GRADED LIE–CARTAN PAIRS I

ARKADIUSZ JADCZYK *

Institute of Theoretical Physics, University of Wroclaw
and

DANIEL KASTLER **

Physics Department, University of Aix-Marseille II (Luminy)
and

Centre de Physique Théorique ***

(Received January 12, 1987)

The Lie–Cartan pairs proposed in [1] as an algebraic frame for the classical
operators of differential geometry are generalized to the Z/2-graded case (graded Lie–
Cartan pairs of a graded Lie algebra and a graded commutative algebra). The generalized
case is reduced to the Abelian case by tensoring with arbitrary graded commutative
algebras.

The Lie–Cartan pairs \((L, A)\) of a Lie algebra and a commutative algebra were
introduced in [1] as a purely algebraic frame for describing the interrelation
between the following classical geometric items: the covariant exterior derivative \(\delta_q\)
attached to a connection \(q\) (resp. the exterior derivative \(\delta\)), the Lie derivative \(\theta(\xi)\),
and the inner product \(i(\xi)\). The (associative) algebra \(A\) in [1] was assumed
commutative because of the need, for a derivation \(\xi\) of \(A\), to yield again a
derivation \(a\xi\) through composition with multiplication from the left by any
element \(a\) of \(A\). However this fact holds more generally for \(A\) graded commutative,
provided we take derivations in the graded commutative sense (thus yielding a

* Postal address: Institute of Theoretical Physics, University of Wrocław, Cybulskiego 36,
50-205 Wrocław.
** Postal address: CPT-CNRS Luminy Case 907-F-13288 Marseille Cedex 9.
*** Laboratoire Propore du Centre National de la Recherche Scientifique.
Lie super algebra structure. With this choice the theory in [1] can be generalized (with identical results up to Z/2-grading to "graded Lie–Cartan pairs" (L, A) of a Lie super algebra L and a graded commutative algebra A, with definition axioms being natural modifications of the definition axioms in [1]: one needs only add, wherever necessary, the characteristic signs (−1)^l concomitant with permutation of factors with respective grades i and j. Now, instead of adapting the proofs in [1] to the Z/2-graded case — which would lead to cumbersome calculations, one can, more interestingly, reduce the generalized Z/2-graded results to the previous (trivially graded) case by means of the following device (possibly of independent interest): given a graded Lie–Cartan pair (L, A) and an arbitrary graded commutative algebra U, the pair (L_U, A_U) obtained from the skew products L_U = U ⊗ L, A_U = U ⊗ A, turns out to be again naturally a graded Lie–Cartan pair, and this in such a way that:

(i) V-connections of (L, A) naturally define corresponding V_U-connections of (L_U, A_U) for V_U = U ⊗ V;
(ii) graded-alternate A-linear V-valued n-forms λ on L naturally yield corresponding A_U-linear V_U-valued forms λ_U on L_U, the operators δ_λ, θ_λ(λ) and i_λ essentially commuting with the map λ → λ_U.

Moreover, since knowledge of all restrictions λ_U to zero grade elements characterizes λ, one has automatic transfer of results from the trivially graded to the Z/2-graded case.

Our paper is organized as follows: Section 1 defines the graded Lie–Cartan pairs, discusses adjunction of a unit, and describes the injective and degenerate special cases. Section 2 defines the classical operators δ_λ, δ_θ, θ_λ(λ) and i_λ and provides the main Theorem 2.3. We formulate the definitions in terms of the graded antisymmetrizer A_λ, so as to obtain operators a priori respecting the graded-alternate property — with the bonus of motivating the complicated signs arising in the explicit formulæ (2.2) specifying these operators. Section 3 describes the passage of pairs (L, A) to pairs (L_U, A_U), the assignment of V_U-connections of (L_U, A_U) to V-connections of (L, A), and the extension λ → λ_U of graded alternate forms, with the ensuing commutation theorem (independent of Sections 1 and 2 in which it is used for the proof of Theorem 2.3). Section 4 discusses derivation properties of δ_λ, θ_λ(λ) and i_λ. We gathered in Appendix A necessary results on graded vector spaces and algebras. Appendix B describes "graded symmetrization" in terms of more general "twisted symmetrization".

In our paper we do not discuss the concepts of graded manifolds or supermanifolds since we do not need to use them. However, for the benefit of the reader, we have given some of the recent references (see Refs. [2–8]) on the subject. In particular, Ref. [6] relates to the tensorization method which we discuss in particular.
of \( V \) behaving as a derivation for the left \( A \)-module structure of \( V \):

\[
\varrho(\xi) X = V_{\xi \otimes x} X,
\]

\[
\varrho(\xi)(\alpha X + \beta Y) = \alpha \varrho(\xi) X + \beta \varrho(\xi) Y,
\]

where

\[
\varrho(\alpha \xi + \beta \xi) = \alpha \varrho(\xi) + \beta \varrho(\xi), \quad \xi \in L', \quad X \in V', \quad \alpha, \beta \in C,
\]

and moreover linear in the sense

\[
\varrho(\alpha \xi, \beta \eta) = \alpha \varrho(\xi) + \beta \varrho(\eta), \quad \xi, \eta \in L, \quad \alpha, \beta \in C.
\]

The \( V \)-connection \( \varrho \) is called local whenever

\[
\varrho(\alpha \xi) = \alpha \varrho(\xi), \quad \alpha \in A, \quad \xi \in L
\]

and flat whenever

\[
\varrho([\xi, \eta]) = 0, \quad \xi, \eta \in L
\]

where \( [\ , \ ] \) is the r.h.s. denotes a graded commutator.

The curvature of the \( V \)-connection \( \varrho \) is the assignment to \( (\xi, \eta) \in L \times L \) of the map: \( V \rightarrow V \) given by

\[
\Omega(\xi, \eta) = \varrho([\xi, \eta]) = \varrho(\xi) \eta - \varrho(\eta) \xi
\]

(1.11)

where \( [\ , \ ] \) is the first bracket on the r.h.s. denotes a graded commutator.

1.2. **Lemma.** Given a \( V \)-connection \( \varrho \) of a graded Lie–Cartan pair \((L, A)\), the value \( \Omega(\xi, \eta) \) of the curvature for \( \xi, \eta \in L' \) behaves as follows w.r.t. the left \( A \)-module structure of \( V \): one has

\[
\Omega(\xi, \eta)(aX) = (-1)^{\alpha \eta + \beta a} a \Omega(\xi, \eta) X, \quad a \in A', \quad X \in V.
\]

(1.13)

**Proof:** We have, writing \( \partial \alpha = i, \partial \eta = j, \partial \xi = k \)

\[
\eta(\xi) \varrho(\eta)(aX) = \varrho(\xi) \{(-1)^{i} a \varrho(\eta) X + (\eta a) X\}
\]

\[
= (-1)^{\alpha \eta + \beta a} \varrho(\xi) \varrho(\eta) X + (-1)^{\beta a} \varrho(\eta) \varrho(\xi) X + \varrho(\eta)(\eta a) X + (-1)^{\eta a} \varrho(\xi) \varrho(\eta) X,
\]

hence

\[
[\varrho(\xi), \varrho(\eta)](aX) = (-1)^{\alpha \eta + \beta a} \varrho(\xi) \varrho(\eta) X + (-1)^{\beta a} \varrho(\eta) \varrho(\xi) X + \varrho(\eta)(\eta a) X + (-1)^{\eta a} \varrho(\xi) \varrho(\eta) X
\]

(1.15)

On the other hand

\[
\varrho([\xi, \eta])(aX) = (-1)^{\alpha + \beta} a \varrho([\xi, \eta]) X + (-1)^{\eta a} \varrho(\eta)(\xi a) X,
\]

(1.17)

whence (1.14) by difference.

1.3. **Remark.** The first example of a real (complex) Lie–Cartan pair \((L, A)\) is obtained as follows: take for \( A \) a graded-commutative algebra, with \( L = \text{Der} \), the set of graded derivations of \( A \), we recall that \( \text{Der} = (\text{Der})^{0} \otimes (\text{Der})^{1} \) with \((\text{Der})^{0}, \text{resp.} (\text{Der})^{1}\) the even, resp. odd endomorphisms \( \xi \) of \( A \) as a graded vector space such that

\[
\xi(ab) = (Xa)b + (-1)^{\alpha \beta} a(\xi b), \quad a, b \in A.
\]

(1.18)

For these facts, we refer to Appendix A.

1.4. **Definition.** A **subpair** of a graded Lie–Cartan pair \((L, A)\) is a couple \((L, A')\) of a subsuper Lie algebra \( L' \) of \( L \) and a graded unital subalgebra \( A' \) of \( A \) such that one has \( \xi \in A' \) and \( \alpha \xi \in L' \) for all \( \xi \in L \) and \( \alpha \in A' \). The subpair \((L, A')\) is itself a graded Lie–Cartan pair for the products (1.1) and (1.2).

Given a real (complex) graded Lie–Cartan pair \((L, A)\) the subpair \((L, R)\) (resp. \((L, C)\)) is called the **deletion** of \((L, A)\).

1.5. **Definition.** We define, for a graded Lie–Cartan pair \((L, A)\)

\[
L^{1} = \{ \xi \in L : \xi a = 0 \text{ for all } a \in A\}
\]

(1.20)

\[
A^{1} = \{ a \in A : a \xi = 0 \text{ for all } \xi \in L\}
\]

(1.21)

The pair \((L, A)\) is called injective whenever \( L^{1} = \{0\} \), and degenerate whenever \( L^{1} = L \) (or equivalently \( A^{1} = \{0\} \)).

1.6. **Lemma.** The Definition 1.5 implies

(i) \( L^{1} \) is a graded ideal of the graded Lie algebra \( L \) and a submodule of the left \( A \)-module \( L \).

(ii) \( A^{1} \) is a graded unital subalgebra of \( A \).

(iii) \( L \) is a graded Lie algebra over \( A^{1} \) (i.e. the bracket \([\ , \ ]\) of \( L \) is graded, \( A^{1}\)-bilinear).

(iv) The products (1.1) and (1.2) are also graded \( A^{1}\)-bilinear.

(v) \((L/L^{1}, A)\) is an injective graded Lie–Cartan pair with the definitions

\[\]
\[
[\zeta, \eta] = [\zeta, \eta]^-, \\
\zeta a = \xi a, \\
a\zeta = a\zeta
\]  
(1.22)

where \(\xi\) denotes the class of \(\xi \in L\) modulo \(L^1\).

(vi) The injective graded Lie–Cartan pairs are pairs of the type \((L, A)\) with \(A\) a graded-commutative unital algebra, and \(L\) a Lie subsuperalgebra of \(\text{Der} A\).

(vii) The degenerate-graded Lie–Cartan pairs are obtained from couples \(L, A\) of a Lie superalgebra \(L\) over \(A\) with \(\zeta a = 0\) for \(\zeta \in L, a \in A\).

Proof: (i) \(L^1\) is a graded ideal of \(L\) as the kernel of the homomorphism \(\zeta \mapsto (a \mapsto \zeta a)\) from \(L\) to \(\text{Der} A\). It is a submodule of the \(A\)-module \(L\) since it is obviously linear and, for \(\zeta \in L^1\), \((a\zeta)b = a(\zeta b) = 0\) for all \(a, b \in A\).

(ii) \(A^+\) is obviously linear. For \(a = a^2 + a^1 \in A^+\), \(\zeta \in L^1\) one has, \(\xi^0 = -\xi a^2 \in A^0 \cap A^1 = \{0\}\) thus \(A^+\) is graded. And, for \(a, b \in A^1\), one has \(\xi(ab) = (\xi a)b + (-1)^{a\xi} a(\xi b) = 0\) for all \(\zeta \in L^1, A^+\) is unital, since \(R \subset A^1(C \subset A^2)\) (cf. footnote 9).

(iii) For \(\zeta, \eta \in L^1\) and \(a \in A^1\) one has by (1.8)
\[
[\zeta, \eta] = (-1)^{a\xi\eta} a[\zeta, \eta], \\
[a\xi, \eta] = a[\zeta, \eta].
\]
(1.23)

(iv) One has, for \(a \in A^1, b \in A^1, \zeta \in L^1\) by (1.3) and (1.5)
\[
(a\zeta)b = a(\zeta b), \\
(\zeta b)(ab) = (-1)^{a\xi}\zeta b(\zeta b)
\]
(1.24)

and (in fact for all \(a, b \in A\))
\[
a(b\zeta) = (-1)^{a\xi b} a\zeta b, \\
(ba)\zeta = b(a\zeta).
\]
(1.25)

(v) The first line in (1.22) defines the bracket of the quotient superalgebra \(L/L^1\).

The two other lines are coherent definitions, since, for \(\eta \in L^1\)
\[
(\zeta + \eta) a = \zeta a \quad \text{and} \quad a(\zeta + \eta) = a\zeta,
\]
since \(a, \eta \in L^1\). Moreover one has, for \(a, b \in A, \zeta \in L\)
\[
(a\zeta)b = a\zeta b = (a\zeta)b = a(\zeta b) = a(\zeta b)
\]
(1.26)

and for \(a \in A, \eta, \zeta \in L\)
\[
[\zeta, \alpha\eta] = [\zeta, \alpha\eta]^- = (-1)^{a\xi\eta} a[\zeta, \eta] + (\zeta(\alpha) \eta)
= (-1)^{a\xi\eta} a[\zeta, \eta] + ((\zeta a) \eta) = (-1)^{a\xi\eta} a[\zeta, \eta] + ((\zeta a) \eta).
\]
(1.27)

(vi) Is obvious.

(vii) Is obvious from (iii). \(\blacksquare\)

1.7. Remark. With \((L, A)\) a graded Lie–Cartan pair and \(Q\) a subalgebra of \(A^+\), Lemma 1.6 shows that the elements of \(Q\) behave like scalars. We say in such case that \((L, A)\) is a graded Lie–Cartan pair over \(Q\).

Remark. Most of the concepts and results of this paper admit an obvious generalization from the category of graded Lie algebras and graded vector spaces to the category of graded Lie algebras over \(Q\) and graded \(Q\)-modules (\(Q\) fixed). This kind of generalization is useful in applications in which anticommuting "parameters" are being used. \(Q\) is then usually taken as a Grassmann algebra with "sufficiently many" (possibly with an infinite number of) generators.

The next lemma shows that our assumptions, in Definition 1.1, that \(A\) is unital and \(L\) a unital \(A\)-module, do not in fact restrict generality.

1.8. Lemma. Define a non-unital graded Lie–Cartan pair as a pair \((L, A)\) satisfying all the axioms in Definition 1.1 except the unital requirement of the algebra \(A\) and the left \(A\)-module \(L\). Then \((L, A)\) is a graded Lie–Cartan pair if one defines:

(i) \(A\) as the algebra obtained from \(A\) by adding a unit \(1\),

(ii) the product (1.1) and (1.2) of \(\zeta \in L\) and \(a1 + a \in \tilde{A}\) as
\[
\zeta (a1 + a) = \zeta a, \\
(a1 + a)\zeta = a\zeta + a\zeta.
\]
(1.28)

Moreover, with \(V\) a linear graded left \(A\)-module, the convention
\[
(a1 + a)X = aX + aX, \quad a1 + a \in \tilde{A}, \quad X \in V
\]
(1.30)

makes \(V\) a unital graded left \(\tilde{A}\)-module, say \(\bar{V}\), each \(V\)-connection \(q\) of \((L, A)\) yielding a \(V\)-connection of \((\tilde{L}, \tilde{A})\).

Proof: Identical to that of Lemma 6 in [1]. \(\blacksquare\)

2. The classical operators attached to a graded Lie–Cartan pair

In this section we present an abstract (purely algebraic) graded version of the classical operators, \(\partial_q, \partial_\alpha, \partial_q(\zeta)\) and \(i(\zeta)\) of differential geometry.
2.1. Definition. We fix a unital graded Lie–Cartan pair \((L, A)\) and a unital graded left \(A\)-module \(V\) and denote by \(A^*(L, V)\) the graded vector space of graded alternate \(V\)-valued \(n\)-linear forms on \(L\), the range of the graded alternator \(\lambda\) acting in \(L^*(L, V)\) (cf. Appendix B).

