Institute of Theoretical Physics
University of Wroclaw
Wroclaw, Cybulskiego 36

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CLASSICAL LIMIT OF THECANONICAL ANTICOMMUTATION RELATIONS AND SELFDUALITY OF THE INFINITE DIMENSIONAL GRASSMANN ALGEBRA(*)

by

A.Z. Jadczyk & K. Pilch

Abstract A classical limit of the twisted group algebra for canonical anticommutation relations (CAR) is constructed and shown to exhibit a kind of selfduality relevant for applications in supersymmetric models.

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Symb.18.(2.3)

1. Introduction

When dealing with Fermi systems it is often convenient to introduce an auxiliary Grassmann algebra Q. Originally Q was introduced to accomodate "eigenvalues" of anticommuting operators of CAR so that a formal analogy between Feynmann integrals for Bose and Fermi systems could be established (see [1,2]). Later on the canonical formalism of classical mechanics was further extended by adding Grassmann coordinates in the configuration space [2-6]. With the invention of supersymmetry [7-9] and establishing its quantum theoretical basis [10] an auxiliary Grassmann algebra was widely applied for integrating of Lie superalgebra relations. Although it is true that one can deal with a quantum theory of supersymmetric systems without "anticommuting variables" (see e.g.[10,11]), nevertheless a possibility that Q can be an indispensable mathematical tool (like the imaginary numbers) should be seriously taken into account. So far, however, the mathematics of Q-numbers is in its early stage of development. In [12] differential supergeometry was developed on a basis of a self-dual Banach--Grassmann algebra Q. In [13] an infinite dimensional algebra (called B_∞ there) was defined. In the present paper we define Q as a classical limit of the CAR twisted group algebra and show that Q defined in such a way is selfdual according to the requirements of [12], and coincides with the B_∞-algebra of A. Rogers [13].

2. GRASSMANN ALGEBRA AS A CLASSICAL LIMIT OF CAR

Let N denotes the set $\{1,2,\dots,N\}$ or $\mathbb{N}$ for $N=\infty$. Let

$$a_i a_j + a_j a_i = 2\delta_{ij}, \quad a_i = a_i^*, \quad i,j \in N \tag{2.1}$$

be a representation of the Canonical Anticommutation Relations (CAR) with $\frac{N}{2} < \infty$ degrees of freedom. It is natural to associate with (2.1) the group $\mathcal{J}$ consisting of all finite subsets $I = \{i_1,\dots,i_k\}$, $i_1 < i_2 < \dots < i_k$ of the set N with the symmetric difference Δ as the group law. The unitary operators

$$U(I) \doteq a_{i_1} \dots a_{i_k}, \quad I = \{i_1,\dots,i_k\} \in \mathcal{J}, \tag{2.2}$$

form a projective representation of the group $\mathcal{J}$ ([14,Ch.XIV,§ 8], see also [15, Sect. 3.10])

$$U(I)U(I') = \omega(I,I')U(I \Delta I') \tag{2.3}$$

with

$$\omega(I,I') = (-1)^{P(I,I')}, \quad P(I,I') = \sum_{j \in I'} P(I,j), \tag{2.4}$$

$$P(I,j) = (\text{the number of } i \in I \text{ such that } i < j). \tag{2.5}$$

The multiplier ω satisfies the standard cocycle relations

$$\omega(e,I) = \omega(I,e) = 1, \tag{2.6}$$

$$\omega(I,I')\omega(I \Delta I', I'') = \omega(I',I'')\omega(I, I' \Delta I''), \tag{2.7}$$

where $e = \emptyset$ is the identity element of the group $\mathcal{J}$. We can also define a one parameter family of multipliers

$$\omega_\lambda(I,I') = \lambda^{|I \cap I'|}\omega(I,I'), \quad \lambda \in [0,1], \tag{2.8}$$

where

$$|I \cap I'| = (\text{the number of elements in } I \cap I').$$

The ω_λ-s may be considered as ω-multipliers (see (2.2) and (2.3)) for the modified CAR

$$a_i(\lambda)a_j(\lambda) + a_j(\lambda)a_i(\lambda) = 2\lambda\delta_{ij} \; . \qquad (2.9)$$

In particular the cocycle relations (2.6),(2.7) are satisfied for all λ and

$$|\omega_\lambda(I,I')| < 1, \qquad \lambda \in [0,1] \; . \qquad (2.9a)$$

Using ω_λ one can construct for J the so called twisted group algebra [16] which is a generalization of the ordinary group algebra. Let μ be the invariant measure on J normalized to $\mu(I) = 1$, $I \in J$. The twisted group algebra $Q^N(\lambda)$ is defined as $L^1(J,\mu)$ with the multiplication

$$(f *_\lambda g)(K) = \sum_{I\in J} f(I)g(I \,\Delta\, K)\omega_\lambda(I,I \,\Delta\, K), \qquad (2.10)$$

$$f,g \in L^1(J,\mu) \; .$$

Since ω_λ are real, Q_N^λ can be real or complex. From now on we assume it is real. Using (2.6),(2.7) and (2.9a) one proves that Q_N^λ is a Banach algebra in the $L^1(J,\mu)$ -norm

$$\|f\| = \sum_{I\in J} |f(I)| \quad , \qquad f \in Q^N(\lambda). \qquad (2.11)$$

Definition 2.1. The algebra $Q^N = Q^N(\lambda=0)$ is the classical limit for CAR.

