (4.43).[6]   □

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[6] The same way one gets (4.43) also for $\mathbf{x}\|\mathbf{n}$. An alternative method of proving (4.42) and (4.43), using $(n + 1)$– dimensional polar coordinates can be found in a previous version of this paper, available as an arxiv preprint [29]

4.2. Stereographic projection

In order to get a better insight into the geometrical nature of our transformations, and also to understand why, in (4.32), following Ref. [20], we have used $\eta/2$, rather than just η as the parameter of the exponential, it is instructive to discuss the action of our transformations on the stereographic projection of the sphere S^n. As before, we fix the vector $\mathbf{n} \in S^n$, and let $s_\mathbf{n}$ be the stereographic projection from S^n onto the hyperplane through the origin of V, orthogonal to $\mathbf{n}$, with the origin at $\mathbf{n}$. Explicitly, we have

$$s_\mathbf{n}(\mathbf{x}) = \frac{\mathbf{x} - (\mathbf{n} \cdot \mathbf{x})\mathbf{n}}{1 - (\mathbf{n} \cdot \mathbf{x})}, \quad \mathbf{x} \in S^n. \tag{4.46}$$

Indeed, the vector $s_\mathbf{n}(\mathbf{x})$ is on the straight line connecting $\mathbf{n}$ and $\mathbf{x}$, and is orthogonal to $\mathbf{n}$, which two properties uniquely characterize the stereographic projection. Let us recall now the action of ϕ_m on S^n. From the formula (4.37) we have:

$$\mathbf{x}' = \frac{(1 - \alpha^2)\mathbf{x} + 2\alpha(1 + \alpha\,(\mathbf{n} \cdot \mathbf{x}))\mathbf{n}}{1 + \alpha^2 + 2\alpha\,(\mathbf{n} \cdot \mathbf{x})}. \tag{4.47}$$

Let us compare now $s_\mathbf{n}(\mathbf{x}')$ with $s_\mathbf{n}(x)$. By a straightforward calculation we obtain:

$$(\mathbf{n} \cdot \mathbf{x}') = \frac{2\alpha + (1 + \alpha^2)\mathbf{n} \cdot \mathbf{x}}{1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x})}, \tag{4.48}$$

$$1 - (\mathbf{n} \cdot \mathbf{x}') = \frac{(1 - \alpha)^2(1 - (\mathbf{n} \cdot \mathbf{x}))}{1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x})}, \tag{4.49}$$

$$\mathbf{x}' - (\mathbf{n} \cdot \mathbf{x}')\mathbf{n} = \frac{(1 - \alpha^2)(\mathbf{x} - (\mathbf{n} \cdot \mathbf{x})\mathbf{n})}{1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x})}, \tag{4.50}$$

and therefore

$$\begin{aligned}
s_\mathbf{n}(\mathbf{x}') &= \frac{\mathbf{x}' - (\mathbf{n}\cdot\mathbf{x}')\mathbf{n}}{1 - (\mathbf{n}\cdot\mathbf{x}')} = \frac{(1-\alpha^2)(\mathbf{x}-(\mathbf{n}\cdot\mathbf{x})\mathbf{n})}{(1-\alpha)^2 + (1-\alpha^2)(\mathbf{n}\cdot\mathbf{x})} \\
&= \frac{1-\alpha^2}{(1-\alpha)^2}\,\frac{\mathbf{x}-(\mathbf{n}\cdot\mathbf{x})\mathbf{n}}{1-(\mathbf{n}\cdot\mathbf{x})} = \frac{1+\alpha}{1-\alpha}\,s_n(\mathbf{x}).
\end{aligned} \tag{4.51}$$

Now, since $\alpha = \tanh(\eta/2)$, we have

$$\frac{1 + \alpha}{1 - \alpha} = \frac{\cosh(\eta/2) + \sin(\eta/2)}{\cosh(\eta/2) - \sinh(\eta/2)} = \frac{2\exp(\eta/2)}{2\exp(-\eta/2)} = \exp(\eta),$$

and therefore

$$s_\mathbf{n}(\mathbf{x}') = e^\eta s_\mathbf{n}(\mathbf{x}), \tag{4.52}$$

so that the family of Möbius transformations $g_\mathbf{n}(\epsilon)$, when parametrized by $\eta = 2\operatorname{arctanh}(\alpha)$, act as a one–parameter group of uniform dilations on the stereographic projection $s_\mathbf{n}(S^n) = \mathbb{R}^n$.

5. Iterated Function Systems of Conformal Maps

Let S be a set, let $\{w_i : i = 1, 2, \ldots, N\}$ be a family of maps $w_i : S \longrightarrow S$, and let $p_i(x)$, $i = 1, 2, \ldots, N$ be positive functions on S satisfying $\sum_{i=1}^{N} p_i(x) = 1$, $\forall x \in S$. The maps w_i and the functions $p_i(x)$ define what is called an *iterated function system (IFS) with place dependent probabilities* - cf. [30]. Starting with an initial point x_0 we select one of the transformations w_i with the probability distribution $p_i(x_0)$. If w_{i_1} is selected, we get the next point $x_1 = w_{i_1}(x_0)$, and we repeat the process again, selecting the next transformation w_{i_2}, according to the probability distribution $p_i(x_1)$. By iterating the process we produce a random sequence of integers $i_0, i_1, \ldots$ and a random sequence of points $x_k = w_{i_k}(x_{k-1}) \in S$, $k = 1, 2, \ldots$. In interesting cases the sequence x_k accumulates on an "attractor set" which has fractal properties. Instead of looking at the points of S we can take a dual look at the functions on S. Let $\mathcal{F}(S)$ be the set of all real–valued functions on S. $\mathcal{F}(S)$ is a vector space, and each transformation $w : S \to S$ induces a *linear* transformation $w^\star : \mathcal{F}(S) \to \mathcal{F}(S)$ defined by $(w^\star f)(x) = f(w(x))$, $x \in S$, $f \in \mathcal{F}(S)$.

5.1. Markov operator

Given an iterated function system $\{w_i, p_i(\,.\,)\}$ on S one naturally associates with it a linear *Markov operator* (sometimes called also the *transfer* operator) $T^* : \mathcal{F}(S) \to \mathcal{F}(S)$ defined by

$$(T^* f)(x) = \sum_{i=1}^{N} p_i(x)(w_i^* f)(x) = \sum_{i=1}^{N} p_i(x) f(w_i(x)). \qquad (5.53)$$

There is a dual Markov operator T_*, acting on measures on S. Suppose S has a measurable structure, w_i and $p_i(\,.\,)$ are measurable, and let $\mathcal{F}(S)$ be the space of all bounded measurable functions on S. Let $\mathcal{M}(S)$ be the space of all finite measures on S. Then $T_* : \mathcal{M}(S) \to \mathcal{M}(S)$ is defined by duality: $(T_*\mu, f) = (\mu, T^* f)$, where $(\mu, f) \doteq \int f d\mu$. Since $T^*(1) = 1$, where $1(x) = 1$, $\forall x \in S$, we have that $\int dT_*\mu = \int d\mu$ and, in particular, T_* maps probabilistic measures into probabilistic measures. In many interesting cases the sequence of iterates $(T_*)^k \mu$ converges, in some appropriate topology, to a limit $\mu_\infty = \lim_{k \to \infty}(T_*)^k \mu$, that is independent of the initial measure μ, and which is the unique fixed point of T_*. The support set of μ_∞ is then the attractor set mentioned above.