With \(\xi, \eta \in L\), \(a \in A\) and \(\phi\) a \(V\)-connection of \((L, A)\) of curvature \(\Omega\), the operators \(\delta_0, \phi \wedge, \delta_g, \delta'_g, \phi(\xi), \phi'(\xi), \Omega(\xi, \eta)\), \(i(\xi), a', \delta_a \wedge\) and \(\Omega \wedge\) are then defined as follows:

\[\delta_0 \lambda = -\frac{n(n+1)}{2} A_{n+1} \lambda^g\]

with \(\lambda^g(\xi_1, \ldots, \xi_{n+1}) = \bar{\lambda}([\xi_1, \xi_2], \xi_3, \ldots, \xi_{n+1})\),

\(\phi \wedge \lambda = (n+1) A_{n+1} \lambda^g\)

with \(\lambda^g(\xi_1, \ldots, \xi_{n+1}) = (-1)^{\lambda^g g} \phi(\xi) \overline{\lambda}(\xi_2, \ldots, \xi_{n+1})\),

\(\delta_g = \delta_0 + \phi \wedge\)

\(\delta'_g(\xi) \lambda = -n A_{n+1} \lambda^g\)

with \(\lambda^g(\xi_1, \ldots, \xi_{n+1}) = (-1)^{\lambda^g g} \phi(\xi) \overline{\lambda}(\xi_2, \ldots, \xi_{n+1})\),

\((\phi(\xi) \lambda)(\xi_1, \ldots, \xi_{n}) = \phi(\xi) \overline{\lambda}(\xi_1, \ldots, \xi_{n})\),

\(\Omega(\xi, \eta)(\xi_1, \ldots, \xi_{n+1}) = \Omega(\xi, \eta) \overline{\lambda}(\xi_1, \ldots, \xi_{n+1})\),

\(i(\xi) \lambda = 0, \quad \lambda \in A^0(L, V)\),

\(a \cdot \overline{\lambda}(\xi_1, \ldots, \xi_{n}) = a \cdot \lambda(\xi_1, \ldots, \xi_{n})\),

\(\lambda \wedge \lambda = (n+1) A_{n+1} \lambda^g\)

with \(\lambda^g(\xi_1, \ldots, \xi_{n+1}) = (-1)^{\lambda^g g} \Omega(\lambda(\xi_2, \ldots, \xi_{n+1})\),

\[\Omega(\lambda \wedge) = \frac{(n+1)(n+2)}{2} A_{n+2} \lambda^g\]

with \(\lambda^g(\xi_1, \ldots, \xi_{n+2}) = (-1)^{\lambda^g g} \Omega(\lambda(\xi_2, \ldots, \xi_{n+2})\).

2.2. Proposition. Let \((L, A), V, \phi, \xi, \eta, a, b\) be as in Definition 2.1. Then:

(i) the operators (2.1) through (2.10) are homomorphisms of the graded vector space \(A^*(L, V)\); specifically, one has for \(\lambda \in A^*(L, V)\) (in fact, for \(\lambda_0, \phi \wedge, \delta_g, \delta'_g, \delta_a \wedge\) and \(\Omega \wedge\), for \(\lambda \in \mathcal{L}(L, V)\))

\[\delta_0 \lambda, \phi \wedge \lambda, \delta_g \lambda \in A^{n+1}(L, V)^{\lambda^g}\]

\[\theta_0 \lambda, \phi(\xi) \lambda, \phi'(\xi) \lambda \in A^{n}(L, V)^{\lambda^g + \lambda^g_0}\]

\[i(\xi) \lambda \in A^{n-1}(L, V)^{\lambda^g + \lambda^g_0}\]

\[a \cdot \lambda \in A^n(L, V)^{\lambda^g + \lambda^g_0}\]

\[\delta_a \wedge \lambda \in A^{n+1}(L, V)^{\lambda^g + \lambda^g_0}\]

\[\Omega \wedge \lambda \in A^n(L, V)^{\lambda^g + \lambda^g_0}\]

\[\Omega(\xi, \eta) \lambda \in A^n(L, V)^{\lambda^g + \lambda^g_0}\]

(ii) one has for \(\delta_0, \phi \wedge, \theta_0, \delta_g, \delta'_g, \delta_a \wedge\) and \(\Omega \wedge\) the following explicit formulae: for \(\xi_i \in L, i = 1, \ldots, n+2\) (indicating omission of the corresponding argument):

\[(\delta_0 \lambda)(\xi_1, \ldots, \xi_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^{\lambda^g g} \lambda([\xi_i, \xi_j], \xi_1, \ldots, \xi_i-1, \xi_j, \ldots, \xi_{n+1})\]

\[(\phi \wedge \lambda)(\xi_1, \ldots, \xi_{n+1}) = \sum_{k=1}^{n+1} (-1)^{\lambda^g g} g(\xi_k) \lambda(\xi_1, \ldots, \xi_k-1, \xi_k, \ldots, \xi_{n+1})\]

\[(\Omega(\xi, \eta) \lambda)(\xi_1, \ldots, \xi_{n+1}) = \sum_{k=1}^{n+1} (-1)^{\lambda^g g} \Omega(\xi_k \lambda)(\xi_1, \ldots, \xi_k-1, \xi_k, \ldots, \xi_{n+1})\]

\[\Omega(\lambda \wedge) = \sum_{k=1}^{n+1} (-1)^{\lambda^g g} \lambda(\xi_1, \xi_2, \ldots, \xi_{n+1})\]

\[\lambda \wedge \lambda = \sum_{k=1}^{n+1} (-1)^{\lambda^g g} \lambda(\xi_1, \xi_2, \ldots, \xi_{n+1})\]

\[\Omega(\xi, \eta) \lambda = \sum_{k=1}^{n+1} (-1)^{\lambda^g g} \Omega(\xi_k \lambda)(\xi_1, \ldots, \xi_{n+1})\]

\[\Omega(\xi, \eta) \lambda = \sum_{k=1}^{n+1} (-1)^{\lambda^g g} \Omega(\xi_k \lambda)(\xi_1, \ldots, \xi_{n+1})\]

\[\Omega(\xi, \eta) \lambda = \sum_{k=1}^{n+1} (-1)^{\lambda^g g} \Omega(\xi_k \lambda)(\xi_1, \ldots, \xi_{n+1})\]

\[\Omega(\xi, \eta) \lambda = \sum_{k=1}^{n+1} (-1)^{\lambda^g g} \Omega(\xi_k \lambda)(\xi_1, \ldots, \xi_{n+1})\]

\[\Omega(\xi, \eta) \lambda = \sum_{k=1}^{n+1} (-1)^{\lambda^g g} \Omega(\xi_k \lambda)(\xi_1, \ldots, \xi_{n+1})\]

\[\Omega(\xi, \eta) \lambda = \sum_{k=1}^{n+1} (-1)^{\lambda^g g} \Omega(\xi_k \lambda)(\xi_1, \ldots, \xi_{n+1})\]

\[\Omega(\xi, \eta) \lambda = \sum_{k=1}^{n+1} (-1)^{\lambda^g g} \Omega(\xi_k \lambda)(\xi_1, \ldots, \xi_{n+1})\]

\[\Omega(\xi, \eta) \lambda = \sum_{k=1}^{n+1} (-1)^{\lambda^g g} \Omega(\xi_k \lambda)(\xi_1, \ldots, \xi_{n+1})\]
(δa ∧ λ)(ξ_1, ..., ξ_{n+1}) = \sum_{i=1}^{n+1} (-1)^i + δ_0(ξ_i) a(ξ_1, ..., \hat{ξ}_i, ..., ξ_{n+1}). \tag{2.15}

(Ω ∧ λ)(ξ_1, ..., ξ_{n+2}) = -\sum_{i=1}^{n+1} (-1)^{ij} + δ_0(ξ_i) Ω(ξ_1, ..., \hat{ξ}_i, ..., ξ_{n+2}). \tag{2.16}

(iii) \text{A}_n^*(L, V) is stable under i(ξ), a, δa ∧ Ω and Ω ∧. 

Proof: (i) δ_0, g ∧ (hence δ_0), θ_0(ξ), δa ∧ and Ω ∧ have range in \text{A}_n^*(E, V) since they are defined by means of the graded alternator. And g(ξ), l(ξ), λ, aδ are graded alternate for λ graded alternate. The inclusions (2.11) are obvious, the intrinsic grade of an operator being the excess of the grade of its value over the sum of grades of its arguments.

(ii) Check of (2.12): we have \sum_{i+1} \bigcup_{i < j < n+1} \text{σ}_{ij} \circ \text{σ}_{ij} with \text{σ}_{ij} the permutation

\text{σ}_{ij} = \begin{pmatrix} 1 & 2 & 3 & \ldots & n+1 \\ i & j & \hat{i} & \hat{j} & \ldots & n+1 \end{pmatrix}, \quad 1 \leq i < j \leq n+1 \tag{2.17}

and \text{σ}_{ij} the subgroup of \text{σ}_{n+1} leaving stable the subset \{i, j\}. Due to the graded antisymmetry of λ and the fact that [ξ_j, ξ_i] = (-1)^{ij} + δ_0(ξ_i) [ξ_j, ξ_i], we have \text{σ}_{n+1} \circ \text{σ}_{ij} λ = σ_{ij} λ for σ_{n+1} ∈ \text{σ}_{n+1}. Formula (2.10) then follows from the fact that Card \text{σ}_{ij} = 2(n-1)! and \chi(ξ, \text{σ}_{ij}) = (-1)^{ij} \tag{2.18}

Check (2.13): we have \sum_{k=1}^{n} \bigcup_{i < n+1} \text{σ}_i \circ \text{σ}_i with

\text{σ}_i = \begin{pmatrix} 1 & 2 & \ldots & n+1 \\ i & 1 & \ldots & \hat{i} & \hat{i} & \ldots & n+1 \end{pmatrix}, \quad 1 \leq i \leq n+1 \tag{2.19}

and \text{σ}_i the subgroup of \text{σ}_{n+1} leaving i invariant. We have \text{σ}_{n+1} \circ \text{σ}_i λ = σ_i λ due to graded antisymmetry of λ. Formula (2.13) then follows from the facts that Card \text{σ}_i = n! and

\chi(ξ, \text{σ}_i) = (-1)^{ij} + δ_0(ξ_i) [ξ_j, ξ_i]. \tag{2.19a}

Check of (2.14): we have now σ_i \circ σ_i λ = σ_i λ for σ_i in the stabilizer \text{σ}_i of i and

σ_i = \begin{pmatrix} 1 & 2 & \ldots & n \\ i & 1 & \ldots & n \end{pmatrix}. \tag{2.19b}

with \chi(ξ, \text{σ}_i) as above and Card \text{σ}_i = (n-1)!. Thus

\text{θ}_0(ξ) λ = \sum_{i=1}^{n} (-1)^{i+j} + δ_0(ξ_1) [ξ_j, ξ_i] \lambda(ξ_1, ..., ξ_i, ..., ξ_j, ..., ξ_{n+2}) \times

\lambda(ξ_1, ..., ξ_i, ..., ξ_j, ..., ξ_{n+2}). \tag{2.19c}

Check of (2.15): we have σ_{n+1} \circ σ_i λ = σ_i λ for σ_{n+1} ∈ \text{σ}_i and σ_i as in (2.18).

Formula (2.15) then follows from the already stated facts, Card \text{σ}_i = n! and (2.18a). Check of (2.16): we have σ_{n+1} \circ σ_i λ = σ_i λ for σ_{n+1} the permutation (2.17), where n → n+1, and σ_i in the subgroup \text{σ}_n of \text{σ}_{n+2} leaving stable the subset {i, j} (this because of the graded antisymmetry of λ and the fact that Card \text{σ}_{ij} = 2(n-1)! and \chi(ξ, σ_{ij}) = (-1)^{ij} \tag{2.19a}).

(iii) Let λ ∈ \text{A}_n^*(L, V); we have, from (2.7),

\text{[i(ξ) λ]}(ξ_1, ..., b_{ξ_k}, ..., ξ_{n+1}) = (-1)^{i(ξ) λ} b_{ξ_k}(ξ_1, ..., ξ_{n+1}) \tag{2.20}

further, from (2.8),

\text{[a(ξ) λ]}(ξ_1, ..., ξ_{n+1}) = (-1)^{a(ξ) λ} b_{a(ξ)}(ξ_1, ..., ξ_{n+1}). \tag{2.21}

With \text{β}_i as in (2.13),

With \text{β}_0 as in (2.12).

Observe that \text{θ}_0(ξ) leaves \text{A}_n^*(L, V) stable also for a non-local connection \phi, this property is not explicitly stated in [1].

See formula (B.7) of Appendix B.
further, from (2.9) using (1.5), since \( V \) is a left \( A \)-module
\[
\lambda^a (b_1^{\xi_1}, \ldots, b_{n+1}^{\xi_{n+1}}) = (-1)^{(b_1+\xi_1)(b_{n+1}+\xi_{n+1})} b (\xi_1, a) \lambda (\xi_2, \ldots, \xi_{n+1})
\]
and for \( k \geq 1 \)
\[
2^a (\xi_1, \ldots, b_1^{\xi_1}, \ldots, \xi_2, a) = (-1)^{\delta b_2 (a + \xi_1) + \delta b_3 (a + \xi_2 + \xi_3) + \cdots + \delta b_{k-1} (a + \xi_k)} b (\xi_2, \ldots, \xi_k, a)
\]
(2.23)
and
\[
\delta (\xi_1, \ldots, b_1^{\xi_1}, \ldots, \xi_2, a) = (-1)^{\delta b_2 (a + \xi_1) + \delta b_3 (a + \xi_2 + \xi_3) + \cdots + \delta b_{k-1} (a + \xi_k)} b (\xi_2, \ldots, \xi_k, a)
\]
(2.24)
hence \( \lambda^a \) is \( A \)-linear, and so is \( \delta a \wedge \), since \( A_{n+1} \) obviously commutes with multiplication from the left by \( b \).

To prove that \( \delta a \wedge \) and \( \delta e \) leave \( A_{\{L, V\}} \) stable, we shall use the intertwining Theorem 3.6 of the next section.17 We denote by \( U \) an arbitrary graded commutative algebra, and observe that, by using Proposition 3.3, \( \delta a \wedge \) is \( A \)-linear if the restriction \( (\delta a \wedge, u) \) of \( (\delta a \wedge, u) \) to \( L_{\{L, V\}} \) is \( (A_{\{L, V\}}) \)-linear. Now, using (3.27) (cf. Proposition 3.6) we may write
\[
\delta a \wedge (a^0) = (\delta a \wedge, a^0) = \delta a \wedge (a^0),
\]
(2.25)
where \( \lambda^a \) is the restriction of \( \lambda_{a^0} \) to \( L_{\{L, V\}} \).

The \( A_{\{L, V\}} \)-linearity of the r.h.s. of (2.25) follows now by the Proposition and Remark 3.5 and Theorem 1.8(iv) of [1]. The same argument proves \( A \)-linearity of \( \delta a \wedge \) and \( \delta e \) follows by difference. We use the same trick to prove \( A \)-linearity of \( \delta a \wedge \).

By (3.30) we have
\[
\theta_a (u \otimes \xi)_{\lambda_{\{a^0\}}} = (u \otimes id) (\theta_a (\xi) \lambda)_{\{a^0\}}
\]
(2.26)
for all \( \xi \in L \) and \( u \in U \). By Theorem 1.8(iv) of [1] the l.h.s. when \( u \otimes \xi \) as well as the arguments of \( \lambda_{\{a^0\}} \) are restricted to \( L^0 \), is \((A_{\{L, V\}})^0\)-linear. This implies \( A_{\{L, V\}}^0 \)-linearity of the r.h.s., and this in turn, owing to the arbitrariness of \( U \) and \( u \in U \), implies \( A \)-linearity of \( \theta_a (\xi) \lambda \) by the same arguments as used in Proposition 3.3. The \( A \)-linearity of \( \theta_a (\xi) \lambda \) and \( (\xi) \lambda \) follows.