It is important to notice that the limit $\lambda \to 0$ can not be taken on representing operators $a_i(\lambda)$, the limiting procedure is performed on the algebra structure of $L^1(J,\mu)$.

The following *Proposition* gives a full characterization of Q^N.

Proposition 2.1. The algebra Q^N is uniquely characterized by the following properties:

There exists a set of generators $e_i \in Q^N$, $i \in N$, such that

(i) $\quad e_ie_j + e_je_i = 0 \quad , \qquad i,j \in N$,

(ii) $\quad$ every $f \in Q^N$ is uniquely represented by an absolutely convergent series

$$f = \sum_{I\in J} f^I e_I \; , \qquad \sum_I |f^I| < \infty \; , \qquad f^I \in \mathbb{R} \; ,$$

where J is the set of all finite subsets $I=\{i_1,\dots,i_k\}$, $i_1 < \dots < i_k$, of N, and

$$e_I = e_{i_1}\cdots e_{i_k} \; , \qquad e_\emptyset = 1 \; ,$$

(iii) $\quad$ for $f \in Q^N$

$$\|f\| = \sum_{I\in J} |f^I| \; .$$

Remark. In this form the algebra Q^N was introduced by A. Rogers [13].

<u>*Proof*</u> Let us denote $\omega_{\lambda=0}(I,I') = \chi(I,I')$. Then (2.8) implies

$$\chi(I,I') = \begin{cases} 0 & \text{for } I \cap I' \neq \emptyset \; , \\ (-1)^{P(I,I')} & \text{for } I \cap I' = \emptyset \; , \end{cases} \qquad (2.12)$$

and, by (2.4), (2.5), in particular

$$\chi(\{i\},\{j\}) = \begin{cases} +1 & \text{for } i < j, \\ 0 & \text{for } i = j, \\ -1 & \text{for } i > j. \end{cases} \tag{2.13}$$

Let e_I be the element of Q^N defined by

$$e_I(J) = \begin{cases} 0 & \text{for } J \neq I, \\ 1 & \text{for } J = I. \end{cases} \tag{2.14}$$

It follows from (2.11) that

$$\| e_I \| = 1 , \tag{2.15}$$

and (2.12) gives

$$e_I e_J \doteq e_I *_{\lambda=0} e_J = \begin{cases} 0 & \text{if } I \cap J \neq \emptyset , \\ \chi(I,J) e_{I \cup J} & \text{if } I \cap J = \emptyset . \end{cases} \tag{2.16}$$

This implies that for $e_i \doteq e_{\{i\}}$

$$e_i e_j = - e_j e_i , \tag{2.17}$$

and for $I = \{i_1,\ldots,i_k\}$, $i_1 < \ldots < i_k$, we have

$$e_I = e_{i_1} \ldots e_{i_k} . \tag{2.18}$$

The proposition follows from an observation that every $f \in Q^N$ can be uniquely written as

$$f = \sum_{J \in \mathcal{J}} f(J) e_J , \tag{2.19}$$

the series being convergent in $L^1(\mathcal{J},\mu)$.

The algebra Q^N admits a natural Z_2-grading $Q^N = Q^N_o \oplus Q^N_1$, where Q^N_1 is the closed linear subspace generated by odd products of generators e_i, and $Q^N_o = \mathbb{R} \oplus Q^{N\prime}_o$ is the direct sum of the number field $\mathbb{R}$ and the closed subalgebra $Q^{N\prime}_o$ generated by even products of elements of Q^N_1. The following norm decompositions hold

$$\| a_o + a_1 \| = \| a_o \| + \| a_1 \| , \quad a_r \in Q^N_r , \quad r=0,1 , \tag{2.20}$$

$$\| \lambda + a'_o \| = |\lambda| + \| a'_o \| , \quad \lambda \in \mathbb{R} , \quad a'_o \in Q^{N\prime}_o . \tag{2.21}$$

The real part of an element $a \in Q^N$ will be denoted by $\sigma(a)$. Therefore $\sigma(\lambda + a') = \lambda$ for $\lambda \in \mathbb{R}$ and $a' \in Q^{N\prime} \doteq Q'_o \oplus Q_1$.