Let μ_0 be a fixed, normalized measure on S, and assume that the maps w_i^{-1} map sets of measure μ zero into sets of measure μ zero. Then, for any finite k, the measure $T_\star^{\,k} \mu_0$ is continuous with respect to μ_0 and therefore can be written as

$$T_\star^{\,k} \mu_0(\mathbf{r}) = f_k(\mathbf{r})\, \mu_0(\mathbf{r}). \qquad (5.54)$$

The sequence of functions $f_k(\mathbf{r})$ gives a convenient graphic representation of the limit invariant measure. In our case, as it follows from the formula (5.54), the maps w_i are bijections, and the functions f_k can be computed explicitly via the

following recurrence formula:

$$f_{k+1}(\mathbf{r}) = \sum_{i=1}^{N} p_i\left(w_i^{-1}(\mathbf{r})\right) \frac{d\mu_0\left(w_i^{-1}(\mathbf{r})\right)}{d\mu_0(\mathbf{r})} f_k\left(w_i^{-1}(\mathbf{r})\right). \tag{5.55}$$

5.2. Conformal maps

In this section the set S is either the sphere S^n, or the closed ball $\bar{B}^{n+1}$, and the maps w are of the form (4.37), and are determined by vectors $\alpha\mathbf{n} \in B^{(n+1)}$. Let us choose one α, $0 < \alpha < 1$, and N unit vectors $\mathbf{n}_i \in S^n$, so that we have N maps

$$w_i(\mathbf{x}) = \frac{(1-\alpha^2)\mathbf{x} + 2\alpha(1+\alpha(\mathbf{n}_i \cdot \mathbf{x}))\mathbf{n}_i}{1+\alpha^2 + 2\alpha(\mathbf{n}_i \cdot \mathbf{x})}, \tag{5.56}$$

as in Proposition 4. In our case we have an additional structure in the set S and in the maps w_i, namely the one stemming from the Clifford algebra realization. First of all to each $\mathbf{x} \in S^n$ we have associated the idempotent $\frac{1}{2}P(\mathbf{x})$, where $P(\mathbf{x}) = (1 + \mathbf{x})$, and then we have a special class of functions on S, namely the functions of the form:

$$f_a(\mathbf{x}) = (P(\mathbf{x}), a),\ a \in \mathcal{C},\ \mathbf{x} \in \bar{B}(n+1). \tag{5.57}$$

We denote by $\mathcal{L}$ the vector space of these functions. Notice that functions in $\mathcal{L}$ separate the points $\mathbf{x} \in \bar{B}^{(n+1)}$. Indeed, for $\mathbf{x}, \mathbf{y} \in \bar{B}^{(n+1)}$ we have $f_\mathbf{y}(\mathbf{x}) = \mathbf{x} \cdot \mathbf{y}/2$, thus our statement reduces to: for any two different vectors $\mathbf{x}_1, \mathbf{x}_2$ one can always find another vector $\mathbf{y}$ such that $\mathbf{x}_1 \cdot \mathbf{y} \neq \mathbf{x}_2 \cdot \mathbf{y}$, which is evident. [7]

Proposition 5. *With the notation as in the beginning of this section, let $0 < \alpha < 1$, $\mathbf{n}_i \in S^n$, $i = 1, 2, \ldots N$ and w_i as in (5.56). Suppose that*

1)
$$\sum_{i=1}^{N} \mathbf{n}_i = 0, \tag{5.58}$$

2)
$$p_i(\mathbf{x}) = \frac{1+\alpha^2 + 2\alpha(\mathbf{n}_i \cdot \mathbf{x})}{Z(\alpha)}, \tag{5.59}$$

   where

$$Z(\alpha) = \sum_{i=1}^{N}(1+\alpha^2 + 2\alpha(\mathbf{n}_i \cdot \mathbf{x})) = N(1+\alpha^2),$$

then the Markov operator T^* of the iterated function system $\{(w_i, p_i)\}$ maps the space $\mathcal{L}$ into itself: $T^* : f_a \mapsto f_V(a)$, where

$$V(a) = \frac{1}{N(1+\alpha^2)} \sum_{i=1}^{N} P(\alpha\mathbf{n}_i)\, a\, P(\alpha\mathbf{n}_i). \tag{5.60}$$

---

[7]The space $\mathcal{L}$ is $(n+2)$–dimensional, as it is clear that $f_a(\mathbf{x}) = 0$, $\forall\, a \in C_p \subset \mathcal{C}$, $p > 1$.

Proof. From (4.36) it follows that if $\sum_i \mathbf{n}_i = 0$, then $Z \doteq \sum_i (1 + \alpha^2 + 2\alpha(\mathbf{n}_i \cdot \mathbf{x})) = N(1 + \alpha^2)/(1 - \alpha^2)$ is a constant, independent of $\mathbf{x}$. From the very definition of the Markov operator, as well as from (5.57), (4.35) it follows then that

$$
\begin{aligned}
(T^* f_a)(\mathbf{x}) &= \sum_i p_i(\mathbf{x}) f_a(w_i(\mathbf{x})) = \sum_i p_i(\mathbf{x}) \Phi(a\, P(w_i(\mathbf{x}))) \\
&= \sum_i p_i(\mathbf{x}) \Phi\left(a \frac{1 - \alpha^2}{(1 + \alpha^2 + 2\alpha(\mathbf{n}_i \cdot \mathbf{x}))} P(\alpha\mathbf{n}_i) P(\mathbf{x}) P(\alpha\mathbf{n}_i) \right) \\
&= \sum_i p_i(\mathbf{x}) \frac{(1 - \alpha^2)}{1 + \alpha^2 + 2\alpha(\mathbf{n}_i \cdot \mathbf{x})} \Phi\left(P(\alpha\mathbf{n}_i) a P(\alpha\mathbf{n}_i) P(\mathbf{x}) \right) \\
&= \frac{1}{Z(\alpha)} \sum_i \Phi\left(P(\alpha\mathbf{n}_i) a P(\alpha\mathbf{n}_i) P(\mathbf{x}) \right) = f_{V(a)}(\mathbf{x}). \qquad \square
\end{aligned}
$$

The Markov operator T^* acts on measures, while its dual T^* acts on functions on S. Every probabilistic measure μ on S determines an algebra element $P(\mu)$ defined by:

$$
P(\mu) = \int_S P(\mathbf{x})\, d\mu(\mathbf{x}) = 1 + \int_S \mathbf{x}\, d\mu(\mathbf{x}) = P\left(\int_S \mathbf{x}\, d\mu(\mathbf{x}) \right), \qquad (5.61)
$$

so that automatically $\Phi(P(\mu)) = 1$. $P(\mu)/2$ is an idempotent if and only if μ is concentrated at just one point on the boundary S^n. In general there are infinitely many measures μ giving rise to the same algebra element $P(\mu)$. The process of integration on one hand leads to simplification (linearization) but, on the other hand, it also leads to the loss of information.