17 Our reader may skip the rest of this proof and wait until Remark 2.5 (ii) for an alternative proof.

18 Graded commutators of operators on \( A^{*} (L, V) \) w.r.t. the grading of the latter defined by the intrinsic grading \( \delta \) of the form \( \lambda \).
Proof: Check of (2.31): \(\delta_0\) evidently commutes with \(a\), as acting “internally” on the arguments of \(A\) without altering \(A\). Further, we have

\[
[i(\xi), a_j] = \lambda(\xi_j, \xi_{j-1}) = (1 - 1)^{i(\xi + a_j)} \lambda(\xi_j, \xi_{j-1}) = (1 - 1)^{i(\xi + a_j) + i(\xi_{j-1})} a\lambda(\xi_j, \xi_{j-1}, \xi_{j-2}).
\]

(2.43)

Analogously,

\[
\{[a, b] \lambda(\xi_1, \xi_2, \ldots, \xi_n) = \lambda'(\xi_1, \xi_2, \ldots, \xi_n).
\]

(2.44)

whence one obtains the vanishing of the \(\delta\)-graded commutator \([\delta_0, a]\), since \(A\) obviously commutes with \(a\).

Further, we have obviously \([a, b] = [a, b] = 0\); on the other hand, from (2.15)

\[
[b, a] = \sum_{i=1}^{n+1} (-1)^{i+1} a_i (\xi_i + a_m) \lambda(\xi_1, \xi_2, \ldots, \xi_{n+1})
\]

(2.45)

Check of (2.32): we have from (2.5), (2.8), using (1.9)

\[
q(\xi) (a - \lambda) = q(\xi) (a\lambda(\xi_1, \xi_2, \ldots, \xi_n)) = (1 - 1)^{1+1} a\lambda(\xi_1, \xi_2, \ldots, \xi_n).
\]

(2.46)

From (2.32) and \([\delta_0, a] = 0\) follows (2.34) (cf. (2.31)). Check of (2.36): we have from (2.27), for \(n \geq 2\)

\[
[i(\xi), \lambda(\xi_1, \ldots, \xi_n)] = (1 - 1)^{i(\xi + a)} [i(\xi), \lambda(\xi_1, \ldots, \xi_n)] = (1 - 1)^{i(\xi + a) + i(\xi_{n+1})} a\lambda(\xi_1, \ldots, \xi_n, \xi_{n+1}).
\]

(2.48)

which, by the graded symmetry of \(\lambda\), changes by a factor \((-1)^{\delta_0 + 1}\) upon exchange of \(\xi\) and \(\eta\).

The rest of our proof is based on the intertwining Theorem 3.6 of the next section. The proofs of the latter are independent of the present section. We begin with the proof of (3.28) which most simply shows how intertwining yields a proof: with \(U\) an arbitrary graded commutative algebra, we have for \(\lambda \in A^*(L, V)\) from (3.30), (3.34)

\[
(\delta_0^2 - \Omega \wedge \lambda) u = (\delta_0 \lambda - \Omega \wedge \lambda) u.
\]

(2.49)

Now, since the pair \((L_0, A_0^0)\) obtained from the zero grade parts of \(L_0\) and \(A_0\) is a Lie–Cartan pair in the sense of [1], we know from Theorem 1.8 formula (52) there, that the l.h.s. of (2.49) vanishes in the restriction to \(V_0^0 \times V_0^0 \times \ldots \times V_0^0\). It then follows by the last assertion in Proposition 3.3 that \((\delta_0 - \Omega \wedge \lambda) u\) vanishes.

Proof of (2.37): We have, for \(u, v \in U\) fulfilling \(u = \delta_0 u, v = \delta_0 v\), from (3.30)

\[
q_0(u \circ \lambda) (v \circ \lambda) = (u \circ \lambda)(v \circ \lambda) = (u \circ \lambda)v = (v \circ \lambda)u = (v \circ \lambda)(u \circ \lambda).
\]

(2.50)

where we used the fact that \(q_0(u \circ \lambda) commutes with \(v \circ \lambda\); this is obviously the case for \(q_0(u \circ \lambda)\) which acts “internally”, with \(\gamma_1\) independent of \(\delta_0\), cf. (2.14); but this holds also for \(q(u \circ \lambda)\). We have indeed, from (3.14) and (A.7)

\[
q(u \circ \lambda)(v \circ \lambda) = (u \circ \lambda)(v \circ \lambda) = (v \circ \lambda)(u \circ \lambda) = (u \circ \lambda)(v \circ \lambda).
\]

(2.51)

thus

\[
q(u \circ \lambda)(v \circ \lambda) = (u \circ \lambda)(v \circ \lambda) = (v \circ \lambda)(u \circ \lambda).
\]

(2.52)

We have on the other hand, from (2.37), (3.4) and (A.8)

\[
q_0([u \circ \lambda, v \circ \lambda]) = (u \circ \lambda)(v \circ \lambda) = (v \circ \lambda)(u \circ \lambda).
\]

(2.53)
hence, from this, (2.50) and (3.35)
\[ [\theta_q (u \otimes \xi), \theta_q (v \otimes \eta) - \theta_q (v \otimes \eta) \theta_q (u \otimes \xi) - \theta_q (u \otimes \xi) \theta_q (v \otimes \eta)] = \Omega (u \otimes \xi, v \otimes \eta)] (\lambda_\nu) = (\nu \otimes id) ([[\theta_q (\xi), \theta_q (\eta) - \theta_q (\eta) \theta_q (\xi)] - \Omega (\xi, \eta)] (\lambda_\nu)\]

(2.54)

The l.h.s. now vanishes in the restriction to \( V_0^0 \times \ldots \times V_0^0 \) by formula (51) in [1], hence the last \{ \} vanishes by the above procedure.

**Proof of (2.39):** We have, from (3.27), (3.31) and (3.30)
\[ i(u \otimes \xi) i(v \otimes \eta) + i(v \otimes \eta) i(u \otimes \xi) = (\nu \otimes \xi) = (\xi \otimes \nu)\]

(2.55)

Now formula (53) of [1] entails the vanishing of the l.h.s. whence the vanishing of the last \{ \}.

**Proof of (2.40):** We have, from (3.30), (3.31), for \( u, v \in U \) with \( \partial u = \partial \xi, \partial v = \partial \eta \), since then, by (A.8): \( [\nu \otimes \xi, \nu \otimes \eta] = (-1)^{\nu \otimes \xi, \nu \otimes \eta} \]
\[ [[i(u \otimes \xi) (v \otimes \eta) - \theta_q (v \otimes \eta) i(u \otimes \xi) - i([u \otimes \xi, v \otimes \eta])] (\lambda_\nu) = i(u \otimes \xi) (v \otimes \eta) i([u \otimes \xi, v \otimes \eta]) (\lambda_\nu) = (\nu \otimes \xi) (\xi \otimes \nu)\]

(2.56)

Observe that we may commute \( \theta_q (v \otimes \eta) \) and \( v \otimes \eta \) by (2.52); and \( i(u \otimes \xi) \) and \( v \otimes \eta \) because, due to (2.7):
\[ i(u \otimes \xi) (v \otimes \eta) = (-1)^{\nu \otimes \xi + \nu \otimes \eta} i(u \otimes \xi) (v \otimes \eta)\]

We then conclude the vanishing of the last \{ \} in (3.56) from the vanishing of the l.h.s. in restriction to \( V_0^0 \times \ldots \times V_0^0 \), as implied by formula (54) in [1].

We have from (2.38) and (2.39)
\[ [\theta_q (\xi), \theta_q (\eta) - \theta_q (\eta) \theta_q (\xi) = i(\xi) \theta_q (\eta) + \delta_q i(\eta) \delta_q - \delta_q i(\eta) \delta_q = [i(\xi), \Omega \wedge .] \]

(2.57)

Formulae (2.38a) through (2.41a) follow from the corresponding formulae (2.38) through (2.41) by replacing the graded Lie–Cartan pair \((L, A)\) by its depletion \((L, R)\) (or \((L, C)\), and then making the choice \( \varphi = 0 \). As for (2.42) it follows from (2.40), (2.40a) due to (2.6).

**2.5. Remark.** The proof of the statements in Theorem 2.4 and in Proposition 2.2 (iii) can either be inferred from the corresponding results of the commuting case (Theorem 1.8 of [1]) by tensorizing by \( U \) as in Proposition 3.5; or else verified directly, which we did here when verification is not complicated. Of course, the first method could be used all through, e.g. (2.32), resp. (2.36) could be inferred from the equalities
\[ [\varphi (u \otimes \xi), (v \otimes a)] - [u \otimes \xi, v \otimes a)] (\lambda_\nu) = (\nu \otimes \xi) (\xi \otimes \nu)\]

(2.58)

resp.
\[ i(v \otimes \eta) i(u \otimes \xi) = (\nu \otimes \xi) (\xi \otimes \nu)\]

(2.59)

valid when \( \partial u = \partial \xi, \partial v = \partial \eta \).

As for the fact that \( \theta_q (\xi) \) and \( \theta_q (\eta) \) leave \( \Lambda^*_A (L, V) \) stable, we gave a proof based on reduction to the commutative case, since verification is cumbersome for \( \partial_q \). However, one could more simply proceed as follows: Let \( \lambda \in \Lambda^*_A (L, V) \):

1. Let \( b \in A \) and \( 1 \leq k \leq n \); we have, using (1.9)
\[ \varphi (\xi) (\lambda) (\xi_1, \ldots, b \xi_k, \ldots, \xi_\ell) = (-1)^{\ell - 1} \sum_{i=1}^{k-1} \varphi (\xi) (\lambda) (\xi_1, \ldots, b \xi_k, \ldots, \xi_\ell) \]

(2.60)

On the other hand, with \( \theta_q (\xi) \) the jth term on the r.h.s. of (2.14) we have, by virtue of \([\xi, b \xi_k] = (-1)^{\nu \otimes \xi, \nu \otimes \eta} \]
\[ \varphi (\xi) (\lambda) (\xi_1, \ldots, b \xi_k, \ldots, \xi_\ell) = \sum_{i=1}^n \varphi (\xi) (\lambda) (\xi_1, \ldots, b \xi_k, \ldots, \xi_\ell) \]

(2.61)

yielding the sum
\[ \{\theta_q (\xi) (\lambda) (\xi_1, \ldots, b \xi_k, \ldots, \xi_\ell)\}

(2.62)
(ii) We proceed by induction w.r.t. $n$: assume that we proved that $\delta_\nu \lambda$ belongs to $A^*_\nu (L, V)$ if $\lambda \in A^*_\mu (L, V)$, with $\lambda \in A^*_\nu (L, V)$, we have, by what precedes, that $i(\delta_\xi \lambda) \in \theta_{\delta_\xi} (\xi \lambda) \in A^*_n (L, V)$ belongs to $A^*_\mu (L, V)$, for all $\xi \in L$, i.e. we have

$$i(\delta_\xi \lambda) (b_{\xi_1}, \xi_2, \ldots, \xi_{n-1}) = (-1)^{\delta_\xi + \delta_\lambda} b_{\xi} (\xi, b_{\xi_1}, \xi_2, \ldots, \xi_{n-1})$$

$$= (-1)^{\delta_\xi + \delta_\lambda} b_{\xi} (\xi, b_{\xi_1}, \xi_2, \ldots, \xi_{n-1}) (2.63)$$

We proved $A$-linearity of $\delta_\nu \lambda$ w.r.t. its second argument, implying multi $A$-linearity, since $\lambda$ is graded symmetric. Now for $\lambda \in A^0 (L, V) = V$, $\delta_\nu (b_{\xi}) = b_{\xi} \delta_\nu$, $\delta_\nu (b_{\xi}) = b_{\xi} \delta_\nu$, if $\nu$ is local.

The next proposition shows how the classical operators acting on $A^*(L, V)$ are obtained from the classical operators acting on $A^*(L, A)$.

2.6. PROPOSITION. Let $(L, A)$, $V$ and $q$ be as in Definition 2.1, $V$ finitely generated and projective, with $\delta_0, \delta, \theta_0, \xi$, $\theta_0 (\xi)$, $\Omega (\xi, \eta), i(\xi)$ acting on $A^*(L, V)$ as defined there. And consider on the other hand $A$ itself as a left $A$-module with linearity, since $A$ is graded symmetric. Now for $\lambda \in A^0 (L, V) = V$, $\delta_\nu (b_{\xi}) = b_{\xi} \delta_\nu$, $\delta_\nu (b_{\xi}) = b_{\xi} \delta_\nu$, if $\nu$ is local.

We then have the following expressions for the classical operators on $A^*(L, V)$ in terms of those on $A^*(L, A)$: for $\xi, \eta \in L$, $V \in V$, $\alpha \in A^*(L, A)$, we have

$$\varrho (\xi X) \alpha = \rho (\xi) X \alpha + \rho (\xi) \delta_\xi X \alpha$$

$$\delta_\nu (X \otimes \alpha) = X \otimes \delta_\nu \alpha$$

$$\delta_\nu (X \otimes \alpha) = \delta_\nu (X) \alpha + X \otimes \delta_\nu \alpha$$

$$\theta_0 (\xi X \otimes \alpha) = (-1)^{\delta_\xi + \delta_\alpha} \left( \theta_0 (\xi) X \otimes \alpha + (-1)^{\delta_\xi + \delta_\alpha} \theta_0 (\xi) \otimes X \alpha \right)$$

$$\theta_0 (\xi X \otimes \alpha) = \left( \theta_0 (\xi) X \right) \otimes \alpha + (-1)^{\delta_\xi + \delta_\alpha} \left( \theta_0 (\xi) \otimes X \alpha \right)$$

and furthermore

$$\{ (\varrho X) \alpha \} (\xi_1, \ldots, \xi_{n+1}) = \sum_{i=1}^{n+1} \left( -1 \right)^{\delta_i + \delta_i} X_{\xi_i} \{ \alpha (\xi_1, \ldots, \xi_{i-1}) \}$$

whence (2.66), using the fact that

$$\varrho (\xi X) \alpha = \rho (\xi) X \alpha + (-1)^{\delta_\xi} X \alpha$$

we have, for $\xi_1, \ldots, \xi_n \in L$, using (2.64) and (2.76) and the corresponding classical operators $\delta_0, \delta, \theta_0, \Omega, \theta_0, \Omega, i(\xi)$ acting on $A^*(L, A)$.

3. Tensorization by graded commutative algebras

Given a unital graded Lie-Cartan pair $(L, A)$ together with a graded commutative algebra $U$, we now show that the skew tensor products naturally yield a graded Lie-Cartan pair $(L_U, A_U)$ in such a way that

(i) each $V$-connection $\varrho$ of $(L, A)$ naturally yields a $V_U$-connection of $(L_U, A_U)$ naturally yields a graded Lie-Cartan pair $(L_U, A_U)$ in such a way that

$$U = U \otimes L$$

$$A_U = U \otimes A$$

and

$$\text{GRADED LIE-CARTAN PAIRS}$$

$$\begin{align*}
\varrho (\xi X) \alpha &= (-1)^{\delta_\xi} X \otimes \left( \varrho (\xi) \alpha \right), \\
\Omega (\xi, \eta) (X \otimes \alpha) &= \{ \Omega (\xi, \eta) X \} \otimes \alpha.
\end{align*}$$

Proof: Check of (2.66): we have, using (B.43)

$$(\varrho X) \alpha (\xi_1, \ldots, \xi_{n+1}) = \sum_{i=1}^{n+1} \left( -1 \right)^{\delta_i + \delta_i} X_{\xi_i} \{ \alpha (\xi_1, \ldots, \xi_{i-1}) \}$$

and thereby

$$(\varrho X) \alpha (\xi_1, \ldots, \xi_{n+1}) = \sum_{i=1}^{n+1} \left( -1 \right)^{\delta_i + \delta_i} X_{\xi_i} \{ \alpha (\xi_1, \ldots, \xi_{i-1}) \}$$

Adding (2.69) and (2.70), one gets (2.71).

The definitions (2.7) and (2.5) immediately imply (2.72) and (2.73).
for the unital graded left $A_U$-module
\[ V_U = U \otimes V. \]  
(3.2)

(ii) each $\lambda \in A^*_0(L, V)$ extends to a $\lambda : A^*_0(U, V)$, the map $\lambda \mapsto \lambda_U$ intertwining the classical operators.

We first describe the module $E_U$, $E$ a unital graded left $A$-module.

3.1. Lemma. Let $A$ and $U$ be real (complex) unital graded-commutative algebras, let $E$ be a unital graded left $A$-module. By letting the skew tensor product $A_U = U \otimes A$ act on 20
\[ E_U = U \otimes E \]
(3.3)
as
\[ (u \otimes a)(v \otimes X) = (-1)^{bh}u \otimes aX \]
(3.4)
we make $E_U$ into a unital graded left $A_U$-module.

