

A COMMENT ON “ON THE ROTATION MATRIX IN MINKOWSKI SPACE-TIME” BY ÖZDEMİR AND ERDOĞDU

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We comment on the article by M. Özdemir and M. Erdoğan [9]. We indicate that the exponential map onto the Lorentz group can be obtained in two elementary ways. The first way utilizes a commutative algebra involving a conjugate of a semi-skew-symmetric matrix, and the second way is based on the classical epimorphism from $SL(2, \mathbb{C})$ onto $SO_0(3, 1)$.

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1. Introduction

The classical Euler–Rodrigues formula gives the explicit form of the rotation matrix in \mathbb{R}^3 in terms of the rotation axis and the rotation angle (see e.g. [1] for a pedagogical introduction by B. Palais and R. Palais). It can be also interpreted as the explicit formula for the exponential of a 3×3 skew-symmetric matrix. Various generalizations to other dimensions have been studied (see e.g. [2]). In physics a generalization to the Minkowski space especially matters. This problem has been mentioned by J. Gallier in his lecture notes on Lie groups [3], sending the reader to the PhD thesis of C. M. Geyer [4]. Geyer indeed has provided such a derivation but not quite optimal nor complete. In a recent paper E. Minguzzi [5] classified standard forms of generators and provided the formula for each class separately¹. The present note emerged as an alternative method to the one proposed recently by M. Özdemir and M. Erdoğan [9]. Our derivation is based on simple algebraic properties of the algebra constructed from the generator and its dual. The singular case of the generator with the quadruple null eigenvalues is also discussed in our note.

¹An introduction to such classification can be found in [6, 7].

2. Minkowski space generalization of the Euler–Rodrigues formula

Let F be a generator of a one-parameter subgroup of the Lorentz group. We write F in the following general form that resembles the form of the electromagnetic field mixed tensor expressed $F^\mu{}_\nu$ in terms of the electric and magnetic field vectors \mathbf{e} and \mathbf{b} ,

$$F = \begin{pmatrix} 0 & b_3 & -b_2 & e_1 \\ -b_3 & 0 & b_1 & e_2 \\ b_2 & -b_1 & 0 & e_3 \\ e_1 & e_2 & e_3 & 0 \end{pmatrix}. \quad (1)$$

The “dual” matrix, denoted by \tilde{F} , is obtained from F by a “dual rotation”, that is, by replacing $\mathbf{e} \rightarrow \mathbf{b}$, $\mathbf{b} \rightarrow -\mathbf{e}$,

$$\tilde{F} = \begin{pmatrix} 0 & -e_3 & e_2 & b_1 \\ e_3 & 0 & -e_1 & b_2 \\ -e_2 & e_1 & 0 & b_3 \\ b_1 & b_2 & b_3 & 0 \end{pmatrix}. \quad (2)$$

We also introduce real numbers u and v defined as

$$u = \frac{1}{4} \text{Tr } F\tilde{F} = \mathbf{e} \cdot \mathbf{b}, \quad v = \frac{1}{4} \text{Tr } F^2 = \frac{1}{2}(\mathbf{e}^2 - \mathbf{b}^2). \quad (3)$$

The characteristic polynomials for F and \tilde{F} can now be expressed in terms of u and v ,

$$\det(F - \lambda I) = \lambda^4 - 2v\lambda^2 - u^2, \quad \det(\tilde{F} - \lambda I) = \lambda^4 + 2v\lambda^2 - u^2. \quad (4)$$

Let σ and θ be defined as

$$\sigma = \sqrt{\sqrt{u^2 + v^2} + v}, \quad \theta = \text{sgn}(u)\sqrt{\sqrt{u^2 + v^2} - v}. \quad (5)$$

It is clear that σ is nonnegative and θ has the sign of u , where the sgn is defined to be right continuous, that is $\text{sgn}(0) = 1$. The eigenvalues of F are $\pm\sigma$ and $\pm i\theta$ while the eigenvalues of \tilde{F} are $\pm\theta$ and $\pm i\sigma$. The following identities follow directly from the definitions:

$$v = \frac{\sigma^2 - \theta^2}{2}, \quad u = \sigma\theta. \quad (6)$$

Let us define T as

$$F^2 + \tilde{F}^2 = T. \quad (7)$$

LEMMA 1. *The matrices F, \tilde{F}, T commute. Moreover, the following identities hold:*