Corollary 2. *Under the assumptions* 1) *and* 2) *of Proposition* 5, *if μ_1 and μ_2 are two probabilistic measures on S such that $P(\mu_1) = P(\mu_2) = P$, then $P(T^*\mu_1) = P(T^*\mu_2) = V(P)$, where $V(P)$ is given by the formula* (5.60), *with a replaced by P.*

Proof. Because functions f_a, $a \in \mathcal{C}$ separate the elements of $\mathcal{C}$, it is enough to show that $f_a(P(T^*\mu)) = f_a(V(P(\mu)))$ for all $a \in \mathcal{C}$. Now, from the very definition of the functions f_a, $f_a(\mathbf{x}) = \Phi(aP(\mathbf{x}))$, and from the linearity of the trace functional Φ, it follows that $(f_a, \mu) \doteq \int f_a(\mathbf{x}) d\mu(x) = \Phi(aP(\mu))$, and so $f_a(V(P(\mu)) = \Phi(aV(P(\mu))) = \Phi(V(a)P(\mu)) = f_V(a)(P(\mu)) = f_a(P(T^*\mu))$. $\square$

Corollary 3. *Under the assumptions* 1) *and* 2) *of Proposition* 5, *the Markov operator recurrence formula* (5.55) *is explicitly given by*

$$
f_{k+1}(\mathbf{r}) = \frac{(1 - \alpha^2)^{n+2}}{N(1 + \alpha^2)} \sum_{i=1}^N \frac{f_k\left(w_i^{-1}(\mathbf{r}) \right)}{(1 + \alpha^2 - 2\alpha(\mathbf{n}_i \cdot \mathbf{x}))^{n+1}}, \qquad (5.62)
$$

where

$$
w_i^{-1}(\mathbf{r}) = \frac{(1 - \alpha^2)\mathbf{r} - 2\alpha(1 - \alpha(\mathbf{n}_i \cdot \mathbf{r}))\mathbf{n}_i}{1 + \alpha^2 - 2\alpha(\mathbf{n}_i \cdot \mathbf{r})}. \qquad (5.63)
$$

Proof. Follows directly by a somewhat lengthy calculation using (5.55), (5.56), (5.59), and (4.43). □

Remark 7. *Iterated function systems for mixed states have been discussed by Łozinski et al. in Ref. [31], while Słomczynski [32] discussed Markov operators and dynamical entropy for general IFS–s on state spaces. In these references the probability distributions assigned to the maps were generic rather than derived geometrically, as is the case in this paper.*

6. Examples

6.1. S^1 – Polygon

As the first example we consider the circle S^1, and unit vectors $\mathbf{n}_i$ pointing to the vertices of a regular polygon. For an illustration we choose the pentagon. Fig. 1 shows the plot of $\log_{10}(f_7 + 1.0)$, the 7–th iteration of the Markov operator – see (5.62), for $\alpha = 0.58$.

6.2. S^2

S^2, the Riemann sphere, is the same as the complex projective line $P^1(\mathbb{C})$ - the space of pure quantum states of the simplest non–trivial quantum system, namely spin 1/2. Examples of quantum fractals on S^2, based on Platonic solids, has been given elsewhere (cf. [33], and references therein). Here we give just one example, namely the octahedral quantum fractal. Fig. 2 shows the 7–th power of the Markov operator: $\log_{10}(f_7 + 1)$, - cf. (5.62) for $\alpha = 0.5$, plotted on the projection of the upper hemisphere of S^2. The emergence of circles on the plot is rather surprising and not well understood.[8]

6.3. S^3 – regular polytopes

There are six regular polytopes in four dimensions: self–dual pentachoron (or 4 simplex), 16 cell (or cross–polytope, or hexadecochoron), dual to it 8 cell (or hypercube or tesseract), self–dual 24 cell (or icositetrachoron), 600 cell (or hexacosichoron), and its dual 120 cell (or hecatonicosachoron) - cf. Fig. 3 and Fig. 8. In our examples of four dimensional quantum fractals we skip the first one. The pentachoron (the four dimensional equivalent of the tetrahedron) leads to rather trivial and uninteresting fractal pattern.

---

[8]The algorithm for generating conformal quantum fractals on S^2 has been included in the CLU-Calc software by Christian Perwass. A video zooming on a quantum fractal based on the regular octahedron, $\alpha = 0.42$, can be seen on the CLUCalc home page: `http://www.clucalc.info/`

6.4. $S^3 - 16$ cell.

Quaternions of the form $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad a, b, c, d \in \mathbb{Z}$ form the so called *Lipschitz* ring. The unit quaternions of this ring form a group of order 8 - the binary dihedral group $D4$. Its eight elements, $\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ form the four-dimensional regular polytope, the so called *cross–polytope*, with Schläfli symbol $\{3, 3, 4\}$. It has 16 tetrahedral cells, each of its 24 edges belongs to 4 cells.

Visualization of quantum fractals that live in four dimensions is difficult. Here we generate 10,000,000 points of the iterated function system described in Sec 5.2, with $\mathbf{n}_i$ being the 8 vertices of the 16 cell, $\alpha = 0.5$, and with probabilities given by (5.59). We plot the three dimensional projections of those 16742 points which fall into the slice of S^3 with the fourth coordinate $0.5 < x^4 < 0.51$ - see Fig. 4.

This pattern, generated by the IFS of conformal maps with place-dependent probabilities should be compared with the plot of the fourth approximation to the density of the limiting invariant measure - Fig. 4. Due to the recursive nature of the formula (5.62) the computation time of f_k grows exponentially with k. With each level new details appear in the graph, at the same time the probability peaks get higher (as in Fig. 6). To present more details in the graph, we are plotting $\log_{10}(f_4(\mathbf{r}) + 1)$, rather than the function $f_4(\mathbf{r})$ itself. Notice that for each k, the integral of $f_k(\mathbf{r})$ over the sphere S^3, with the natural $SO(4)$ invariant measure, is constant and equal to the volume of S^3.

6.5. $S^3 - 8$ cell.

Dual to the 16 cell is the 8 cell, also known as *cross polyhedron hypercube*, or *tesseract*. Its 16 vertices are the unit quaternions $\frac{1}{2}(\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k})$. Its Schläfli symbol is $\{4, 3, 3\}$, which means that its cells are $\{4, 3\}$ - that is cubes, each face belongs to 2 cells, and each edge belongs to 3 cells. The hypercube is built of two 3 dimensional cubes, their edges being connected along the fourth coordinate. The projection of the hypercube is shown in Fig. 3.

We choose 16 unit vectors $\mathbf{n}_i$ pointing to the vertices of the hypercube. Fig. 5 shows the plot of f_5, the 5–th iteration of the Markov operator (given by (5.62)), for $\alpha = 0.60$, restricted to the section $x^3 = 0.8$, projected onto (x^1, x^2) plane.

6.6. $S^3 - 24$ cell.

Quaternions of the form $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad a, b, c, d \in \mathbb{Z}$ or $a, b, c, d \in \mathbb{Z} + \frac{1}{2}$ form a ring, called the *Hurwitz ring*. Its additive group is the F_4 lattice. The unite quaternions of this ring form a group, the *binary tetrahedral group* $T_2 4$, isomorphic to the group $SL(2, 3)$ - with generators the same as for $SL(2, 5)$, - cf. (8.79), except that the multiplications are carried in $\mathbb{Z}_3$. 24 cell has Schläfli symbol $\{3, 4, 3\}$, which means that its 24 cells are octahedrons, with each edge belonging to three cells [34, p. 68]. Each of its 16 vertices is common to 6 cells - cf. Fig. 3. Fig. 6 shows the plots of $\log(f_k + 1)$ for $k = 2, 3, 4$, for $x^4 = 0.5$, and $\alpha = 0.6$. With each power of the Markov operator more details of the limit measure appear.

6.7. $S^3 - 600$ **cell.**

Here we provide an example of a quantum fractal on S^3, based on the regular polytope in four dimensions, namely the 600 cell, with Schläfli symbol $\{3,3,5\}$. The vertices of the 600 cell are given in the Appendix 8.3. (cf. also [34, p. 74–75].) Fig. 3 shows a two dimensional projection of the 600 cell as viewed from the direction of the center of one of its cells, while Fig. 7 (top) shows the more perfect (all 120 vertices can bee seen) Coxeter's projection. The inner ring, consisting of 30 vertices is on the torus. We show the functions $\log_{10}(f_1 + 1)$ and $log_1 0(f_2 + 1)$ plotted at the surface of this torus. The 30 highest peaks that can be seen on the bottom plots are located at the vertices.