Proof: The grade of the r.h.s. of (3.4) is $\partial u + \partial v + \partial aX = \partial (u \otimes a) + \partial v \otimes X$. On the other hand, (3.4) obviously specifies an $E_U$-valued bilinear product such that

\[ \alpha(\beta X) = (\alpha \beta) X, \quad \alpha, \beta \in A_U, \quad X \in E_U \]
(3.5)
by (A.7) (cf. footnote 2 in Appendix A).

3.2. Remark. The left $A_U$-module $E_U$ can be obtained as the tensor product
\[ A_U = A_U \otimes A \]
(3.6)of the left $A_U$-right $A_U$-bimodule $A_U$ by the left $A$-module $E$.

Proof: Indeed, one has $U \otimes A \otimes A \sim U \otimes E$ with $u \otimes a \otimes X = u \otimes aX$, $u \in U$, $a \in A$, $X \in E$. And, with $v \in U$, $u \in V$, $b \in B$
\[ (v \otimes b)(u \otimes aX) = (v \otimes b)[u \otimes aX] = [(v \otimes b)(u \otimes a)] \otimes X = (-1)^{bh}[(vu \otimes ba)] \otimes X = (-1)^{bh}v \otimes baX. \]
(3.7)
We now describe the map $\lambda \mapsto \lambda_U$ acting on $A^*_0(E, F)$, $E$ a unital graded left $A$-modules, $A$, $U$ unital graded commutative algebras. In order to motivate definition (3.9) to follow, let us note that, with $\mu$ a $F_U$-valued $A_U$-linear form on $E_U$, and for $u_i \in U$, $\xi_i \in E$ writing $u_i \otimes \xi_i = (u_i \otimes 1)(1 \otimes \xi_i)$, $A_U$-linearity implies
\[ \mu = (-1)^{\sum_{j=1}^i \partial | j \in \phi} \mu(1 \otimes \xi_1, \ldots, 1 \otimes \xi_i) \]
(3.8)

3.3. Proposition. Let $A$ and $U$ be unital graded commutative algebras and let $E$ and $F$ be unital graded left $A$-modules. 21 Then defining $\lambda_U$ as follows for $\lambda \in \mathcal{L}^n(E, F)$
\[ \lambda_U(u_1 \otimes \xi_1, \ldots, u_i \otimes \xi_i) = (-1)^{\sum_{j=1}^{i-1} \partial | j \in \phi} u_1 \cdots u_{i-1} \otimes \lambda_1(\xi_1, \ldots, \xi_i) \]
(3.9)we define a map $\lambda \mapsto \lambda_U$ from $\mathcal{L}^n(E, F)$ to $\mathcal{L}^n(E_U, F_U)$ preserving grade and commuting with the $\sigma_\sigma$, $\sigma \in \Sigma_n$ and yielding a homomorphism: 22 $A^*_A(E, F) \rightarrow A^*_A(E_U, F_U)$.

Furthermore, for any choice of $\xi_i \in \mathbb{Z}/2$, $i = 1, \ldots, n$ the specification of the restrictions of $\lambda_U$ to $E^*_0(E, F)$ is that $\partial \lambda_U = \partial \lambda$.

Proof: The grade of the r.h.s. of (3.9) is $\partial \lambda + \sum_{i=1}^n (\partial i + \partial \xi_i)$; thus $\partial \lambda_U = \partial \lambda$.

We check that $\lambda_U \in \mathcal{L}^n(E_U, F_U)$ if $\lambda \in \mathcal{L}^n(A, F)$: we have by (3.9), for $u_i \in U$, $\xi_i \in E$, using the short hands $\delta_i = \delta_i(\xi_i, \xi_i)$, $\epsilon = (-1)^{\sum_{j=1}^i \partial | j \in \phi} \mu(1 \otimes \xi_1, \ldots, 1 \otimes \xi_i)$ and taking account of the fact that $(u \otimes a)(u \otimes \xi_i) = (-1)^{pa}u \otimes a \otimes \xi_i$
\[ \lambda_U(u_1 \otimes \xi_1, \ldots, u_i \otimes \xi_i)(u_1 \otimes \xi_i, \ldots, u_i \otimes \xi_i) = \epsilon(-1)^{pa}\sum_{j=1}^i \partial | j \in \phi}
\]
(3.10)
We verify that \( \sigma_{\mathcal{L}} \lambda_\mathcal{V} = (\sigma_{\mathcal{L}}) \lambda_\mathcal{V} \), \( \sigma \in \Sigma_n \). It is enough to check this property for \( \sigma \) a transposition of neighbouring elements, since \( \Sigma_n \) is generated by these transpositions. Let \( \sigma \) be the transposition \([i \leftrightarrow i+1]\). With the above shorthands we have \( \chi(\hat{\xi}, \sigma) = (-1)^{\nu_1+1} \) and \( \chi(u \otimes \hat{\xi}, \sigma) = (-1)^{\nu_1+\nu_2+1} \). Now, from (3.9) we have

\[
(\sigma \lambda_\mathcal{V})(u_1 \otimes \hat{\xi}_1, \ldots, u_n \otimes \hat{\xi}_n) = (-1)^{\eta(n+\eta_1+\nu_1+1)} \lambda_\mathcal{V}(u_1 \otimes \hat{\xi}_1, \ldots, u_i \otimes \hat{\xi}_i, u_{i+1} \otimes \hat{\xi}_{i+1}, u_{i+2} \otimes \hat{\xi}_{i+2}, \ldots, u_n \otimes \hat{\xi}_n).
\]

(3.9a)

Now given a choice of \( \alpha \in \mathbb{Z}/2 \) for \( i = 1, \ldots, n \), assume that the restriction of \( \lambda_\mathcal{V} \) to \( \mathcal{E}^{\alpha}_{i+1} \) vanish for all \( U \). We may take for \( U \) a Grassmann algebra and choose \( u_i \in U \hat{\mathcal{E}}_{i+1} \), \( i = 1, \ldots, n \) with \( u_i, u_2, \ldots, u_n \neq 0 \). The vanishing of the expression (3.9) for all \( \xi_i \in \mathcal{E} \) then implies the vanishing of \( \lambda \). A similar argument proves the last statement of the proposition.

3.4. Remark. (i) Arguing exactly as before, we would obtain a homomorphism of left \( \mathcal{A} \)-modules \( \lambda \rightarrow \lambda_\mathcal{V} \) from \( \mathcal{E}^{\alpha}_{1}(\mathcal{E}^{\beta}_{1}, \ldots, \mathcal{E}^{\beta}_{n}, \mathcal{F}) \) to \( \mathcal{E}^{\alpha}_{n}(\mathcal{E}^{\beta}_{1}, \ldots, \mathcal{E}^{\beta}_{n}, \mathcal{F}) \) for arbitrary unitary graded left \( \mathcal{A} \)-modules \( \mathcal{E}^{1}(\mathcal{E}^{\beta}_{1}, \ldots, \mathcal{E}^{\beta}_{n}, \mathcal{F}) \).

(ii) Using the identification of ordered-permuted tensor products of graded vector spaces discussed in Appendix (A.1), (3.9) can be written in the form

\[
\lambda_{\mathcal{V}}(u_1 \otimes \hat{\xi}_1 \otimes \ldots \otimes u_n \otimes \hat{\xi}_n) = (-1)^{\nu_1} u_1 \otimes \lambda(\hat{\xi}_1, \ldots, \hat{\xi}_n).
\]

(3.9a)

We now apply the foregoing results to extensions of graded Lie-Cartan pairs by a graded commutative algebra.

3.5. Proposition. With \( (L, \mathcal{A}) \) a unital graded Lie-Cartan pair, \( U \) a graded unital commutative algebra \(^{23}\) and \( \mathcal{A}_U \), \( \mathcal{L}_U \) as in (3.1), (3.2), the pair \( (L_U, \mathcal{A}_U) \) becomes a unital graded Lie-Cartan pair over \( U \) if we define as follows the products of \( u \otimes \hat{\xi} \in \mathcal{L}_U \) by \( v \otimes a \in \mathcal{A}_U \).

\[
(u \otimes \hat{\xi})(v \otimes a) = (-1)^{\nu_1 \mathcal{V}} u \mathcal{V} \otimes \hat{\xi} a,
\]

(3.12)

\[
(v \otimes a)(u \otimes \hat{\xi}) = (-1)^{\nu_1 \mathcal{V}} u \mathcal{V} \otimes \hat{\xi} a, \quad u, v \in U, \quad \hat{\xi} \in \mathcal{L}, \quad a \in \mathcal{A}.
\]

(3.13)

Furthermore, with \( V \) a unital graded left \( \mathcal{A} \)-module and \( \varrho \) a \( V \)-connection, letting \( \varrho(u \otimes \hat{\xi}) \), \( u \otimes \hat{\xi} \in \mathcal{L}_U \) act on \( V_U \) as a graded tensor product of endomorphisms:

\[
\varrho(u \otimes \hat{\xi}) = u \otimes \varrho(\hat{\xi}),
\]

(3.14)

\(^{23}\) Both real or complex.

3.5.1. Proposition (i.e.)

\[
\varrho(u \otimes \hat{\xi})(v \otimes \eta) = (-1)^{\nu_1 \mathcal{V}} u \mathcal{V} \otimes \varrho(\hat{\xi})(v \otimes \eta), \quad \xi \in \mathcal{L}, \quad v \in U;
\]

(3.14a)

we define a \( V_U \)-connection \( \varrho \), which is local iff the original \( V \)-connection is local, and whose curvature \( \Omega \) is given by

\[
\Omega(u \otimes \hat{\xi}, v \otimes \eta) = (-1)^{\nu_1 \mathcal{V}} u \mathcal{V} \otimes \varrho(\hat{\xi})(v \mathcal{V} \otimes \hat{\eta}), \quad u \in U, \quad v \in V, \quad \xi \in \mathcal{L}, \quad \eta \in \mathcal{L},
\]

(3.15)

where \( \otimes \) on the r.h.s. denotes a graded tensor product of endomorphisms, i.e. one has, for \( v \in V, \quad X \in V, \eta \in \mathcal{L} \):

\[
\varrho(u \otimes \hat{\xi})(v \otimes \eta)(w \otimes X) = (-1)^{\nu_1 \mathcal{V} + \nu_1 \mathcal{V} + \nu_1 \mathcal{V}} u \mathcal{V} \otimes \varrho(\hat{\xi})(v \mathcal{V} \otimes \hat{\eta})(w \otimes X).
\]

(3.15a)

Proof: We check that \( (L_U, \mathcal{A}_U) \) is a graded Lie-Cartan pair. As noted above, \( L_U = U \otimes \mathcal{L} \) is a graded Lie algebra whilst \( \mathcal{A}_U = U \otimes \mathcal{A} \) is a unital graded commutative algebra. Let \( \hat{\xi}, \eta \in \mathcal{L}, \quad a, b \in \mathcal{A}, \quad u, v, w, z \in U \) and \( X \in V \) be homogeneous.

Check of property (1.1): we have, using (3.12),

\[
\varrho((u \otimes \hat{\xi})(b \otimes a)) = \varrho((b \otimes a)(u \otimes \hat{\xi})) = \varrho(u \otimes \hat{\xi}) + \varrho(b \otimes a).
\]

(3.16)

Check of property (1.3): we have, using (1.3), (3.12),

\[
(w \otimes \hat{\xi})(u \otimes \hat{\xi})(v \otimes \eta) = (-1)^{\nu_1 \mathcal{V} + \nu_1 \mathcal{V} + \nu_1 \mathcal{V}} (w \otimes \hat{\xi})(v \mathcal{V} \otimes \hat{\eta})(u \otimes \hat{\xi}) +
\]

\[
= (-1)^{\nu_1 \mathcal{V} + \nu_1 \mathcal{V} + \nu_1 \mathcal{V}} u \mathcal{V} \otimes \varrho(\hat{\xi})(v \mathcal{V} \otimes \hat{\eta})(w \otimes \hat{\xi})
\]

(3.17)

\[
= (-1)^{\nu_1 \mathcal{V}} u \mathcal{V} \otimes \varrho(\hat{\xi})(v \mathcal{V} \otimes \hat{\eta})(w \otimes \hat{\xi})
\]

(3.18)
Check of property (1.5): one has
\[(u \otimes a) ((v \otimes b)(w \otimes c)) = (u \otimes a)(v \otimes b)(w \otimes c)\] (3.19)
by (A.7).

Check of property (1.6): one has, by (1.6)
\[I \otimes (z \otimes t) = (I \otimes I)(z \otimes t).\] (3.20)

Check of property (1.7): one has, using (3.4), (1.7), (3.8), (3.9)
\[(u \otimes a) ((w \otimes z)(v \otimes b)) = (-1)^{\delta v a} (u \otimes a)(w \otimes z)(v \otimes b)\]
\[= (-1)^{\delta w a + \delta a + \delta v a} uv (w \otimes z) a b\]
\[= (-1)^{\delta a + \delta v a + \delta w a} (u \otimes a)(w \otimes z)(v \otimes b)\]
\[= (u \otimes a)(w \otimes z)(v \otimes b).\] (3.21)

Check of property (1.8): one has, using (1.8), (3.4)
\[a(v \otimes b)(w \otimes c) = (-1)^{\delta w a} (u \otimes a)(v \otimes b)(w \otimes c)\]
\[= (-1)^{\delta w a + \delta w b} uv (w \otimes z) a b\]
\[= (-1)^{\delta w a + \delta w b} (u \otimes a)(v \otimes b)(w \otimes c)\]
\[= (u \otimes a)(v \otimes b)(w \otimes c).\] (3.22)

To check that \((L_u, A_u)\) is a graded Lie-Cartan pair over \(U\) we must show that \(\zeta u = 0\) for \(\zeta \in L_u, \ u \in U < A).\ This follows from
\[(v \otimes z)(u \otimes I) = (-1)^{\delta v a} uv (\otimes z) I = 0\]
by (3.12) and footnote 9 in Section 1.

We now check that \(g\) specified by (3.14) is a \(V_u\)-connection. The two first properties (1.9) are obvious, we check the last one: we have, from (3.15), (A.8)
\[g(u \otimes v)(w \otimes a) = [u \otimes q(v) (v \otimes a)\]
\[= (-1)^{\delta v w} uv \otimes q(v) a\]
\[= (-1)^{\delta w a + \delta w b} uv \otimes q(v) a\]
\[= (-1)^{\delta a + \delta v a} (u \otimes a)(v \otimes b) (w \otimes c)\] (3.23)
Finally we check (3.15): we have, using (A.8) from (3.13)
\[g(u \otimes v), (v \otimes w) \otimes q(u \otimes v) \otimes w = (u \otimes v, v \otimes w) \otimes q(u \otimes v) \otimes w\]
\[= [u \otimes q(v), v \otimes q(w)] - (-1)^{\delta w a} q(u \otimes v) \otimes w\]
\[= (-1)^{\delta w a} uv \otimes [q(v), q(w)] - q([u \otimes v, v \otimes w]).\] (3.24)

We now show how the map \(\lambda \mapsto \lambda u\) applied to \(A^*_\lambda (L, V)\) intertwines the classical operators of Section 2.

3.6. Proposition. Let \((L, V)\) be a unital graded Lie-Cartan pair, with \(q\) a \(V_u\)-connection of \((L, A_u)\), and \(\delta_v, \varphi, \dot{\theta}_v, \dot{\theta}_v, \delta_v, \theta_v, \xi, a, \delta_v + \Omega \wedge \cdot\) the corresponding classical operators acting on \(A^*_\lambda (L, V)\).