$$F\tilde{F} = \tilde{F}F = \sigma\theta I, \quad (8)$$

$$F^2 - \tilde{F}^2 = (\sigma^2 - \theta^2) I, \tag{9}$$

$$F^3 = (\sigma^2 - \theta^2)F + \sigma\theta\tilde{F}, \tag{10}$$

$$F^2 = \frac{T + (\sigma^2 - \theta^2)I}{2}, \tag{11}$$

$$\tilde{F}^2 = \frac{T - (\sigma^2 - \theta^2)I}{2}, \tag{12}$$

$$FT = (\sigma^2 - \theta^2)F + 2\sigma\theta\tilde{F}, \tag{13}$$

$$\tilde{F}T = 2\sigma\theta F - (\sigma^2 - \theta^2)\tilde{F}, \tag{14}$$

$$T^2 = (\sigma^2 + \theta^2)^2 I. \tag{15}$$

Proof: Eqs. (8)–(10) follow by a routine matrix algebra from the identities $\sigma\theta = \mathbf{e} \cdot \mathbf{b}$ and $\sigma^2 - \theta^2 = \mathbf{e}^2 - \mathbf{b}^2$. Eq. (11) (resp. (12)) follows by adding (resp. subtracting) Eq. (7) and Eq. (9). Eq. (13) results in a similar way. In order to show Eq. (14) we first multiply Eq. (9) by \tilde{F} and use Eq. (8). Finally, Eq. (15) can be derived from Eq. (13) multiplied by F , and by using (8) and (9). \square

PROPOSITION 1 (Generalized Euler–Rodrigues formula). *Assume that $\sigma^2 + \theta^2 > 0$. Then the following general formula holds,*

$$\begin{aligned} \exp(Ft) = & \frac{\cosh(t\sigma) + \cos(t\theta)}{2} I + \frac{\sigma \sinh(t\sigma) + \theta \sin(t\theta)}{\sigma^2 + \theta^2} F \\ & + \frac{\theta \sinh(t\sigma) - \sigma \sin(t\theta)}{\sigma^2 + \theta^2} \tilde{F} + \frac{\cosh(t\sigma) - \cos(t\theta)}{2(\sigma^2 + \theta^2)} T. \end{aligned} \tag{16}$$

Equivalently, using F^2 instead of T ,

$$\begin{aligned} \exp(Ft) = & \frac{\theta^2 \cosh(t\sigma) + \sigma^2 \cos(t\theta)}{\sigma^2 + \theta^2} I + \frac{\sigma \sinh(t\sigma) + \theta \sin(t\theta)}{\sigma^2 + \theta^2} F \\ & + \frac{\theta \sinh(t\sigma) - \sigma \sin(t\theta)}{\sigma^2 + \theta^2} \tilde{F} + \frac{\cosh(t\sigma) - \cos(t\theta)}{\sigma^2 + \theta^2} F^2. \end{aligned} \tag{17}$$

If $\sigma = \theta = 0$, then

$$\exp(tF) = I + tF + \frac{t^2}{4}T = I + tF + \frac{t^2}{2}F^2. \tag{18}$$

Proof: In the proof we use the following theorem about generators of one-parameter matrix groups: If $\gamma(t)$ is a one-parameter group of matrices, then $\gamma(t) = \exp(Xt)$, where $X = \gamma'(0)$.² We consider first the case of at least one of the numbers σ, θ being nonzero, i.e. $\sigma^2 + \theta^2 > 0$.

²The proof of this classical theorem can be found, for instance, in *An Introduction to Matrix Groups and their Applications* by Andrew Baker, Springer 2002, Theorem 2.17, also available online, the same title and author, Theorem 2.5: <http://www.maths.gla.ac.uk/~ajb/dvi-ps/lie-bern.pdf>.

Let $L(t)$ denote the right-hand side of Eq. (16). Immediately, $L(0) = I$, $L'(0) = F$. We aim to show that $L(t)$ is a one-parameter matrix group, i.e. that

$$L(t+s) = L(t)L(s). \quad (19)$$

The proof is somewhat tedious but straightforward. We write $L(t+s)$ and expand the functions $\sin(x+y)$, $\cos(x+y)$, $\sinh(x+y)$, $\cosh(x+y)$ in terms of products of functions of the corresponding arguments x, y (i.e. products of t, s and σ, θ). This way we get a long expression with coefficients at the matrices I, F, \tilde{F}, T .

On the other hand, we multiply $L(t)L(s)$ and obtain coefficients in front of the products of I, F, \tilde{F}, T . All of these products can be reduced to I, F, \tilde{F}, T using Lemma 1. Comparing the coefficients in front of I, F, \tilde{F}, T establishes the result.

Suppose now that $\sigma = \theta = 0$. Then, from Lemma 1, we have that

$$FT = T^2 = 0, \quad F^2 = T/2. \quad (20)$$

The group property of $L(t)$ given by Eq. (19) follows then by the following observation:

$$\begin{aligned} L(t)L(s) &= \left(I + tF + \frac{t^2}{4}T\right)\left(I + sF + \frac{s^2}{4}T\right) \\ &= I + sF + \frac{s^2}{4}T + tF + \frac{ts}{2}T + \frac{t^2}{4}T \\ &= I + (s+t)F + \frac{1}{4}(s+t)^2T. \end{aligned} \quad (21)$$

On the other hand $L(0) = I$, $L'(0) = F$, which completes the proof. Eq. (17) follows from Eq. (16) and Eq. (11). \square

REMARK 1. Since we are dealing with commuting matrices, the problem reduces to a simple commutative symbolic algebra. It can be handled more efficiently by an adequate software, capable of commutative symbolic operations.