6.8. $S^3 - 120$ **cell.**

The last example is the 120 cell, with 600 vertices. Fig. 8 (top) shows a particular projection of this polytope, with one of its 120 octahedral cells plotted in bold. Below is the plot of $\log_{10}(f_2+1)$, for $\alpha = 0.9$, at the upper hemisphere circumscribing this cell.

7. Summary and Conclusions

In the standard formulation of the quantum theory the imaginary unit i plays an important yet somewhat mysterious role: it appears in front of the Planck constant $\hbar$, and provides a one–to–one formal correspondence between Hermitian "observables" and anti–Hermitian generators of one–parameter groups of unitary transformations. In particular it is necessary in order to write the time evolution equation for the wave function, with the energy operator (the Hamiltonian) defining the evolution. But the imaginary "i" is not needed for quantum jumps. In a theory where quantum jumps are the driving force of the evolution, the real algebra structure, with a real trace functional suffices. In the present paper we have studied the simplest case of real Clifford algebras of Euclidean spaces and demonstrated that from the algebra and from the geometry a natural family of iterated function systems of conformal maps leads to fractal structures and pattern formation on spheres S^n. In this way we open a way towards algebraic generalizations of quantum theory that are based on discrete, algebraic structure, as expressed in the late Einstein's vision quoted in the Introduction.

Among the open problems we would like to point out particularly the following ones.

7.1. Existence and uniqueness of the invariant measure

While numerical simulations (see the next section), suggest that for the class of iterated function systems discussed in this paper, the attractor set and the invariant measure exists and is unique, we are not able to provide a mathematical proof. Even if the spheres S^n and balls B^{n+1} are compact, the Möbius transformations of these spheres are non-contractive. The question of existence and uniqueness of invariant measures for non-contractive iterated function systems has been discussed

in the mathematical literature [35, 36, 37], yet none of the sufficient conditions seems to be easily applicable to our case. Apanasov has a whole book devoted to conformal maps, yet we find that his criteria, esp. Theorem 4.16 of Ref. [37], are abstract and difficult to apply. Therefore the problem of existence and uniqueness of the invariant measure for IFS–s discussed in the present paper remains open at this time.

7.2. Fractal dimension as a function of the parameter ϵ.

Anticipating a positive answer to the above problem, the next important question is the exact nature of the fractal attractor as a function of the parameter ϵ. The numerical simulations seem to suggest that the fractal dimension of the attractor of our IFSs on S^n decreases, starting from n, for $\epsilon = 0$. Yet our attempt to determine its behavior, even for the simplest case of S^1, met an obstacle. We tried to calculate the correlation dimension for the pentagon case, described in Example 1. To this end we generated $N = 10,000,000$ points, using the algorithm of Sec. 5, and plotted, on the log–log scale the function $C(N,r)$, where r is the distance between two points, and $C(r)$ is the relative number of pairs, out of N points, within this distance. More precisely, the correlation dimension D is defined as

$$D = \lim_{r \to 0} \log(C(r))/\log(r), \qquad (7.64)$$

where

$$C(r) = \frac{1}{N^2} \lim_{N \to \infty} \sum_{i,j}^{N} \Theta(|r - |x_i - x_j||), \qquad (7.65)$$

Θ being the unit step function. For the standard Cantor set the correlation dimension algorithm gives the correct fractal dimension, namely $D = 0.63 \approx \log(2)/\log(3)$. For the pentagon, with $\epsilon = 0.58$, (cf. Fig. 1) we get a reasonable straight line with the slope $D \approx 0.9$, but with $\epsilon = 0.925$, when the expected fractal dimension should be close to zero, we get a staircase. It is not clear whether this is due to numerical artifacts, or is it a pointer towards the possible multifractality of quantum fractals for high values of ϵ.

8. Appendices

8.1. The boosts in $SO(1, n+1)$

Let e_μ, $\mu = 0, 1, 2, \ldots, \mu_{n+1}$, $e_0 = 1 \in \mathbb{R}$, $e_i \in V$, $i = 1, 2, \ldots n+1$ be an orthonormal basis in V^1. Then the two–fold covering homomorphism $\Lambda : \mathcal{G} \to SO^+(1, n+1)$, $g \mapsto \Lambda(g)$, can be written as $g e_\mu g^\tau = \Lambda^\nu{}_\mu$, or, more explicitly:

$$g e_0 g^\tau = \Lambda^0{}_0 e_0 + \Lambda^i e_i, \quad g e_i g^\tau = \Lambda^0{}_i e_0 + \Lambda^j{}_i e_j. \qquad (8.66)$$

If $x \in V^1$ is written in terms of the basis e_μ, $x =^\mu e_\mu$, then $x' = g x g^\tau = x'^\mu e_\mu$ is given by $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. It is then easy to see that the map $\phi_g : S^n \to S^n$, given by

(4.38), when written in terms of the representing its matrix $\Lambda(m) \in SO^+(1, n+1)$, is

$$\phi_\Lambda(\mathbf{x})^i = x'^i/x'^0 = \frac{\Lambda^i{}_0 + \Lambda^i{}_j x^j}{\Lambda^0{}_0 + \Lambda^0{}_j x^j}, \quad i, j = 1, 2, \dots, n+1 \tag{8.67}$$

where $\mathbf{x}^2 = \sum_i^{n+1} (x^i)^2 = 1$.

Proposition 6. *The map $\phi_\Lambda : S^n \longrightarrow S^n$ given by:*

$$\phi_\Lambda(\mathbf{x})^i = x'^i/x'^0 = \frac{\Lambda^i{}_0 + \Lambda^i{}_j x^j}{\Lambda^0{}_0 + \Lambda^0{}_j x^j}, \quad i, j = 1, 2, 3, \tag{8.68}$$

transforms the normalized S^n invariant measure dS on S^n into a new measure $dS' = \phi_\Lambda^*(dS)$, where $\phi_\Lambda^*(dS)$ is the pullback, (or the "inverse image", cf. e.g. [27, Ch. 16.20.8]) of dS by ϕ_Λ. For $\mathbf{x} \in S^n$ we have

$$(\phi_\Lambda^*(dS))(\mathbf{x}) = \frac{1}{(\Lambda^0{}_0 + \Lambda^0{}_i \mathbf{x}^i)^n} \, dS(\mathbf{x}). \tag{8.69}$$

To prove (8.69) we will need a couple of lemmas.