Given a graded-commutative algebra \(U\), we consider the corresponding graded Lie-Cartan pair \((L_u, A_u)\) and \(V_u\)-connection \(q\) (cf. Proposition 3.5) and the classical operators attached to the latter acting on \(A^*_\lambda (L, V)\). With \(\lambda \mapsto \lambda u\) the map of Proposition 3.3 for \(E = L\) and \(F = V\) we then have the following intertwining properties: for \(\lambda \in A^*(L, V), \xi, \eta \in L, u \in U, a \in A\)
\[\delta_v (\lambda u) = (\delta_v \lambda) u,\]
\[q \wedge (\lambda u) = (q \wedge \lambda) u,\]
\[\theta_v (u \otimes \xi) (\lambda u) = (u \otimes \xi) (\delta_v \lambda) u,\]
\[\theta_v (u \otimes \xi) (\lambda u) = (u \otimes \xi) (\delta_v \lambda) u,\]
\[\theta_v (u \otimes a) (\lambda u) = (u \otimes a) (\delta_v \lambda),\]
\[\theta_v (u \otimes a) (\lambda u) = (u \otimes a) (\delta_v \lambda),\]
\[\delta_v (u \otimes a) \wedge \lambda u = (u \otimes a) (a \wedge \lambda) u,\]
\[\Omega \wedge (\lambda u) = (\Omega \wedge \lambda) u,\]
\[q(u \otimes \xi, v \otimes \eta) (\lambda u) = (-1)^{\delta v w} (uv \otimes \Omega) (\Omega \lambda) u.\] (3.25)

Proof: Let \(u \in U, \xi_i \in L, i = 1, 2, \ldots, n+1, \)

Check of (3.25): we have, since \([u \otimes \xi_i, u_j \otimes \xi_j] = (-1)^{\delta_{ij} \delta k} u_k \otimes [\xi_i, \xi_j]\) (cf. (A.28)), using (3.9) and the equalities \(\delta \lambda = \delta \lambda u = \delta \lambda^3\)
\[(\lambda u)^3 (u_1 \otimes \xi_1, \ldots, u_{n+1} \otimes \xi_{n+1})\]
\[= (-1)^{\delta_{ij} \delta k} (u_1 \otimes \xi_1, u_2 \otimes \xi_2, \ldots, u_{n+1} \otimes \xi_{n+1})\]. (3.26)
Hence $(\lambda_\nu)^p = (\lambda_\nu)_p$. By Proposition 3.3 it follows that
\[
\frac{2}{n(n+1)} \delta_0(\lambda_\nu) = A_{n+1} \left( (\lambda_\nu)^p \right) = A_{n+1} \left( (\lambda_\nu)_p \right)
\]
\[
= \left( A_{n+1} (\lambda_\nu)_p \right)_V = \frac{2}{n(n+1)} (\delta_0 \lambda_\nu)_V.
\]
(3.37)

Check of (3.26): we have now
\[
(\lambda_\nu)^p (u_1 \otimes \xi_1, \ldots, u_{n+1} \otimes \xi_{n+1})
\]
\[
= (-1)^{\sum_{i=1}^{n+1} \delta_i + \lambda_\nu} (u_1 \otimes \xi_1) \lambda_\nu (u_2 \otimes \xi_2, \ldots, u_{n+1} \otimes \xi_{n+1})
\]
\[
= (-1)^{\sum_{i=1}^{n+1} \delta_i + \lambda_\nu} \sum_{2 \leq i < j \leq n+1} \delta_i \delta_j + \delta_1 \delta_2 + \cdots + \delta_n \delta_{n+1}
\]
\[
\times \varphi (u_1 \otimes \xi_1) [u_2 \otimes \ldots \otimes u_{n+1} \otimes \lambda (\xi_2, \ldots, \xi_{n+1})
\]
\[
= (-1)^{\sum_{i=1}^{n+1} \delta_i + \lambda_\nu} \sum_{2 \leq i < j \leq n+1} \delta_i \delta_j + \delta_1 \delta_2 + \cdots + \delta_n \delta_{n+1}
\]
\[
\times u_1 u_2 \cdots u_{n+1} \otimes \lambda (\xi_2, \ldots, \xi_{n+1})
\]
\[
= (-1)^{\sum_{i=1}^{n+1} \delta_i + \lambda_\nu} \sum_{2 \leq i < j \leq n+1} \delta_i \delta_j + \delta_1 \delta_2 + \cdots + \delta_n \delta_{n+1}
\]
\[
\times u_1 u_2 \cdots u_{n+1} \otimes \lambda (\xi_2, \ldots, \xi_{n+1})
\]
\[
= (-1)^{\sum_{i=1}^{n+1} \delta_i + \lambda_\nu} \sum_{2 \leq i < j \leq n+1} \delta_i \delta_j + \delta_1 \delta_2 + \cdots + \delta_n \delta_{n+1}
\]
\[
\times (\lambda_\nu)_V (u_1 \otimes \xi_1, \ldots, u_{n+1} \otimes \xi_{n+1})
\]
(3.38)

hence $(\lambda_\nu)^p = (\lambda_\nu)_p$, and
\[
\frac{1}{n+1} q \wedge \lambda_\nu = A_{n+1} \left( (\lambda_\nu)^p \right) = A_{n+1} \left( (\lambda_\nu)_p \right)
\]
\[
= \left( A_{n+1} (\lambda_\nu)_p \right)_V = \frac{1}{n+1} (q \wedge \lambda)_V.
\]
(3.39)

(3.27) then follows from (3.25), (3.26).
28 A. JADCZYK and D. KASTLER

\[ \delta(\omega + \eta) = \delta(\omega + \eta) + \delta(\omega + \eta) + \omega + \eta \]

hence \((\omega, \eta) = (u \otimes id)(\lambda^2)\).

Then

\[ \frac{1}{n+1} \delta(u \otimes \omega \wedge (\lambda_v) = A_{n+1}(\lambda_v)^{\omega \otimes \omega} = (u \otimes id) A_{n+1} (\lambda^2)_v \]

\[ = (u \otimes id) \{ A_{n+1} (\lambda^2) \}^v = \frac{1}{n+1} (u \otimes id) (\delta \lambda \wedge \lambda)_v. \]  

Check of (3.34): from (3.9), (3.15a), we have

\[ \Omega \wedge (\lambda_v) \}

\[ = (u \otimes id) (\lambda v)_v \}

\[ = (u \otimes id) (\lambda^2) \}

\[ = (u \otimes id) \{ (\lambda^2)_v \}^v = \frac{1}{n+1} (u \otimes id) (\delta \lambda \wedge \lambda)_v. \]  

Check of (3.35): from (2.5), (3.15), we have

\[ \Omega \wedge (\lambda_v) \}

\[ = (u \otimes id) (\lambda^2) \}

\[ = (u \otimes id) \{ (\lambda^2)_v \}^v = \frac{1}{n+1} (u \otimes id) (\delta \lambda \wedge \lambda)_v. \]  

3.7. Remark. Restricting the above classical operators to the arguments from the even part \(L_0\), we thus arrive at the classical (Abelian) situation discussed in [1].
4. Derivation properties

In this section we describe two types of derivations properties fulfilled by the classical operators attached to a V-connection (cf. Section 2). On the other hand, if V carries a bilinear 24 (not necessarily associative) product, this product extends as a graded wedge product of $A^* (L, V)$ which becomes an algebra 25 for which the classical operators are derivations. On the other hand, for a general A-module V (which we then prefer to consider as a right A-module, cf. (A.40) in Appendix A), $A^* (L, V)$ is a right $A^* (L, A)$-module and the classical operators are derivations of this module. Both are particular cases of a more general situation when there is given a bilinear map $V \times V' \rightarrow V''$ (see Proposition 4.4 below).

4.1. DEFINITION. Let $(L, A)$ be a graded Lie–Cartan pair, with $V, V', V''$ unital left $A$-modules and let there be given a graded bilinear product

$$V \times V' \otimes V'' \rightarrow V''', \quad i, j \in \mathbb{Z}/2.$$ 

(4.1)

Given $\lambda \in \mathcal{L}^n (L, V), \mu \in \mathcal{L}^m (L, V')$, their graded tensor product $\lambda \otimes \mu \in \mathcal{L}^{n+m} (L, V'')$ is defined by

$$\lambda \otimes \mu (\xi_1, \ldots, \xi_{n+m}) = (-1)^{i_1 \cdots i_n} \lambda (\xi_1, \ldots, \xi_i) \mu (\xi_{i+1}, \ldots, \xi_{n+m}),$$

(4.2)

their graded wedge product $\lambda \wedge \mu \in \Lambda (L, V'')$ then being given by

$$\lambda \wedge \mu = ((n+m)!/m! m!) A_{n+m} (\lambda \otimes \mu),$$

(4.3)

where $A_\chi$ denotes the graded antisymmetrizer determined by the graded alternate character $\chi$ (cf. Definition B.3).

4.2. Remark. The elements of

$$V \otimes_R \Lambda^n (L, R) / (V \otimes_C \Lambda^n (L, C),$$

resp. $V \otimes_A \Lambda^n (L, A)$ can be considered as $V$-valued $R$-bi (in $C$-bi) linear (C-$n$-linear) forms, on $L$, resp. $V$-valued $n$-linear forms on $L$, the identification being in both cases given by

$$\alpha (\otimes \phi (\xi_1, \ldots, \xi_n) = (-1)^{\alpha \otimes + \xi_1 \cdots \xi_n} \phi (\xi_1, \ldots, \xi_n) X,$$

(4.4)

$$X \otimes \phi \in V \otimes_R \Lambda^n (L, R) / (V \otimes_C \Lambda^n (L, C),$$

resp. $V \otimes_A \Lambda^n (L, A)$.

4.3. PROPOSITION. With the notations and definitions in Definition 4.1 we have that

(i) If $\lambda$ and $\lambda'$ belong to $\Lambda^n (L, V)$, and $\Lambda^m (L, V)$, then $\lambda \otimes (\lambda')'$ and $\lambda \wedge (\lambda')'$ are in $\Lambda_{n+m} (L, V')$.

(ii) A $\Lambda^n (L, V)$-linear $\phi$, $\phi' \in \mathcal{L}^n (L, V)$, $\phi (\xi_1, \ldots, \xi_n) \phi' (\xi_1, \ldots, \xi_n) X = \phi (\xi_1, \ldots, \xi_n) \phi' (\xi_1, \ldots, \xi_n) X$, cf. (A.42),

$$X \otimes \phi \in V \otimes_R \Lambda^n (L, R) / (V \otimes_C \Lambda^n (L, C),$$

resp. $V \otimes_A \Lambda^n (L, A)$. 

24 $(C)$-bilinear or $A$-bilinear. In the latter case, if the connection is local, one may consider $\Lambda^* (L, V)$.

25 In general non-associative.
and \( \land \) defined in (4.2), resp. (4.3) \((\mathcal{L}^* (L, V) \otimes \mathcal{A}^* (L, V, \land))\) are then associative algebras with respective subalgebras \(\mathcal{L}^*_\Lambda (L, V)\) and \(\mathcal{A}^*_\Lambda (L, V)\).

(iii) If \( V = V' = V''\) and the product of \( V \) is graded commutative, the graded wedge product of \(\mathcal{A}^* (L, V)\) fulfills

\[
\mu \land \lambda = (1)^{\mu + \nu + 1} \lambda \land \mu,
\]

\(\lambda \in \mathcal{A}^* (L, V), \quad \mu \in \mathcal{A}^* (L, V)\).

Proof: (i) The proof is the same as that of the \( A\)-linearity of \(\lambda \otimes \mu \) in Proposition B.6, cf. equations (B.38), (B.39) in Appendix B.

(ii) We have, for \( X, X', X'' \in V', \varphi, \varphi', \psi' \in \mathcal{L}^* (L, R) \{\mathcal{L}^* (L, C)\},

\[
\{ (X \otimes \varphi) \wedge (X' \otimes \varphi') \} \wedge (X'' \otimes \varphi'') = \left( -1 \right)^{\nu \otimes \psi} \left( (X \otimes X') \otimes (\varphi \wedge \varphi') \right)
\]

whilst

\[
(X \otimes \varphi) \{ (X' \otimes \varphi') \wedge (X'' \otimes \varphi'') \} = \left( -1 \right)^{\nu \otimes \psi} \left( (X \otimes X') \otimes (\varphi \wedge \varphi') \right)
\]

showing that the product \( \wedge \) is associative if this holds for the product. The proof for the tensor product \( \otimes \) is identical (or can be checked directly as in (B.40)).

(iii) One has then, for \( X, X' \in V, \varphi, \varphi' \in \mathcal{A}^* (L, V)\)

\[
\{ (X \otimes \varphi) \wedge (X' \otimes \varphi') \} \wedge (X'' \otimes \varphi'') = \left( -1 \right)^{\nu \otimes \psi} \left( (X \otimes X') \otimes (\varphi \wedge \varphi') \right)
\]

4.4. Proposition. With the graded Lie–Cartan pair \((L, A)\) and the \( A\)-modules \(V_i (i = 1, 2, 3)\) as in Definition 4.1, and with \( q_i (i = 0, V)\), being \( L\)-connections, we consider the corresponding classical operators (acting on \( \mathcal{L}^* (L, V_i) \)) as specified in Definition 2.1 along with their restrictions to \(\mathcal{A}^* (L, V_i)\) resp. \( q_i \) local, to \(\mathcal{A}^*_\Lambda (L, V_i)\) (cf. Proposition 2.2). These operators fulfil the following derivation properties: one has for \( \xi \in L, \lambda_i \in \mathcal{A}^*(L, V_i) \), \( i = 1, 2 \)

\[
i (\xi) (\lambda_1 \wedge \lambda_2) = [i (\xi) \lambda_1] \wedge \lambda_2 + (-1)^{\mu \otimes \lambda_2} \lambda_1 \wedge [i (\xi) \lambda_2]
\]

and, if \( q_3 \) is the "tensor product" of \( q_1 \) and \( q_2 \), i.e. if

\[
q_3 (X_1 \wedge X_2) = q_1 (X_1) \cdot X_2 + (-1)^{\nu \otimes \lambda_2} \lambda_1 \wedge q_2 (X_2) \]

\( X_1 \in V_1, X_2 \in V_2 \).

then one has

\[
\theta_{q_3} (\xi) (\lambda_1 \wedge \lambda_2) = [i (\xi) \lambda_1] \wedge \lambda_2 + (-1)^{\mu \otimes \lambda_2} \lambda_1 \wedge [i (\xi) \lambda_2] \]

\[
\theta_{q_3} (\lambda_1 \wedge \lambda_2) = (\delta_0 (\lambda_1) \wedge \lambda_2 + (-1)^{\mu \otimes \lambda_2} \lambda_1 \wedge \delta_0 (\lambda_2) \]

\[
\omega_{q_3} (\xi) (\lambda_1 \wedge \lambda_2) = (\Omega_1 \wedge \lambda_1) \wedge \lambda_2 + \lambda_1 \wedge (\Omega_2 \wedge \lambda_2) \]

In fact, one has, separately

\[
\theta_{q_0} (\xi) (\lambda_1 \wedge \lambda_2) = [i (\xi) \lambda_1] \wedge \lambda_2 + (-1)^{\mu \otimes \lambda_2} \lambda_1 \wedge [i (\xi) \lambda_2] \]

\[
\theta_{q_0} (\lambda_1 \wedge \lambda_2) = (\delta_0 (\lambda_1) \wedge \lambda_2 + (-1)^{\mu \otimes \lambda_2} \lambda_1 \wedge \delta_0 (\lambda_2) \]

resulting from

\[
\theta_{q_0} (\xi) (\lambda_1 \wedge \lambda_2) = [i (\xi) \lambda_1] \wedge \lambda_2 + (-1)^{\mu \otimes \lambda_2} \lambda_1 \wedge [i (\xi) \lambda_2] \]

\[
\theta_{q_0} (\lambda_1 \wedge \lambda_2) = (\delta_0 (\lambda_1) \wedge \lambda_2 + (-1)^{\mu \otimes \lambda_2} \lambda_1 \wedge \delta_0 (\lambda_2) \]

and

\[
\delta_0 (\lambda_1 \wedge \lambda_2) = (\delta_0 (\lambda_1) \wedge \lambda_2 + (-1)^{\mu \otimes \lambda_2} \lambda_1 \wedge \delta_0 (\lambda_2) \]

\[
\omega_{q_0} (\xi) (\lambda_1 \wedge \lambda_2) = (\Omega_1 \wedge \lambda_1) \wedge \lambda_2 + \lambda_1 \wedge (\Omega_2 \wedge \lambda_2) \]

Proof: Check of (4.12) for \( V_i = A \). By linearity, it is enough to check that (4.12) holds for \( \lambda = q_i \). Hence, one has

\[
i (\xi) (a \wedge \mu) = (1)^{\nu \otimes \lambda} \lambda \wedge [i (\xi) \mu] \]

\(\xi = \xi_1, \ldots, \xi_m \in L\).