2.1. Alternative derivation via $\text{SL}(2, \mathbb{C})$

With the four Hermitian matrices³ $\sigma_\mu = \sigma^\mu$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (22)$$

the group homomorphism $A \mapsto \Lambda(A)$ from the group of unimodular matrices $\text{SL}(2\mathbb{C})$ onto the connected component of the identity $\text{SO}(3, 1)_0$ of the homogenous Lorentz

³By abuse of notation σ_μ and σ^μ constitute exactly the same set matrices. Their components are $\sigma_{\mu AB}$ and $\sigma^{\mu AB}$, ($\mu = 1, \dots, 4$), ($A, B = 1, 2$).

group is given by

$$\Lambda(A)^\mu{}_\nu = \frac{1}{2} \text{Tr}(A\sigma^\mu A^\dagger\sigma_\nu), \quad (\mu, \nu = 1, \dots, 4). \quad (23)$$

The completeness relations for σ matrices

$$\sum_{\mu=1}^4 \sigma^{\mu AB} \sigma_{\mu CD} = 2\delta_D^A \delta_C^B, \quad (A, B, C, D = 1, 2), \quad (24)$$

entail

$$\text{Tr}(\Lambda(A)) = |\text{Tr}(A)|^2. \quad (25)$$

Taking the derivative of Eq. (23) we arrive at the linear relation (isomorphism) between infinitesimal generators f (traceless 2×2 complex matrices) from the Lie algebra $\text{sl}(2, \mathbb{C})$ to the Lie algebra of elements F in $\text{so}(3, 1)$:

$$F^\mu{}_\nu = \frac{1}{2} \text{Tr}(f\sigma^\mu\sigma_\nu + \sigma^\mu f^\dagger\sigma_\nu). \quad (26)$$

With f defined by

$$f \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i=1}^3 (e_i + ib_i)\sigma_i, \quad (27)$$

we arrive at F given by (1), while $\tilde{f} \stackrel{\text{def}}{=} -if$ gives \tilde{F} . The characteristic polynomial for f , $\det(f - \lambda I) = \lambda^2 - \frac{1}{2}(v + iu)$, entails two roots $\pm\omega$. There is a simple relation between ω and σ, θ : $\omega = \frac{1}{2}(\sigma + i\theta)$. Every 2×2 complex matrix X determines a vector in the complex Minkowski space with complex coordinates $x^\mu = \text{Tr}(\sigma^\mu X)/2$. There are two scalar products in this space: $(x, y) = x^T J y$ and $\{x, y\} = x^\dagger J y$. The first one is bilinear, while the second one is Hermitian. Both are $\text{SO}(3, 1)$ invariant. X is Hermitian if and only if x^μ are real, moreover $\text{Tr}(X^\dagger \epsilon Y \epsilon)/2 = \{x, y\}$ and $\text{Tr}(X^T \epsilon X \epsilon)/2 = \det(X) = (x, y)$, where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If ξ_\pm are eigenvectors of f belonging to eigenvalues $\pm\omega \neq 0$, and if $X_\pm = \xi_\pm \otimes \xi_\pm^\dagger$, then x_\pm are real isotropic (i.e. $(x, y) = \{x, y\} = 0$) eigenvectors of F corresponding to real eigenvalues $\pm 2\Re(\omega)$. Vectors y_\pm corresponding to $\xi_+ \otimes \xi_-^\dagger$ and $\xi_- \otimes \xi_+^\dagger$ are Hermitian space-like (we have $\{y, y\} = 2(|\xi_+|^2 |\xi_-|^2 - |\xi_+^\dagger \xi_-|^2) > 0$), bilinear isotropic (i.e. $(y_\pm, y_\pm) = 0$), and J -orthogonal to x_\pm , resp. They are eigenvectors of F corresponding to imaginary eigenvalues $\pm 2\Im(\omega)$. Since $f^2 = (v + iu)I/2 = \omega^2$, $\exp(tf)$ is easily computed

$$e^{tf} = \cosh(\omega t) I + \frac{\sinh \omega t}{\omega} f, \quad (28)$$

where it does not matter which of the two possible signs of ω is chosen. If $\omega = 0$, then f has just one eigenvector ξ , vector x , corresponding to ξ is real isotropic, and F annihilates 2-dimensional plane in the three dimensional hyperplane orthogonal to x . Moreover, when $\omega = 0$, which happens if and only if $f^2 = 0$, we get instantly

$$e^{tf} = I + tf, \quad (29)$$

which can be also obtained by taking the limit of $\omega \rightarrow 0$ in Eq. (28). We can now expand the functions $\cos(t\omega)$, $\sinh(t\omega)$ of the complex argument $t\omega = t(\sigma + i\theta)$ and use Eq. (23) to recover the results of Proposition 1 by straightforward though somewhat lengthy calculations.

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