Lemma 4. *Let r be a real number, and let $f_r : SO^+(1, n+1) \times S^n \longrightarrow \mathbb{R}$ be defined as*

$$f_r(\Lambda, \mathbf{x}) = \left(\Lambda^0{}_0 + \Lambda^0{}_i \mathbf{x}^i\right)^r. \tag{8.70}$$

Then f_r has the following cocycle property:

$$f_r(\Lambda\Lambda', \cdot) = \phi_{\Lambda'}^*\left(f_r(\Lambda, \cdot)\right) f_r(\Lambda', \cdot). \tag{8.71}$$

Proof. It is enough to consider the case $r = 1$. We set, during the course of this proof, $f_1 = f$. We have

$$\begin{aligned}
f(\Lambda\Lambda', \mathbf{x}) &= (\Lambda\Lambda')^0{}_0 + (\Lambda\Lambda')^0{}_i \mathbf{x}^i \\
&= \Lambda^0{}_0 \Lambda'^0{}_0 + \Lambda^0{}_i \Lambda'^i{}_0 + \Lambda^0{}_0 \Lambda'^0{}_i \mathbf{x}^i + \Lambda^0{}_k \Lambda'^k{}_i \mathbf{x}^i \\
&= \Lambda^0{}_0 \left(\Lambda'^0{}_0 + \Lambda'^0{}_i \mathbf{x}^i\right) + \Lambda^0{}_k \left(\Lambda'^k{}_0 + \Lambda'^k{}_i \mathbf{x}^i\right) \\
&= \left(\Lambda'^0{}_0 + \Lambda'^0{}_i \mathbf{x}^i\right) \left(\Lambda^0{}_0 + \Lambda^0{}_k \frac{\Lambda'^k{}_0 + \Lambda'^k{}_j \mathbf{x}^j}{\Lambda'^0{}_0 + \Lambda'^0{}_j \mathbf{x}^j}\right) \\
&= f(\Lambda', \mathbf{x}) f(\Lambda, \phi_{\Lambda'}(\mathbf{x})). \qquad \square
\end{aligned}$$

Lemma 5. *Let $m \in \mathcal{G}_+$, and let $\Lambda = \Lambda(m) \in SO^+(1, n+1)$ be the matrix representing m. Then (8.69) holds for Λ.*

Proof. It is enough to consider the case of $m \neq I$. Let us write m in the form $m = \frac{1}{\sqrt{1-\alpha^2}}$, $0 < \alpha < 1$, $\mathbf{n}^2 = 1$, as in (4.31). From (4.36), and the general formula $gx^\mu e_\mu g^\tau = x^\mu \Lambda(g)^\nu{}_\mu e^\mu$ it follows that $\Lambda^0{}_0 + \Lambda^0{}_i x^i$ is the coefficient in front of e_0 on the right hand side of (4.36), which is $(1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x}))/(1-\alpha^2)$. Comparing now (4.42) and (8.69) we see that the two formulas coincide. $\qquad \square$

Proof of Proposition 6. Let $g \in \mathcal{G}$ and let $g^t a u = m u$ be the decomposition of g^τ into a spin–boost and a rotation as in (4.34), so that $g = u\tau m$. Let $R = \Lambda(u^\tau)$. Since $R \in SO(n+1)$, and dS is rotation invariant, we have $\phi_R^*(dS) = dS$. Notice that (8.69) can also be written as

$$dS' = f_{-n}(\Lambda(g), \cdot) \, dS. \tag{8.72}$$

Now,

$$\phi_{\Lambda(g)}^*(dS) = \phi_{\Lambda(mu^\tau)}^* = \phi_{\Lambda(m)}(\phi_R^*(dS)) = \phi_{\Lambda(m)}^*(dS) = f_{-n}(\Lambda(m), \cdot) \, dS, \tag{8.73}$$

where we have used Lemma 5.

Now, from the definition (8.70) of the cocycle f_k we have that $f(R^{-1}, \cdot) = 1$, $\forall\, R \in SO(n+1)$. Therefore

$$f_{-n}(\Lambda(m), \cdot) = f_{-n}((R^{-1}R)\Lambda(m), \cdot) = f_{-n}(R^{-1}(R\Lambda(m)), \cdot) = f_{-n}(R^{-1}\Lambda(g), \cdot),$$

and from Lemma 4, and the rotational invariance of f_{-n} mentioned before, we find $f_{-n}(R^{-1}(\Lambda(g)), \cdot) = \phi_{\Lambda(g)}^*(f_{-n}(R^{-1}, \cdot)f_{-n}(\Lambda(g), \cdot) = f_{-n}(\Lambda(g), \cdot)$. This proves that the formula (8.72) for a general $\Lambda \in SO^+(1, n+1)$, which is the same as (8.69). $\qquad\square$

8.2. Hamilton's Icosian Calculus

Hamilton's "Icosian Calculus" dates back to his communication to the Proc. Roy. Irish Acad. of November 10, 1856 [38, p.609], followed by several papers, the last one in 1863. According to the contemporary terminology Hamilton proposes a particular presentation of the alternating group $A5$ - the symmetry group of the icosahedron.

<div align="center">

Account of the Icosian Calculus
Communicated 10 November 1856.

Proc. Roy. Irish Acad. vol. vi (1858), pp. 415-16.

</div>

Sir William Rowan Hamilton read a Paper on a new System of Roots of Unity, and of operations therewith connected: to which system of symbols and operations, in consequence of the geometrical character of some of their leading interpretations, he is disposed to give the name of the "ICOSIAN CALCULUS". This Calculus agrees with that of the Quaternions, in three important respects: namely, 1st that its three chief symbols ι, κ, λ are (as above suggested) roots of unity, as i, j, k are certain fourth roots thereof: 2nd, that these new roots obey the associative law of multiplication; and 3rd, that they are not subject to the commutative law, or that their places as factors must not in general be altered in a product. And it differs from the Quaternion Calculus, 1st, by involving roots with different exponents; and 2nd by not requiring (so far as yet appears) the distributive property of multiplication. In fact, $+$ and $-$, in these new calculations, enter only as connecting exponents, and not as connecting terms: indeed, no terms, or in other words, no polynomes, nor even binomes, have hitherto presented themselves, in these late researches of the author. As regards the exponents of the new roots, it may be

mentioned that in the principal system - for the new Calculus involves a family of systems-there are adopted the equations,

$$1 = \iota^2 = \kappa^3 = \lambda^5,\ \lambda = \iota\kappa; \quad (A)$$

so that we deal, in it, with a new square root, cube root, and fifth root, of positive unity; the latter root being the product of the two former, when taken in the order assigned, but not in the opposite order. From these simple assumptions (A), a long train of consistent calculations opens itself out, for every result of which there is found a corresponding geometrical interpretation, in the theory of two of the celebrated solids of antiquity, alluded to with interest by Plato in the Timaeus; namely the Icosahedron, and the Dodecahedron: whereof the angles may now be unequal. By making $\lambda^4 = 1$, the author obtains other symbolical results, which are interpreted by the Octahedron and the Hexahedron. The Pyramid is, in this theory, almost too simple to be interesting: but it is dealt with by the assumption, $\lambda^3 = 1$, the other equations (A) being untouched. As one fundamental result of those equations (A), which may serve as a slight specimen of the rest, it is found that if we make $\iota\kappa^2 = \mu$, we shall have

$$\mu^5 = 1,\ \mu = \lambda\iota\lambda,\ \lambda = \mu\iota\mu;$$

so that this new fifth root mu has relations of perfect reciprocity with the former fifth root lambda. But there exist more general results, including this, and others, on which Sir W. R. H. hopes to be allowed to make a future communication to the Academy: as also on some applications of the principles already stated, or alluded to, which appear to be in some degree interesting.