Further, using (B.43),

\[
i (\xi) (\phi \wedge \mu) = (1)^{\nu \otimes \lambda} \lambda \wedge [i (\xi) \mu] \]

\(\xi = \xi_1, \ldots, \xi_m \in L\).

(4.23)

(4.24)

(4.25)

(4.26)

27 (4.12) can be proved by direct computation. Since this is cumbersome we use tensorial notation for the proof. (4.12) corresponds to Theorem 3.3 in [18].
Assuming now (4.12) to hold for $\lambda$ and $\mu$, we show that it holds for $\varphi \land \lambda$ and $\mu$ we have, by what precedes
\[ i(\zeta) \left. \varphi \right| \lambda \land \mu \left| -(-1)^{\text{deg}(\varphi) + \text{deg}(\lambda) + \text{deg}(\mu)} i(\zeta) \right| \lambda \land \mu \mid \left| +(-1)^{\text{deg}(\varphi) + \text{deg}(\lambda) + \text{deg}(\mu)} i(\zeta) \right| \lambda \land \mu \mid \leq \left( \begin{array}{c} \text{deg}(\varphi) + \text{deg}(\lambda) + \text{deg}(\mu) \end{array} \right) \}
\]

We checked (4.12) on $A^\ast(L, A)$. Now $i(\zeta)$ on $A^\ast(L, V) = V \otimes A^\ast(L, A)$ is given by
\[ i(\zeta) = i(\zeta) \]

in the sense (A.7): indeed, for $X \in V'$, $\varphi \in A^\ast(L, A)$,
\[ i(\zeta) \left. \varphi \right| X \otimes \varphi \left| \zeta_{n-1} \right| = (-1)^{\text{deg}(\varphi) + \text{deg}(X) + \text{deg}(\varphi) + \text{deg}(\zeta_{n-1})} \varphi(\zeta, \zeta, ..., \zeta_{n-1}) X
\]

Now, for $X, X' \in V'$, $\varphi, \varphi' \in A^\ast(L, A)$, we have
\[ i(\zeta) \left. \varphi \right| X \otimes \varphi' \left| \zeta_{n-1} \right| = (-1)^{\text{deg}(\varphi) + \text{deg}(X) + \text{deg}(\varphi) + \text{deg}(\zeta_{n-1})} \varphi(\zeta, \zeta, ..., \zeta_{n-1}) X
\]

Now, we check (4.19) directly: for $\zeta_{1}, ..., \zeta_{n+m} \in L'$, we have
\[ -\theta(\zeta) \left[ \left( \begin{array}{c} \text{deg}(\varphi) + \text{deg}(X) + \text{deg}(\varphi) + \text{deg}(\zeta_{n-1}) \end{array} \right) \}
\]

We proved (4.12). We now check (4.19) directly: for $\zeta_{1}, ..., \zeta_{n+m} \in L'$, we have
\[ -\theta(\zeta) \left[ \left( \begin{array}{c} \text{deg}(\varphi) + \text{deg}(X) + \text{deg}(\varphi) + \text{deg}(\zeta_{n-1}) \end{array} \right) \}
\]

Now, (4.14) follows from (4.17) and (4.20). We now prove (4.15) by induction w.r.t. $n+m$, using (4.12), (4.14) and the Cartan identity (2.39): we have, for $\zeta \in L'$
\[ i(\zeta) \left[ \left( \begin{array}{c} \text{deg}(\varphi) + \text{deg}(X) + \text{deg}(\varphi) + \text{deg}(\zeta_{n-1}) \end{array} \right) \}
\]

Since $\zeta$ is arbitrary, we see that (4.15) holds for $n+m = k+1$ if it holds for $k$. Now (4.15) holds for $n = m = 0$, because for $X_{1} \in A^\ast(L, V_{1}) = V_{1}$, $X_{2} \in A^\ast(L, V_{2}) = V_{2}$
\[ \{\zeta \otimes (X_{1} \otimes X_{2})\}(\zeta) = \{\zeta \otimes (X_{1} \otimes X_{2})\}(\zeta) = (-1)^{\text{deg}(X_{1}) + \text{deg}(X_{2})} \{(\zeta) X_{1} \otimes X_{2}\}
\]

Now, (4.17) follows from (4.19) and (4.3) since $\theta(\zeta)$ (defined on $A^\ast(L, V)$) commutes with $A_{4}$. To prove this, it is enough to check that $\theta(\zeta)$ commutes with the action of the transposition $\sigma_{\zeta} = \{\zeta_{\zeta} \leftrightarrow \zeta_{\zeta+1}\}$. Now, with $\theta(\zeta)$ the $i_{th}$ term in the r.h.s. of (2.14) $\sigma_{\zeta}$ leaves all $\theta(\zeta)$ with $i \neq k, i \neq k+1$ unaffected and exchanges $\theta(\zeta)$ with $\theta^{(k+1)}(\zeta)$.

We now check (4.20), from which (4.17) then follows, since $\varphi(\zeta)$ evidently commutes with all $\sigma_{\zeta} \in \Sigma_{\zeta}$. We have, for $\zeta_{1}, ..., \zeta_{n+m} \in L'$ using (4.13)
\[ \{\zeta \otimes (\lambda_{1} \otimes \lambda_{2})\}(\zeta_{1}, ..., \zeta_{n+m})
\]

Now (4.14) follows from (4.17) and (4.20). We now prove (4.15) by induction w.r.t. $n+m$, using (4.12), (4.14) and the Cartan identity (2.39): we have, for $\zeta \in L'$
\[ i(\zeta) \left[ \left( \begin{array}{c} \text{deg}(\varphi) + \text{deg}(X) + \text{deg}(\varphi) + \text{deg}(\zeta_{n-1}) \end{array} \right) \}
\]

Since $\zeta$ is arbitrary, we see that (4.15) holds for $n+m = k+1$ if it holds for $k$. Now (4.15) holds for $n = m = 0$, because for $X_{1} \in A^\ast(L, V_{1}) = V_{1}$, $X_{2} \in A^\ast(L, V_{2}) = V_{2}$
\[ \{\zeta \otimes (X_{1} \otimes X_{2})\}(\zeta) = \{\zeta \otimes (X_{1} \otimes X_{2})\}(\zeta) = (-1)^{\text{deg}(X_{1}) + \text{deg}(X_{2})} \{(\zeta) X_{1} \otimes X_{2}\}
\]

Now, (4.17) follows from (4.19) and (4.3) since $\theta(\zeta)$ (defined on $A^\ast(L, V)$) commutes with $A_{4}$. To prove this, it is enough to check that $\theta(\zeta)$ commutes with the action of the transposition $\sigma_{\zeta} = \{\zeta_{\zeta} \leftrightarrow \zeta_{\zeta+1}\}$. Now, with $\theta(\zeta)$ the $i_{th}$ term in the r.h.s. of (2.14) $\sigma_{\zeta}$ leaves all $\theta(\zeta)$ with $i \neq k, i \neq k+1$ unaffected and exchanges $\theta(\zeta)$ with $\theta^{(k+1)}(\zeta)$.

We now check (4.20), from which (4.17) then follows, since $\varphi(\zeta)$ evidently commutes with all $\sigma_{\zeta} \in \Sigma_{\zeta}$. We have, for $\zeta_{1}, ..., \zeta_{n+m} \in L'$ using (4.13)
\[ \{\zeta \otimes (\lambda_{1} \otimes \lambda_{2})\}(\zeta_{1}, ..., \zeta_{n+m})
\]

Now (4.14) follows from (4.17) and (4.20). We now prove (4.15) by induction w.r.t. $n+m$, using (4.12), (4.14) and the Cartan identity (2.39): we have, for $\zeta \in L'$
\[ i(\zeta) \left[ \left( \begin{array}{c} \text{deg}(\varphi) + \text{deg}(X) + \text{deg}(\varphi) + \text{deg}(\zeta_{n-1}) \end{array} \right) \}
\]

Since $\zeta$ is arbitrary, we see that (4.15) holds for $n+m = k+1$ if it holds for $k$. Now (4.15) holds for $n = m = 0$, because for $X_{1} \in A^\ast(L, V_{1}) = V_{1}$, $X_{2} \in A^\ast(L, V_{2}) = V_{2}$
\[ \{\zeta \otimes (X_{1} \otimes X_{2})\}(\zeta) = \{\zeta \otimes (X_{1} \otimes X_{2})\}(\zeta) = (-1)^{\text{deg}(X_{1}) + \text{deg}(X_{2})} \{(\zeta) X_{1} \otimes X_{2}\}
\]

Now, (4.17) follows from (4.19) and (4.3) since $\theta(\zeta)$ (defined on $A^\ast(L, V)$) commutes with $A_{4}$. To prove this, it is enough to check that $\theta(\zeta)$ commutes with the action of the transposition $\sigma_{\zeta} = \{\zeta_{\zeta} \leftrightarrow \zeta_{\zeta+1}\}$. Now, with $\theta(\zeta)$ the $i_{th}$ term in the r.h.s. of (2.14) $\sigma_{\zeta}$ leaves all $\theta(\zeta)$ with $i \neq k, i \neq k+1$ unaffected and exchanges $\theta(\zeta)$ with $\theta^{(k+1)}(\zeta)$.
Setting \( q = 0 \) in the preceding calculation, we obtain the proof of (4.21), without recourse to (4.19). Finally (4.16) follows from (4.15) via (2.34): we have
\[
\Omega_3 \wedge (\lambda_1 \wedge \lambda_2) = \delta_{\Omega_3}^2 (\lambda_1 \wedge \lambda_2) = \delta_{\Omega_3} (\delta_{\Omega_3} (\lambda_1 \wedge \lambda_2) + (-1)^{\Omega_3} \lambda_1 \wedge \lambda_2)
\]
\[
= (\delta_{\Omega_3}^2 \lambda_1 \wedge \lambda_2) + \lambda_1 \wedge \delta_{\Omega_3} \lambda_2. 
\]
(4.33)
Returning now the case of a general left \( A \)-module for which we chose a \( \nabla \)-connection, we shall describe the module-derivation properties of the corresponding classical operators. To this end we prefer to look at Appendix A, cf. (4.40) as an explanation in Appendix A, as a right \( A^* (L, V) \)-module, and we consider \( A^* (L, A) \) as a right \( A^* (L, A) \)-module (cf. (4.34) below). The classical operators are then derivations of this module according to the general

4.5. DEFINITION. Let \( \mathcal{A} \) be a \( Z/2 \)-graded complex algebra, with \( \mathcal{A} \) a graded right \( \mathcal{A} \)-module; and let \( \delta \) be a derivation of \( \mathcal{A} \) (in the \( Z/2 \)-graded sense, cf. (A.22) in Appendix A).

A \( \delta \)-derivation of grade \( p \) is a linear map \( D \) of degree \( p \) fulfilling
\[
D(\lambda a) = (D \lambda)a + (-1)^{\mathcal{A} \lambda} \delta \lambda a \tag{4.34}
\]
for all \( \lambda \in \mathcal{A} \) of grade \( \mathcal{A} \lambda \) and \( a \in \mathcal{A} \).

This will apply to our situation, making \( \delta = A^* (L, V) \) and \( \mathcal{A} = A^* (L, A) \) in

4.6. COROLLARY. With \( (L, A) \) a graded Lie–Cartan pair, we consider the classical operators attached to a \( \nabla \)-connection \( \nabla \), \( V \), a graded left \( A \)-module (cf. Definition 2.1). Moreover,

(i) we consider \( V \) as a right \( A \)-module (cf. (A.40));
(ii) we equip \( A^* (L, A) \) with the graded wedge product \( \wedge \) obtained by specializing (4.3) to the case \( V = A \) with its graded-commutative product and with the classical operators defined in Definition 2.1 where we make \( V = A \) and \( \varphi (\xi) = \xi, \xi \in L \); and
(iii) we consider \( A^* (L, V) \) as a right \( A^* (V, A) \)-module by the formulae (4.2), (4.3), setting \( V = V, V^* = A, V'' = V \).

We then have the following derivation properties for the classical operators: for \( \xi \in L, \lambda \in A^* (E, V), a \in A^* (E, A) \):
\[
i (\xi) (\lambda a) = [i (\xi) \lambda] a + (-1)^{\mathcal{A} \lambda} \delta [i (\xi) a], \tag{4.37}
\]
\[
\varphi (\xi) (\lambda a) = [\varphi (\xi) \lambda] a + (-1)^{\mathcal{A} \lambda} \delta [\varphi (\xi) a], \tag{4.38}
\]
\[
\theta_\varphi (\xi) (\lambda a) = [\theta_\varphi (\xi) \lambda] a + (-1)^{\mathcal{A} \lambda} \delta [\theta_\varphi (\xi) a]. \tag{4.39}
\]

REFERENCES


APPENDIX A. GRADED VECTOR SPACES, GRADED HOMOMORPHISMS, GRADED ALGEBRAS, GRADED MODULES

Throughout this appendix all vector spaces, or algebras are either real or complex; with one of these alternatives holding throughout. Graded means throughout \( Z/2 \)-graded.

A1. Graded vector spaces

A graded vector space is a vector \( E \), together with a direct sum decomposition
\[
E = E^0 \oplus E^1. \tag{A.1}
\]
in two vector subspaces \( E^0, E^1 \) consisting of the even, resp. odd elements of \( E \).
This entails, for the composition rule \( = \) the grading composition of \(-\text{homomorphisms as a product} \) build a category. We denote furthermore by trivially graded, the we see that the graded vector spaces together with their homomorphisms resp. spaces corresponding to and we set, for we define. Now with \( \text{Hom}(E, F) \) from \( E \) to \( F \) as the vector space of linear maps: \( E \rightarrow F \) equipped with the grading
\[
\text{Hom}(E, F)^{0} = \{ a \in \text{Hom}(E, F); aE^{i} \subset F^{i}, i \in \mathbb{Z}/2 \},
\]
\[
\text{Hom}(E, F)^{1} = \{ a \in \text{Hom}(E, F); aE^{i+1} \subset F^{i}, i \in \mathbb{Z}/2 \}. \tag{A.4a}
\]
The usual notion of vector space appears as the specialization of graded vector spaces corresponding to trivial grading, i.e., \( E = E^{0}, E^{1} = \{ 0 \} \). Considering \( R(C) \) as trivially graded, the dual of the graded vector space \( E \) is then \( E^{*} = \text{Hom}(E, R) \) \( (= \text{Hom}(E, C)) \), in other terms \( E^{0*} \) and \( E^{1*} \) are the respective annihilators of \( E^{1} \), resp. \( E^{0} \) in \( E^{*} \).

Note that, since we have, with \( E, F \) graded vector spaces
\[
\text{Hom}(E, G) = \text{Hom}(F, G) \odot \text{Hom}(E, F) \tag{A.5}
\]
we see that the graded vector spaces together with their homomorphisms (with composition of homomorphisms as a product) build a category.

With \( E, F \) graded vector spaces their tensor product \( E \otimes F \) is that of \( E, F \) as vector spaces, taken as a graded vector space with the grading
\[
(E \otimes F)^{i+j} = \bigoplus_{i+j=k} E^{i} \otimes F^{j}. \tag{A.6}
\]
Now with \( E, F, E', F' \) graded vector spaces and \( S \in \text{Hom}(E, E'), T \in \text{Hom}(F, F') \) we define graded tensor product \( S \otimes T \in \text{Hom}(E \otimes F, E' \otimes F') \) by
\[
(S \otimes T)(\zeta \otimes \eta) = (-1)^{pq} S \zeta \otimes T \eta, \quad \zeta \in E', \eta \in F'. \tag{A.7}
\]
This entails, for \( S \in \text{Hom}(E', E''); T \in \text{Hom}(F', F'') \) \( (E'', F'' \) graded vector spaces), the composition rule
\[
(S \otimes T)(S' \otimes T') = (-1)^{pq} S' S' \otimes T' T \tag{A.8}
\]
where we have the following formula for graded commutators:
\[
[S \otimes T, S' \otimes T'] = (-1)^{pq} SS' \otimes [T, T'] \tag{A.9}
\]
if \( S' S = (-1)^{pq} SS', \quad S, S' \in \text{Hom}(E, E), \quad T, T' \in \text{Hom}(F, F) \).

With \( E_{1}, E_{2}, E_{3} \) graded vector spaces, we shall identify \( (E_{1} \otimes E_{2}) \otimes E_{3} \) and \( E_{1} \otimes (E_{2} \otimes E_{3}) \) (denoted \( E_{1} \otimes E_{2} \otimes E_{3} \)) by making the identification \(1\)
\[
\xi_{1} \otimes (\xi_{2} \otimes \xi_{3}) = (\xi_{1} \otimes \xi_{2}) \otimes \xi_{3}, \quad \xi_{1} \in E_{1}, i = 1, 2, 3. \tag{A.10}
\]
This matches the definition (A.7) in as much as one has, for \( S \in \text{Hom}(E_{i}, F_{j}) \), \( i = 1, 2, 3 \)
\[
S \otimes (S' \otimes S'' \otimes S') = (S \otimes S' \otimes S'') (S \otimes S' \otimes S''), \tag{A.11}
\]
which acting on \( \xi_{1} \otimes \xi_{2} \otimes \xi_{3}, \quad \xi_{1} \in E_{1}, i = 1, 2, 3 \), yields \(2\)
\[
(S \otimes S \otimes S) (\xi_{1} \otimes \xi_{2} \otimes \xi_{3}) = (-1)^{i_{1}i_{2}+i_{1}i_{3}+i_{2}i_{3}} S_{1} \xi_{1} \otimes S_{2} \xi_{2} \otimes S_{3} \xi_{3}. \tag{A.12}
\]
With \( E, F \), graded vector spaces, we may also identify \( F \otimes E \) with \( E \otimes F \) by deciding that
\[
\eta \otimes \xi = (-1)^{pq} \xi \otimes \eta, \quad \xi \in E', \eta \in F'. \tag{A.13}
\]
With \( E', F' \) other graded vector spaces, identifying like this \( F' \otimes E' \) with \( E' \otimes F' \), and identifying \( \text{Hom}(F \otimes E, F' \otimes E') \) with \( \text{Hom}(E \otimes F, E' \otimes F') \), by deciding that
\[
T \otimes S = (-1)^{pq} S \otimes T, \quad S \in \text{Hom}(E, E'), T \in \text{Hom}(F, F') \tag{A.14}
\]
we then have that
\[
(T \otimes S)(\eta \otimes \xi) = (-1)^{pq} \xi \otimes \eta. \tag{A.15}
\]
These identifications allow us to alter the order of the factors in tensor products according to convenience.

We close this section by noting that, with \( E \) and \( F \) graded vector spaces, the set \( \mathcal{L}^{e}(E, F) \), of \( F \)-valued \( n \)-linear forms on \( E \), identifies with \( \text{Hom}(E^{*n}, F) \) by requiring
\[
\lambda(\xi_{1}, ..., \xi_{n}) = \lambda(\xi_{1} \otimes ... \otimes \xi_{n}), \quad \xi_{1}, ..., \xi_{n} \in E \tag{A.16}
\]
with the implication that
\[
\partial \{ \lambda(\xi_{1}, ..., \xi_{n}) \} = \partial \lambda + \sum_{i=1}^{n} \partial \xi_{i}, \quad \lambda \in \mathcal{L}^{e}(E, F), \quad \xi_{1}, ..., \xi_{n} \in E. \tag{A.17}
\]

\(^{1}\) Identification compatible with the graded vector space structure.

\(^{2}\) We then write \( \xi_{1} \otimes \xi_{2} \otimes \xi_{3} \) and \( S \otimes S \otimes S \) to mean the common value (A.9), resp. (A.10).
A.2 Graded algebras

A graded algebra is a graded vector space $A = A^0 \oplus A^1$ with a bilinear product $A \times A \to A$ such that

$$A^i A^j \subset A^{i+j}, \quad i, j \in \mathbb{Z}/2.$$  \hfill (A.18)

The graded algebra $A$ is associative whenever

$$(ab)c = (a(bc)), \quad a, b, c \in A,$$  \hfill (A.19)

it is graded commutative whenever it is associative and such that

$$ba = (-1)^{ab} ab, \quad a, b \in A.$$  \hfill (A.20)

A graded Lie algebra$^3$ is a graded algebra $L$ whose product, called the bracket and denoted $(\xi, \eta) \mapsto [\xi, \eta] \in L$ is such that

$$[\eta, \xi] = (-1)^{\eta \xi + 1} [\xi, \eta],$$  \hfill (A.21)

and with the skew product, bilinear extension of

$$(a \otimes b)(a' \otimes b') = (-1)^{ab'} a a' \otimes b b',$$  \hfill (A.22)

where $E$ is a graded vector space, $\text{End}(E) = \text{Hom}(E, E)$ with the grading (A.4) is a graded commutative algebra under the composition of endomorphisms, and $A$ is a graded Lie algebra under the graded commutator — bilinear extension of

$$[a, b] = ab - (-1)^{ab} ba, \quad a, b \in \text{End}(E).$$  \hfill (A.22)

In fact, each associative graded algebra $A$ becomes a graded Lie algebra (called the commutator Lie algebra of $A$) under the graded commutator defined as in (A.22) for $a, b \in A$.

With $A$ a graded algebra, the graded vector space $\text{Der} A$ of graded derivations of $A$ is defined as

$$\text{Der} A = (\text{Der} A)^0 \oplus (\text{Der} A)^{1}$$  \hfill (A.23)

with, for $i \in \mathbb{Z}/2$

$$(\text{Der} A)^i = \{ \xi \in \text{End}(A)^i : \xi(ab) = (\xi a)b + (-1)^{a\xi} a(\xi b) \}, \quad a \in A^i, \quad b \in A,$$  \hfill (A.24)

where $\text{End}(A)$ denotes the set of endomorphisms of $A$ as a vector space. With this definitions, $\text{Der} A$ is a sub-graded Lie algebra of the commutator Lie algebra of $\text{End}(A)$. If furthermore $A$ is graded commutative, the composition $a_\xi$ of $\xi \in (\text{Der} A)^i$ and multiplication from the right by $a \in A^i$

$$(a_\xi) b = a(\xi b), \quad b \in A$$  \hfill (A.25)

belongs to $(\text{Der} A)^{i+1}$, and we have that

$$[\xi, \eta] = (-1)^{ab} a [\xi, \eta] + (\xi a) \eta,$$  \hfill (A.26)

so that $(A, \text{Der} A)$ is a graded Lie-Cartan pair, cf. Definition 1.1.

Given two graded algebras $A$ and $B$, their skew tensor product $A \otimes B$ is, as a vector space, the usual tensor product of $A$ and $B$ as vector spaces, equipped with the grading

$$(A \otimes B)^i = \sum_{j+k = i} A^j \otimes B^k$$  \hfill (A.27)

and with the skew product, bilinear extension of

$$(a \otimes b)(a' \otimes b') = (-1)^{ab'} a a' \otimes b b',$$  \hfill (A.28)

as graded algebra, for $A, B, C$ graded algebras.

Furthermore, the skew product of two associative algebras is associative, the skew product of two graded commutative algebras is graded commutative; and the skew product of a graded commutative algebra and of a graded Lie algebra (in either order) is a graded Lie algebra.

Given a graded associative algebra $A$, one defines as follows the graded associative algebra $\tilde{A}$ obtained from $A$ by adjoining a unit 1. We set

$$\tilde{A} = R \oplus A \quad (C \oplus A)$$  \hfill (A.30)

(direct sum of vector spaces), with the grading

$$\tilde{A}^0 = R \oplus A^0 \quad (C \oplus A^0), \quad \tilde{A}^1 = O \oplus A^1$$  \hfill (A.31)

and the product given by

$$(x, a)(\beta, b) = (x \beta, ab + \beta x + ab),$$  \hfill (A.32)

which is bilinear associative, graded in the sense (A.6), and has the unit $I = (1, 0)$, so that one can write

$$(x, a) = x I + a.$$  \hfill (A.33)

$^3$ Or Lie super algebra.

---

$^4$ Note that we would obtain (A.26) from (A.6) by letting $a, a'$ and $b, b'$ act by multiplication from the left on $A$, resp. $B$, and constructing $a \otimes b \in \text{End}(A \otimes B)$ from $a \in \text{End}(A), b \in \text{End}(B)$. 

---

1. We set

$$\tilde{A} = R \oplus A \quad (C \oplus A)$$  \hfill (A.30)

(direct sum of vector spaces), with the grading

$$\tilde{A}^0 = R \oplus A^0 \quad (C \oplus A^0), \quad \tilde{A}^1 = O \oplus A^1$$  \hfill (A.31)

and the product given by

$$(x, a)(\beta, b) = (x \beta, ab + \beta x + ab),$$  \hfill (A.32)

which is bilinear associative, graded in the sense (A.6), and has the unit $I = (1, 0)$, so that one can write

$$(x, a) = x I + a.$$  \hfill (A.33)
Note that $A \sim O \oplus A$ becomes an ideal in $\hat{A}$, generated by the idempotent $(0, e)$ if $A$ was unital with unit $e$ to begin with.

### A.3 Linear modules over graded commutative algebras

Throughout this section $A$ denotes a graded commutative algebra (as a special case, with trivial grading, we have $A = R$ ($A = C$)).

A graded linear left $A$-module (resp. right $A$-module) $E$ is a graded vector space $E = E^1 + E^2$ endowed with a bilinear product

$$A \times E \ni (a, \xi) \mapsto a\xi \in E,$$

(resp. $E \times A \ni (\xi a) \mapsto \xi a \in E$)

fulfilling

$$A^p E^q \subset E^{p+q},$$

(resp. $E^p A^q \subset E^{p+q}$),

$$a(\xi b) = (ab)\xi, \quad a, b \in A, \quad \xi \in E,$$

and

$$a(\xi b) = (ab)\xi, \quad a, b \in A, \quad \xi \in E.$$  

If $A$ is unital with unit $I$, the module $E$ is called unital whenever

$$I\xi = \xi, \quad \xi \in E,$$

(resp. $\xi I = \xi$).

Note that a graded linear left (right) $A$-module over a non-unital algebra $A$ can be made into a unital module over $\hat{A}$ by defining

$$(a \xi + a') \xi = a\xi + a'\xi,$$

(resp. $\xi (a \xi + a') = a\xi + \xi a$)

$a I + a \xi = a \xi + a' \xi$.

Recall that, with $A$ and $B$ (graded commutative) algebras a linear left $B$, right $A$-bimodule is a graded vector space which is both a left $B$-module and a right $A$-module in such a way that

$$(b \xi) a = b (\xi a), \quad \xi \in E, \quad a, b \in B.$$  

With such an $E$ and a left $A$-module $F$ the tensor product $E \otimes_A F$ of $E$ and $F$ over $A$ is then defined as the quotient of $E \otimes F$ through the vector space generated by the elements

$$\xi a \otimes \eta - \xi \otimes a \eta, \quad \xi \in F, \quad \eta \in E, \quad a \in A.$$  

$E \otimes_A F$ then becomes a left $B$-module, left multiplication by $b$ arising by linear extension from

$$b (\xi \otimes \eta) = b \xi \otimes \eta.$$  

Suppose now $A$ is graded commutative, each linear left $A$-module (or linear right $A$-module) $E$ is automatically a linear $A$-bimodule, by defining

$$\xi a = (-1)^{oA o \xi} a \xi, \quad a = A', \xi \in E;$$

indeed, for $b \in A'$

$$(b \xi) a = (-1)^{oA o \xi} ab \xi = (-1)^{oA o \xi} ba \xi = b (\xi a)$$

and

$$(\xi a) b = (-1)^{oA o \xi} ab \xi = ((-1)^{oA o \xi} b) a \xi = (b \xi) a.$$

This allows us, given linear left $A$-modules $E$ and $F$, to build the tensor product $E \otimes_A F$ as before, or to consider the tensorial powers

$$E \otimes_A E \otimes_A \ldots \otimes_A E,$$

with the property

$$\lambda_1 \otimes \ldots \otimes \lambda_{i-1} \otimes a \lambda_i \otimes \lambda_{i+1} \otimes \ldots \otimes \lambda_n = (-1)^{oA o \lambda_1 + \ldots + oA o \lambda_{i-1}} a \lambda_i \otimes \lambda_{i+1} \otimes \ldots \otimes \lambda_n,$$

$$a \in A', \quad \lambda_1, \ldots, \lambda_n \in E.$$  

If, for arbitrary linear left $A$-modules, $E, F$ we define $\text{Hom}_A(E, F)$ as the graded subspace of $\text{Hom}(E, F)$ (as vector spaces), given by

$$\text{Hom}_A(E, F)^k = \{ \lambda \in \text{Hom}(E, F) : \lambda(a) = (-1)^{oA o \lambda} a \lambda, \quad a = A', \xi \in E \},$$

we then have that $\text{Hom}_A(E^{\otimes n}, F)$ identifies, as a graded vector space, with the set $\mathcal{L}_n(E, F)$ of $F$-valued graded $n$-linear forms on $E$, setting

$$\lambda(\xi_1, \ldots, \xi_n) = \lambda(\xi_1 \otimes \ldots \otimes \xi_n), \quad \xi_1, \ldots, \xi_n \in E.$$  

The elements of $\mathcal{L}_n(E, F)$ are thereby characterized as the $\lambda \in \mathcal{L}(E, F)$, fulfilling, for $a \in A', \xi_1, \ldots, \xi_n \in E$

$$\lambda(\xi_1, \ldots, \xi_{i-1}, a \xi_i, \xi_{i+1}, \ldots, \xi_n) = (-1)^{oA o \xi_1 + \ldots + oA o \xi_{i-1}} a \lambda(\xi_1, \ldots, \xi_n).$$  

We shall write $\mathcal{L}_n(E, F)$ to mean $\mathcal{L}_n(E, F) (\mathcal{L}_n(E, F).$
APPENDIX B. TWISTED AND GRADED ALTERNATION
THE SPACE, THE ALGEBRAS

B.1 DEFINITIONS AND NOTATION. \( I_n \) denotes the set of the first \( n \) integers and \( \Sigma_n \) the group of permutations of \( I_n \).

(i) Given a set \( E \), we denote by \( E^n \) the set of maps \( I_n \to E \)
\[
i \in I_n \to \xi_i \in E
\]
alternatively denoted by listing the \( \xi_i \) with increasing indexes\n\[
\xi = \{ \xi_1, \ldots, \xi_n \}.
\]
Setting \( \xi_0' = \xi \in \Sigma_n, \xi'E \in E^n \), we get a right action of \( \Sigma_n \) on \( E^n \):\n\[
(\xi'O')E \in \Sigma_n \times \Sigma_n \to \xi'O'E \in \Sigma_n
\]
\[
(\xi'O')L = \xi'(\xi'O'), \xi \in \Sigma_n, \xi'O' \in \Sigma_n,
\]
\[
\xi O' = \xi, \xi \in E^n.
\]

The Cartesian product \( E^n \times \Sigma_n \) thereby acquires a groupoid structure, for which \( (\xi, \xi'O') \) and \( (\xi'O', \xi') \) are multiply able, with product \( (\xi, \xi'O') \) whenever \( \xi \xi'O' = \xi' \).

\[\text{GRADED LIE-CARTAN PAIRS}\]

We then define\n\[
(\sigma f)(\xi) = \chi_{\sigma}(\xi', \sigma f), \xi \in E^n,
\]
i.e.
\[
(\sigma f)(\xi_1, \ldots, \xi_n) = \chi_{\sigma}(\xi', \sigma f)(\xi_1, \ldots, \xi_n), \xi_1, \ldots, \xi_n \in E
\]
with average over \( \Sigma_n \)
\[
A_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma
\]
(called the \( \text{nth antisymmetrizer} \)).