8.3. The Binary Icosahedral Group

Putting $R = \iota$, $S = \kappa$, $T = \lambda^4$, we can equivalently write Hamilton's equations (A) (Sec. 8.2) as

$$R^2 = S^3 = T^5 = RST = 1. \tag{8.74}$$

Removing the last equality we get the code for the *binary icosahedral group:*

$$R^2 = S^3 = T^5 = RST. \tag{8.75}$$

It is evident from the definition that $Z = RST$ is a central element of the group, and it can be shown [39, p. 69 and references therein] that Z is of order 2: $Z^2 = 1$. This group if order 120, denoted as $2.A5$, and it is a double cover of the icosahedral group $A5$. The group has a particularly simple representation in terms of the quaternions. Let

$$\phi = \frac{1 + \sqrt{5}}{2} = 1.61803\ldots, \quad \Phi = \frac{-1 + \sqrt{5}}{2} = \phi^{-1} = 0.61803\ldots, \tag{8.76}$$

be the Golden Ratio and its inverse, respectively. Consider the group G consisting of 120 elements given by Table 1 below:

Table 1. 120 vertices of the 600 cell

$2 \times 4 = 8$	elements of the form $(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0),$ $(0, 0, \pm 1, 0), (\pm 0, 0, 0, \pm 1)$
$2^4 = 16$	elements of the form $(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$
$3! \times 2^3 = 96$	elements that are even permutations of elements of the form $\frac{1}{2}(\pm \phi, \pm 1, \pm \Phi, 0)$.

These 120 elements form a group of unit icossians (cf. Appendix 8.2) that is a finite subgroup of the group $Spin(3)$. For generators R, S we can take, for instance[9],

$$S_1 = \frac{1}{2}(1 - \Phi i - \phi k), \quad T_1 = \frac{1}{2}(\Phi - i - \phi j), \quad R_1 = S_1 T_1 = -i, \qquad (8.77)$$

or an inequivalent set

$$S_2 = \frac{1}{2}(1 + \phi i + \Phi j), \quad T_2 = \frac{1}{2}(-\phi - i - \Phi k). \quad R_2 = S_2 T_2 = -i. \qquad (8.78)$$

In both cases we have $RST = -1$, but the two sets of generators are geometrically inequivalent (they are related by an *outer* automorphism of G), the angle between S_1 and T_1 is $\pi/5$ while the angle between S_2 and T_2 is $3\pi/5$.

The binary icosahedral group is isomorphic to $SL(2, 5)$, the group of unimodular 2×2 matrices over the field Z_5, as can be seen by taking for the generators R, S, T the matrices:

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ S = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \ T = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}. \qquad (8.79)$$

Fig. 3 shows the vertices of the 600 cell as viewed from the direction of the center of one of its cells. There is another realization of the 600 cell as a polytope, due to Coxeter [40, p.247], where all of the 120 vertices are organized on four different tori within the sphere S^3. Let

$$a = \sqrt{(1 + 3^{-1/2}5^{-1/4}\phi^{3/2})/2} \approx 0.947274,$$

$$b = \sqrt{(1 + 3^{-1/2}5^{-1/4}\phi^{-3/2})/2} \approx 0.770582,$$

---

[9]One can check that there are 120 possible choices of triples of quaternionic generators R, S, T satisfying (8.75).

$$c = \sqrt{(1 - 3^{-1/2}5^{-1/4}\phi^{-3/2})/2} \approx 0.637341,$$

$$d = \sqrt{(1 - 3^{-1/2}5^{-1/4}\phi^{3/2})/2} \approx 0.320426,$$

let $\theta = \pi/30$, and let the four families, each of 30 vertices, be given by:

$$
\begin{aligned}
a[k] &= \{a\cos(k\theta),\ a\sin(k\theta),\quad d\cos(11k\theta),\quad d\sin(11k\theta)\}, \\
b[k] &= \{d\cos(k\theta),\ d\sin(k\theta),\quad -a\cos(11k\theta),\quad -a\sin(11k\theta)\},
\end{aligned}
\tag{8.80}
$$

where

$$k = 0, k < 60, k = k + 2,$$

and

$$
\begin{aligned}
a[k] &= \{b\cos(k\theta),\ b\sin(k\theta),\quad c\cos(11k\theta),\quad c\sin(11k\theta)\}, \\
b[k] &= \{c\cos(k\theta),\ c\sin(k\theta),\quad -b\cos(11k\theta),\quad -b\sin(11k\theta)\},
\end{aligned}
\tag{8.81}
$$

where

$$k = 1, k \leq 60, k = k + 2.$$

9. Figures
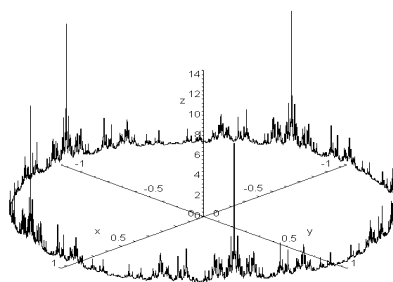
FIGURE 1. Pentagon. 7–th power of the Markov operator applied to $f = 1$.
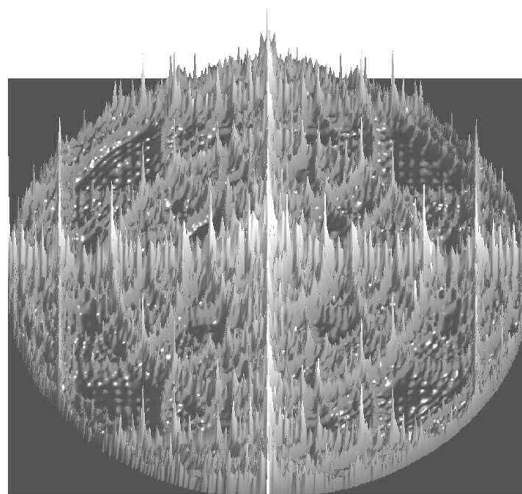
FIGURE 2. Octahedron – {3,4}. 7–th power of the Markov operator, $\alpha = 0.5$.

16 cell

8 cell
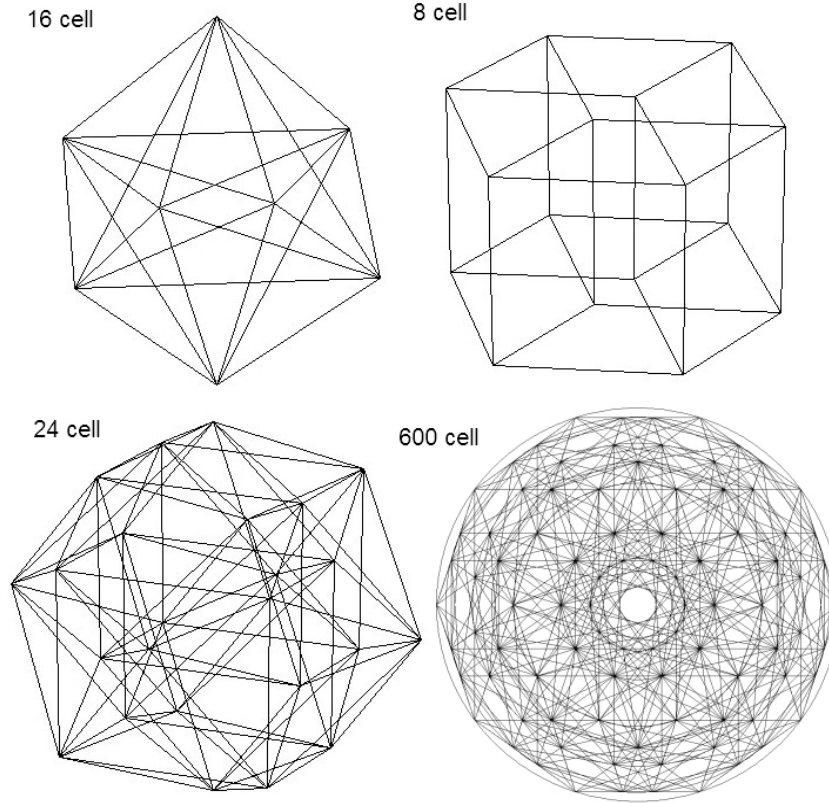
24 cell

600 cell

FIGURE 3. a) 16 cell - {3,3,3}. 8 vertices, 24 edges, 32 triangular faces, 16 tetrahedral cells. b) 8 cell or Hypercube - {4,3,3}. 16 vertices, 32 edges, 24 square faces, 8 cubic cells. c) 24 cell - {3,4,3}. 24 vertices, 96 edges, 96 triangular faces, 24 octahedral cells. d) 600 cell - {3,3,5}. 120 vertices, 720 edges, 1200 triangular faces, 600 tetrahedral cells. The graphics were generated by choosing the tetrahedral cell with vertices $t_0 = (1,0,0,0)$, $t_1 = (\phi, \Phi, 0, 1)/2$, $t_2 = (\phi, 0, 1, \Phi)/2$, $t_3 = (\phi, 1, \Phi, 0)/2$, and choosing the unit vector f_1 in the direction of the center of this cell $(t_0 + t_1 + t_2 + t_3)/4$. The second unit vector f_1 was chosen in the direction of $f_0 * t_1$, (the quaternionic product). Then the frame $(f_0, f_1, f_2 = (0,0,1,0), f_3 = (0,0,0,1))$ was orthonormalized to (e_0, e_1, e_2, e_3) via Gram-Schmidt procedure, and the 720 edges of the 600 cell have been projected onto (e_2, e_3) plane.
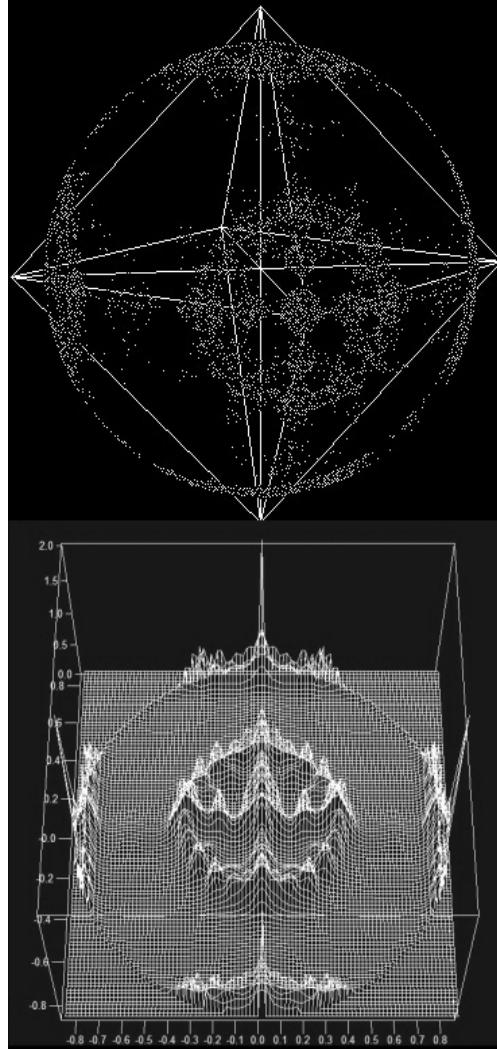
FIGURE 4. 16 cell – $\{3,3,4\}$. Generated 10,000,000 random points of the IFS system of conformal maps with $\alpha = 0.5$. Plotted are 16742 points whose fourth coordinate is in the slice $0.5 < x^4 < 0.51$. The picture is superimposed on the projection of the edges of the 16 cell. Below: Plotted the fourth power of the Markov operator, more precisely of the function $\log_{10}(f_4(\mathbf{r}) + 1)$, with f_4 function defined in (5.62), calculated for $\alpha = 0.5$ and $x^4 = 0.5$.
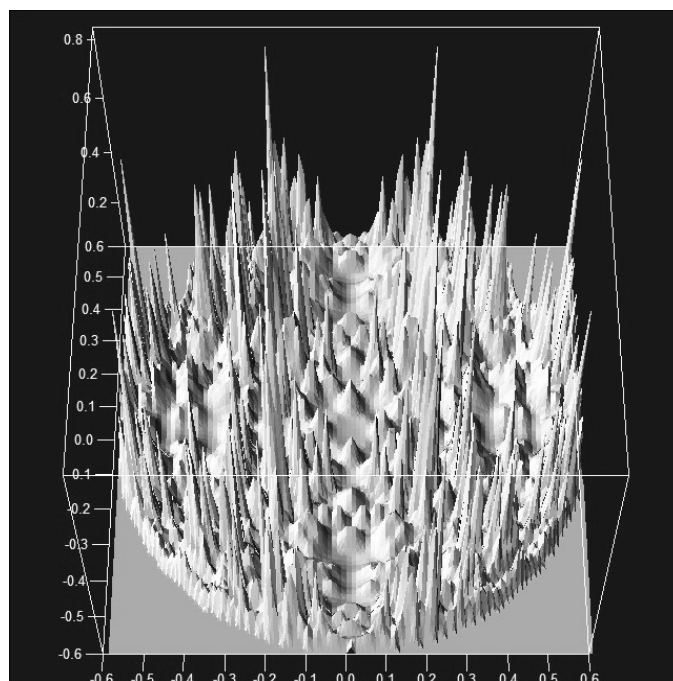
FIGURE 5. Hypercube – {4,3,3}. 5th power of the Markov operator, (5.62), with $\alpha = 0.6$, computed at the section $x^4 = 0.8$. Plotted is the $\log_{10}((f_5) + 1)$.
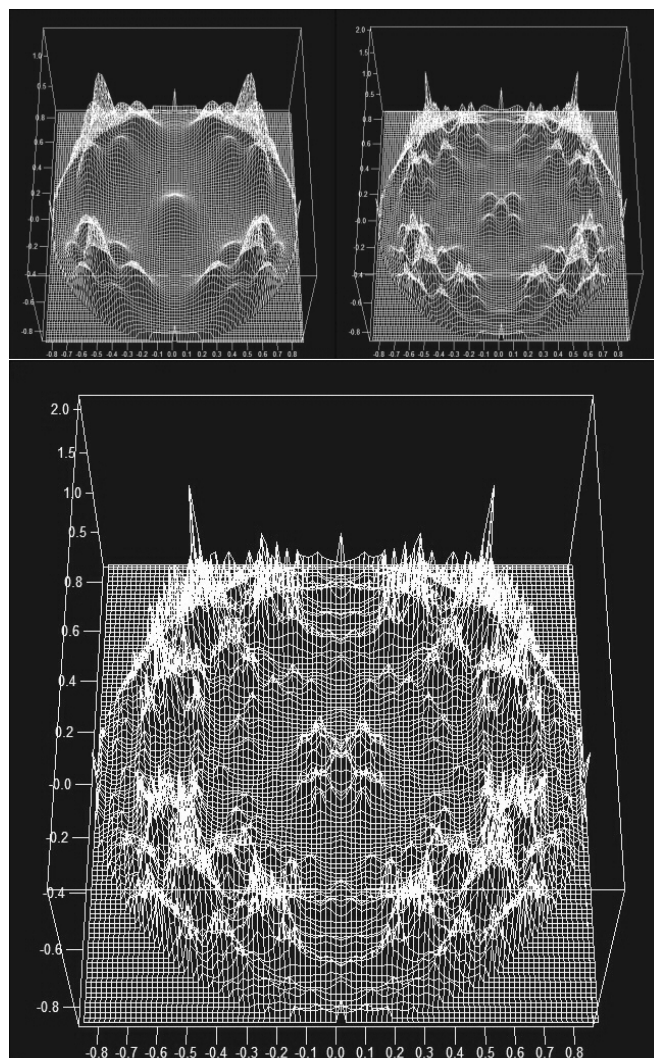
FIGURE 6. 24 cell – $\{3,4,3\}$. Markov operator levels 2,3 and 4, for $\alpha = 0.6$, plotted at $x^4 = 0.5$.
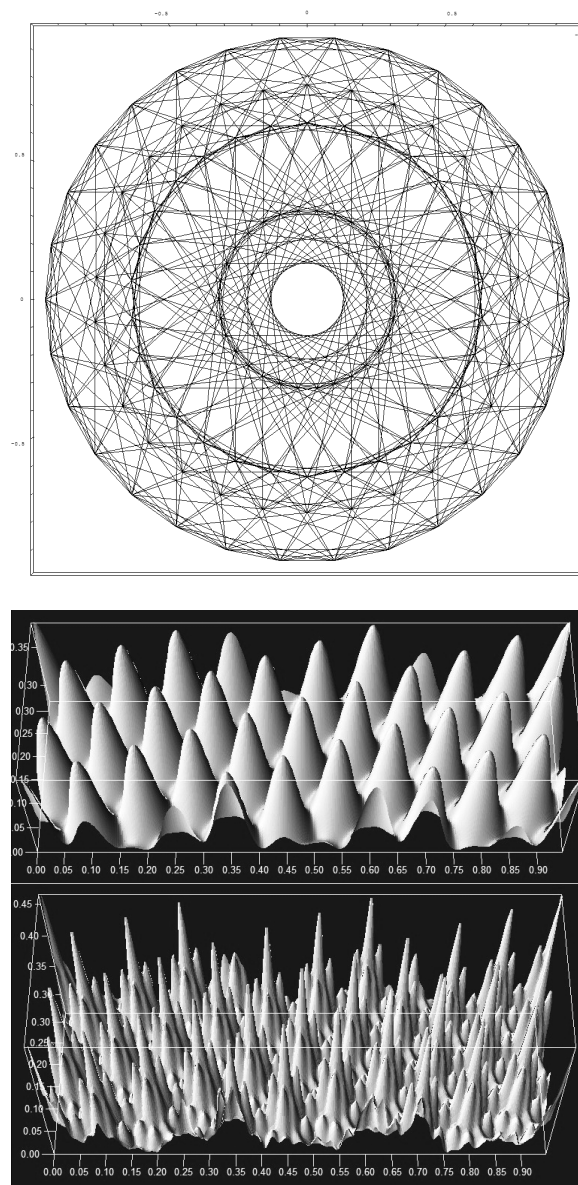
Figure 7. 600 cell - {3,3,5}. Top: Coxeter's projection. Below
1st and 2nd powers of the Markov operator, for $\alpha = 0.6$ plotted
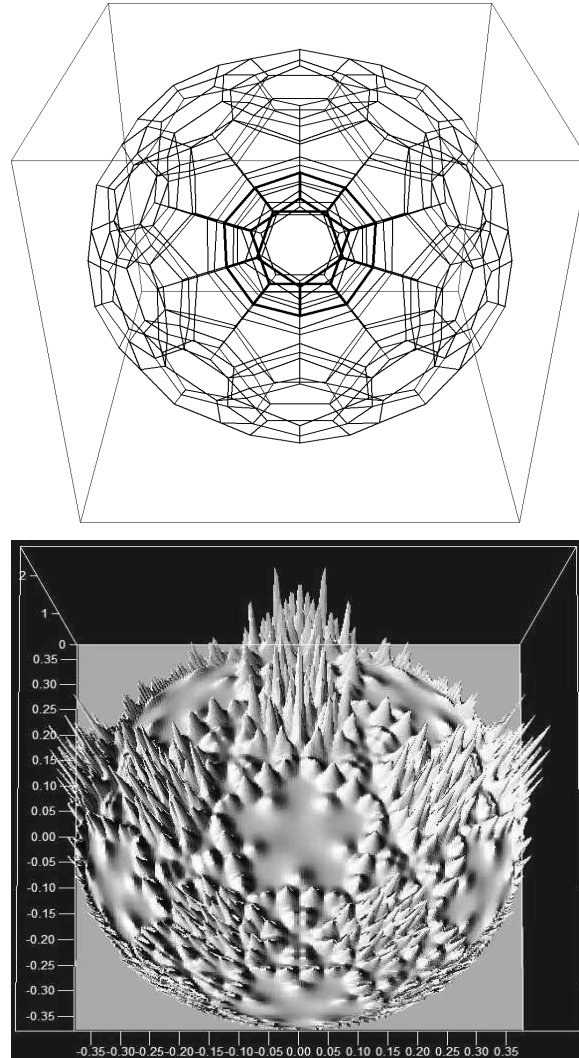at the surface of the most inner torus.