B.2 THEOREM. With the definitions and notation of B.1 we have that \( \sigma \mapsto \sigma_n \) is a linear representation of \( \Sigma_n \) on \( \mathbb{F}^n \). One has\n\[
\sigma \mapsto \sigma_n \text{ is a linear representation of } \Sigma_n \text{ on } \mathbb{F}^n.
\]
\[
\sigma n \circ \tau n = (\sigma \tau)n, \quad (\sigma_n)^{-1} = (\sigma^{-1})_n, \quad \sigma \in \Sigma_n.
\]
\[
\tau n A_n = A_n \tau n = A_n, \quad \tau \in \Sigma_n.
\]
i.e. for \( f \in \mathbb{F}^n, \xi_1, \ldots, \xi_n \in E \)
\[
(A_n f)(\xi_1, \ldots, \xi_n) = \chi_n(\xi', \tau)(A_n f)(\xi_1, \ldots, \xi_n).
\]
\[
A_n \text{ is an idempotent of } \mathbb{F}^n
\]
with range of the fixpoznts of \( \mathbb{F}^n \) under all \( \sigma_n, \sigma \in \Sigma_n: \)
\[
A_n \mathbb{F}^n = \{ f \in \mathbb{F}^n : f(\xi_1, \ldots, \xi_n) = \chi_n(\xi', \tau)(f(\xi_1, \ldots, \xi_n)) \}.
\]
\[
\text{Proof:} \ (i) \text{ One has, for } f \in \mathbb{F}^n, \xi_1, \ldots, \xi_n \in E, \text{ using (B.3)}
\]
\[
\sigma_n(\tau_n f)(\xi_1, \ldots, \xi_n) = \chi_n(\xi', \tau)(\tau_n f)(\xi_1, \ldots, \xi_n)
\]
\[
= \chi(\xi', \tau)(\chi(\xi', \sigma)f(\xi_1, \ldots, \xi_n))
\]
\[
= \chi(\xi', \tau)f(\xi_1, \ldots, \xi_n)
\]
\[
= \chi(\tau_n f)(\xi_1, \ldots, \xi_n)
\]
and\n\[
\sigma_n(\sigma^{-1})_n = (\sigma^{-1})_n \sigma_n = (id)_n = id.
\]
\[
(i) \text{ One has then}
\]
\[
\tau_n A_n = \frac{1}{n!} \tau_n \sum_{\sigma \in \Sigma_n} \sigma = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (\tau_n \sigma) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma = A_n
\]
and
\[ A_n \cdot \tau = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot \tau = \frac{1}{n!} \sum_{\sigma \in S_n} (\sigma \tau)_n = A_n. \quad (B.17) \]

(iii) One then has
\[ A_n^2 = \frac{1}{n!} \sum_{\sigma, \tau} \sigma \tau A_n = \frac{1}{n!} \sum_{\sigma} A_n = A_n. \quad (B.18) \]

On the other hand, for \( f = A_n g \), \( g \in \mathbb{F}^m \), we have, by (B.10)
\[ \tau f = \tau A_n g = A_n g = f \quad (B.19) \]
and conversely, if \( f \in \mathbb{F}^m \) fulfils \( \sigma f = f \) for all \( \sigma \in S_n \), we have
\[ A_n f = \frac{1}{n!} \sum_{\sigma} \sigma f = \frac{1}{n!} \sum_{\sigma} f = f. \quad (B.20) \]

We now describe the linear variety of twisted antisymmetrization relevant to our graded Lie-Cartan pairs: the latter is obtained by taking for \( E \) and \( F \) graded \( A \)-modules, \( A \) graded-commutative algebra.

**B.3 Definition.** Let \( A = A^0 \otimes A^1 \) be a graded-commutative algebra, with \( E = E^0 \otimes E^1 \) and \( F = F^0 \otimes F^1 \) graded left \( A \)-modules. We denote by \( \mathcal{L}_n^E(A, F) \) the set of graded \( F \)-valued \( n \)-linear forms on \( E \), i.e. \( n \)-linear maps \( \lambda : E \to F \) fulfilling, for each \( i \in S_n \), \( a \in A \), \( \xi_i \in E^i \), \( k = 1, \ldots, n \)
\[ \lambda(a^{(1)}, \ldots, a^{(k)}, \ldots, a^{(n)}) = (-1)^{\sigma_1(k_1 + \cdots + k_{i-1})} \beta(a^{(1)}, \ldots, a^{(k)}, \ldots, a^{(n)}). \quad (B.21) \]

where \( \beta(a) \) is the graded-source of \( \lambda \).

On the other hand, we define as follows the \( Z/2 \)-valued graded alternate character \( \chi \), \( \chi \in (E)^* \) such that
\[ \{ \xi_1, \ldots, \xi_n \} = \{ \xi_1, \ldots, \xi_m \} \cup \{ \xi_{m+1}, \ldots, \xi_n \}, \]
\[ \xi_i \in E^0, \quad 1 \leq i_1 < \cdots < i_p \leq n, \quad (B.22) \]
\[ \xi_{i_1} \in E^1, \quad 1 \leq j_1 < \cdots < j_q \leq n \]
we set \( \chi_n(a, \sigma) = \chi(a) \cdot \chi_n^0(\xi_1, \sigma) \) with \( \chi_n^0(\xi, \sigma) = (-1)^{\sigma(\xi)} q(\xi, \sigma) \) the number of pairs \( l, m \in I_n \) with \( l < m \) and \( \sigma_l > \sigma_m \).

**B.4 Proposition.** With the definitions and notation in B.1 through B.3, we have that
\[ a_1 \cdots a_n = \chi_n(a, \sigma) a_1 \cdots a_n. \quad (B.23) \]

(i) If \( a_i \in A \), \( a_i \) is a graded-commutative algebra, then
\[ a_1 a_2 \cdots a_n = \chi_n^0(a, \sigma) a_1 \cdots a_n. \quad (B.24) \]

(iii) \( \chi_n \) is a character of the groupoid \( E^0 \times S_n \); \( \chi_n \in X^N(E) \). Moreover the hierarchy \( \chi_n \) is tensorial.\(^{11}\)

(iii) \( \sigma_n \) as defined in (B.7a) with \( \chi_n \) the graded alternate character in Definition B.3 leaves stable the subset \( \mathcal{L}_n^E(A, F) \) of \( \mathcal{L}(E^0, \mathcal{F}) \). By restricting the \( \sigma_n \) to \( \mathcal{L}_n^E(A, F) \), we get a representation of \( S_n \) by zero grade endomorphisms (still denoted \( \sigma_n \)) of \( \mathcal{L}_n^E(A, F) \) as a graded vector space. We denote by \( A_n^* (E, F) \) the corresponding set of fixpoints:
\[ A_n^* (E, F) = A_n \mathcal{L}_n^E(A, F). \quad (B.24) \]

and call the elements of \( A_n^* (E, F) \) the \( F \)-valued graded alternate \( n \)-\( A \)-linear forms on \( E \).

(iv) With \( E^0 = \mathbb{F}^0 \) the graded nth tensor power of \( E \),\(^{12}\) and identifying \( \mathcal{L}_n^E(A, F) \) with \( \text{Hom}_n(E^0, F) \) as follows:
\[ \lambda(\xi_1, \ldots, \xi_n) = \lambda(\xi_1 \otimes \cdots \otimes \xi_n), \quad (B.25) \]

\( \sigma_n \) acting on \( \mathcal{L}_n^E(A, F) \), \( \sigma_n \) is the transpose of \( \sigma \) acting as follows on \( E^0 \):
\[ \sigma^n(\xi_1 \otimes \cdots \otimes \xi_n) = \chi(\xi_1, \ldots, \xi_n) \chi_1 \otimes \cdots \otimes \chi_n. \quad (B.25) \]

**Proof:** (i) Since \( A \) is graded-commutative, \( \chi(a, \sigma) \) as in (B.23) is the sign obtained as follows: decompose \( a \in E^0 \) as in (B.22), write \( \sigma \) as a product of transpositions and set \( \chi(a, \sigma) = (-1)^s \), \( s \) the number of transpositions affecting elements \( a_k \) of odd degree: we then have \( s = g(a, \sigma) \).

(ii) Since \( \chi \) is a character of \( S_n \), \( \chi_n \in X^N(E) \). Now with \( a \in A^0 \) and \( \tau \in S_n \) we have, by (B.23)
\[ a_{\tau_1} \cdots a_{\tau_n} = \chi_n^0(a, \sigma) a_1 \cdots a_n \]
\[ = \chi_n(a, \sigma) a_1 \cdots a_n \quad (B.27) \]

(iii) Let \( \lambda \in \mathcal{L}_n^E(A, F) \). Since \( S_n \) is generated by transpositions of neighbouring elements of \( I_n \), it is enough to show that, for each \( k \in I_n \), one has \( \lambda_k \in \mathcal{L}_n^E(A, F) \), \( \lambda_k \) the transformation of \( \lambda \) by the transposition \( (k \leftrightarrow k+1) \). Now if \( \xi_i \in E^1 \), \( i = 1, \ldots, n \) we have \( \chi(\xi, k \leftrightarrow k+1) = (-1)^{i+k+1} \), hence, assuming \( \lambda \) of grade \( s \)
\[ \lambda(\xi_1, \ldots, \xi_n) = (-1)^{s+k+1} \chi(\xi_1, \ldots, \xi_{k-1}, \xi_k+1, \xi_{k+1}, \ldots, \xi_n). \quad (B.28) \]

Now property (B.21) obviously holds for \( i = k+1 \) and \( i \geq k+2 \). Let \( a \in A^0 \). For
\[ \chi^+(\xi, \sigma) = (-1)^{\sigma(\xi)} \chi(\xi, \sigma) \xi_n \] is also a character of the groupoid \( E_n \times S_n \) (called the graded symmetric character) and the hierarchy \( \chi^+ \) is also tensorial.

\(^{11}\) Cf. Appendix A.
with \(\lambda (\otimes \mu)\) the graded antisymmetrizer defined in term of the graded alternate character (cf. Definition B.3), we get a graded wedge product \(\wedge\) which is associative

\[
\lambda \wedge (\mu \wedge v) = (\lambda \wedge \mu) \wedge v = (n+m+r)! \quad \text{for } \lambda \in \mathcal{L}^m(E, A), \quad \mu \in \mathcal{L}^m(E, A), \quad v \in \mathcal{L}^r(E, A).
\]

Hence, under the bilinear extension of the product, (B.36), \(\Lambda^n(E, A)\) is an associative algebra with subalgebra \(\Lambda^n_\ast(A, E)\).

**Proof:** For \(a \in A^\ast\), writing \(\delta a = s, \delta a_i = t, \delta a_{i}^{-} = \alpha_i, 1 \leq i \leq n\)

\[
(\lambda \otimes \mu)(\xi_1, \ldots, \xi_{i-1}, a_{i}, \xi_{i+1}, \ldots, \xi_{n+m}) = (-1)^{s+t+a_{i}^{-}+\sum a_{1}+\ldots+a_{i-1}} a \lambda(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_{n+m})
\]

and for \(i > n\)

\[
(\lambda \otimes \mu)(\xi_1, \ldots, \xi_{i-1}, a_{i}, \xi_{i+1}, \ldots, \xi_{n+m})
\]

We proved the \(A\)-linearity of \(\lambda \otimes \mu\). We now check (B.35): we have, for \(\lambda \in \mathcal{L}^m(E, A), \mu \in \mathcal{L}^m(E, A)\)

\[
[(\lambda \otimes \mu) \otimes v](\xi_1, \ldots, \xi_{n+m} + )
\]

then we get an associative-graded tensor product \(\otimes\): one has

\[
\lambda \otimes (\mu \otimes v) = (\lambda \otimes \mu) \otimes v, \quad \lambda, \mu \in \mathcal{L}^m(E, A), \quad v \in \mathcal{L}^r(E, A).
\]

Moreover, \(\lambda \otimes \mu \in \mathcal{L}^m_\ast(E, A)\) if \(\lambda \in \mathcal{L}^m_\ast(E, A)\) and \(\mu \in \mathcal{L}^m_\ast(E, A)\)."
We now check (B.27) in this context, from which (B.18) follows as in the proof of Theorem B.2 (iv), entailing (B.37). We have, indeed, from (B.34), for $A, g \mapsto A, g \mapsto J$ and the $\tilde{\sigma}_i$ as above

\[ (\sigma_\lambda \otimes \mu)(\xi_1, \ldots, \xi_{n+m}) = (-1)^{\alpha_1 + \cdots + \alpha_n} \chi(\xi', \sigma) \lambda(\xi_{n+1}, \ldots, \xi_{n+m}) \]

and

\[ (\lambda \otimes \tau_\mu)(\xi_1, \ldots, \xi_{n+m}) = (-1)^{\alpha_1 + \cdots + \alpha_n} \chi(\xi', \tau) \lambda(\xi_{n+1}, \ldots, \xi_{n+m}) \]

whence our conclusion (using (B.5) we denoted $\{\xi_1, \ldots, \xi_n\} = \xi', \{\xi_{n+1}, \ldots, \xi_{n+m}\} = \xi'').$

Remark. We note the expression of the graded wedge product of a one-form and an $n$-form: for $(\omega \wedge \A)(\xi_1, \ldots, \xi_{n+1}) = \sum_{i=1}^n (-1)^{i-1} \omega_i \otimes \xi_i \wedge \lambda(\xi_{n+1}, \ldots, \xi_{n+m})$, (B.43) follows from

\[ (\omega \wedge \lambda)(\xi_1, \ldots, \xi_{n+1}) = (-1)^{\alpha_1 + \cdots + \alpha_n} \chi(\xi', \sigma) \lambda(\xi_{n+1}, \ldots, \xi_{n+m}), \]

where $\tilde{\sigma}_i$ means omission.

Proof: Use the fact that $\Sigma_{n+1} = \bigcup_{i \in \Sigma_{n+1}} \Sigma_i \circ \sigma_i$ where

\[ \sigma_i = \left( \begin{array}{ccc} 1 & 2 & \ldots & n+1 \\ i & 1 & \ldots & n+1 \end{array} \right) \]

and $\Sigma_i$ is the subgroup of $\Sigma_{n+1}$ leaving $i$ invariant. (B.43) then follows from Card $\Sigma_i = n!$ and

\[ \chi(\xi', \sigma_i) = (-1)^{i+1+\alpha_1 + \cdots + \alpha_i} \tilde{\sigma}_i \cdot \tilde{\sigma}_k \]

(8.45)

B.8 Remark. Applied to the trivially graded commutative algebra $A = R (= C)$ with $E$ a real (complex) vector space, Proposition B.6 describes the algebra $A^*(E) = \bigoplus_n A^*(E)$ of real (complex) graded alternate multilinear forms on $E$. $A^*(E)$ is a subalgebra of the algebra $A^{\mathbb{F}^*}$ in Theorem B.2 (iv). $A^*(E)$ gives rise to the commutation property

\[ \mu \wedge \lambda = (-1)^{nm+\delta_\lambda \mu} \lambda \wedge \mu, \quad \lambda \in A^*(E), \mu \in A^*(E), \]

hence it is graded commutative for a trivially graded $E$.

Proof of (B.46): We first look at elements $\lambda, \mu \in A^*(E)$. Now we have, for $\xi_1 \in E^{\xi_1}, \xi_2 \in E^{\xi_2}$. $\tilde{\lambda} = s, \tilde{\mu} = t$

\[ (\mu \wedge \lambda)(\xi_1, \xi_2) = (A_2(\mu \otimes \lambda))(\xi_1, \xi_2) \]

\[ = \frac{1}{2} (\mu \otimes \lambda)(\xi_1, \xi_2) - \frac{1}{2} \lambda(\xi_1, \xi_2)(-1)^{\xi_1 + \xi_2} \mu(\xi_2), \]

\[ = \frac{1}{2} (1 - (-1)^{\xi_1 + \xi_2} \mu(\xi_2)(-1)^{\xi_1 + \xi_2} \lambda(\xi_1)), \]

where we used the facts that, by definition, with $\delta_\lambda$ the Kronecker symbol

\[ \chi(\xi_1) = \delta_{a_1} \mu(\xi_1), \mu(\xi_1) = \delta_{a_2} \mu(\xi_2), \lambda(\xi_1) = \delta_{a_1} \lambda(\xi_1), \lambda(\xi_1) = \delta_{a_2} \lambda(\xi_2). \]

Exchange of $\lambda$ and $\mu$ then shows that we have

\[ \mu \wedge \lambda = (-1)^{nm+\delta_\lambda \mu} \lambda \wedge \mu, \quad \partial \lambda = \partial \mu = \partial. \]

Now, by linearity, it is enough to check (B.46) for $\lambda = \phi_1 \wedge \cdots \wedge \phi_n, \mu = \psi_1 \wedge \cdots \wedge \psi_m, \psi_i \in \Lambda^i(E), i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$. Applying (B.44) $nm$ times then brings about a factor $(-1)^{nm+\delta_\phi \mu}(\delta_\phi \mu)$.

GRADED LIE-CARTAN PAIRS

51