FIGURE 8. 120 cell – {5,3,3}. 600 vertices, 1200 edges of length $(1-\phi)/\sqrt{(2)}$, 720 pentagonal faces, 120 dodecahedral cells. One of its dodecahedral cells in bold. Below the 2nd power of the Markov operator, for $\alpha = 0.9$, plotted at the upper hemisphere of this particular cell.
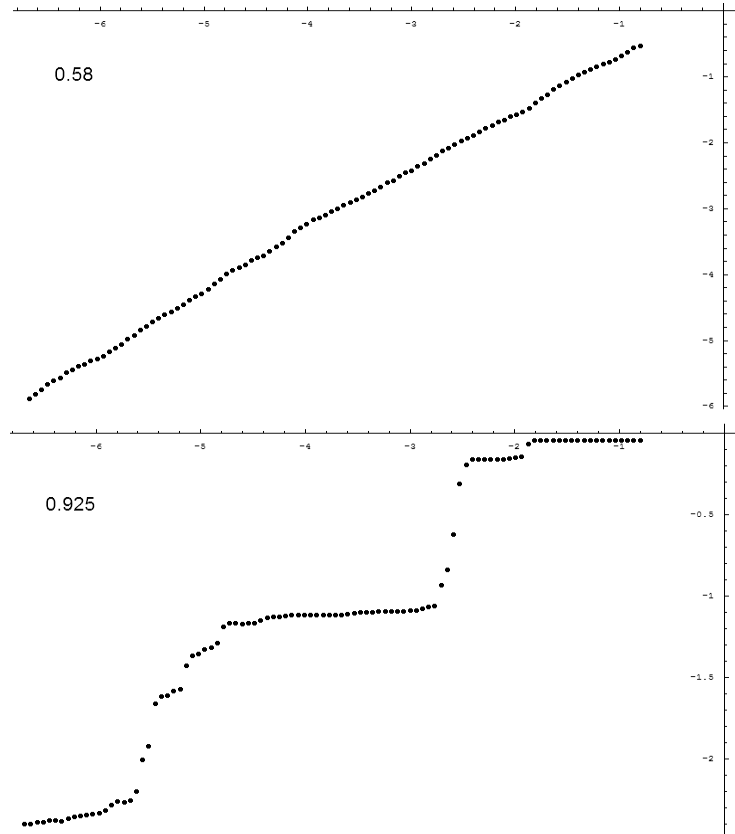
FIGURE 9. Correlation dimension plots for the pentagon, for $\alpha = 0.58$, and $\alpha = 0.925$. Plotted is the function $\log(C(r))$, defined in (7.65), versus $\log(r)$. The slope of the graph should give the correlation dimension D – (7.64). For $\epsilon = 0.58$, (cf. Fig. 1) we get a reasonable straight line with the slope $D \approx 0.9$, but with $\epsilon = 0.925$, when the expected fractal dimension should be close to zero, we get a staircase.

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