3. SELFDUALITY OF THE INFINITE DIMENSIONAL GRASSMANN ALGEBRA Q^∞

When N is finite Q^N is 2^N-dimensional and coincides with the standard Grassmann algebra of $\mathbb{R}^N$. The *Proposition 2.1* implies then

Proposition 3.1. If $N < \infty$ then

(i) There exists $e \in Q^N$ such that for every $a \in Q^N_1$

$$ae = ea = 0$$

(ii) For every $a_1,\ldots,a_{N+1} \in Q^N_1$

$$a_1 a_2 \ldots a_{N+1} = 0 \quad \blacksquare$$

The fact that for finite N the equation $ax = 0$, $a \in Q^N_1$, admits non-zero solutions makes it impossible to define partial derivatives of supersmooth function (see [12]) $f: Q_1^N \to Q^N$ in such a way that the Leibnitz rule is satisfied (even modulo an ideal). The "superanalysis " formulated in [12] is based on the assumption of selfduality of Q. We shall now prove that the algebra $Q \doteq Q^\infty(\lambda=0)$ defined in Section 2 is selfdual in the required sense.

First of all we observe that by the properties (ii) and (iii) of the *Proposition 2.1.* for every $a \in Q$ we have

$$\| a \| = \lim_{i\to\infty} \| e_i a\| = \lim_{i\to\infty} \|a e_i\| \quad . \tag{3.1}$$

It follows that for each $x \in Q$, if $ax = 0$ (resp. $xa = 0$) for all $a \in Q_1$, then $x = 0$.
Let Q_r^* denote the space of continuous Q_o-linear maps from Q_r $(r=0,1)$ to Q. Every element $q \in Q$ determines $F_q \in Q_r^*$ by

$$F_q(a) \doteq aq \quad , \qquad a \in Q_r. \tag{3.2}$$

Since Q_o is an algebra with unit, it follows that every $F \in Q_o^*$ is of the form F_q for $q = F(1)$.

Proposition 3.2. (Selfduality of Q). For each continuous Q_o-linear map $F : Q_1 \to Q$ there exists a unique element $q \in Q$ such that

$$F(a) = aq$$

for all $a \in Q_1$.

To prove the above property of Q we first establish two Lemmas.

Lemma 3.1. Let for each $a \in Q$, $\mathcal{J}(a) = \{I \in \mathcal{J}: a^I \neq 0\}$, and let $I_o \subset a$ (resp. $I_o \perp a$) denotes the fact that $I_o \not\subset I$ (resp. $I_o \cap I = \emptyset$) for all $I \in \mathcal{J}(a)$. Then

a) $I_o \not\subset a$ implies

$$\| a e_{I_1} + b e_{I_o} \| = \| a e_{I_1} \| + \| b e_{I_o} \|$$

for $I_1 \cap I_o = \emptyset$, and for all $b \in Q$. If $I_o \perp a$ then $I_o \not\subset a$ and $\|a e_{I_o}\| = \|a\|$.

b) if $a \in Q$, $I \in \mathcal{J}$ and $a e_i = 0$ for all $i \in I$, then there exists a unique $v \in Q$ with $I \perp v$ and such that $a = v e_I$. We also have $\|a\| = \|v\|$.

Proof Follows in a straightforward way from the *Proposition 2.1.* ∎

Lemma 3.2. There is a one-to-one correspondence between continuous Q_o-linear maps $F : Q_1 \to Q$ and bounded sequences (f_i) from Q satisfying

$$f_i e_j + f_j e_i = 0 \qquad \text{for all } i,j \in \mathbb{N} \tag{3.3}$$

Moreover, $\|F\| = \sup\{\|f_i\| : i \in \mathbb{N}\}$ and $\sigma(f_i)=0$.

Proof If $F: Q_1 \to Q$ is Q_o-linear then, with $x,y,z \in Q_1$, we have $F(x)yz = F(xyz) = -F(yxz) = -F(y)xz$, and therefore $F(x)y=-F(y)x$ for all $x,y \in Q_1$. With $x=e_i$, $y=e_j$ and $f_i \doteq F(e_i)$ we get (3.3). In particular $f_i e_i = 0$ so that $\sigma(f_i)=0$. Conversely given a bounded sequence $f_i \in Q$ satisfying (3.3) we define $F(e_I) \doteq f_{i_1} e_{i_2} \dots e_{i_k}$ and $F(a) \doteq \Sigma\, \{a^I F(e_I) : I \in \mathcal{J}\}$. It is immediate to see that F so defined is Q_o-linear and continuous. Since

$$\| F(a)\| = \lim_i \| F(a) e_i\| = \lim_i \| F(e_i) a\| < \sup\{\| f_i\| \cdot \|a\| : i \in \mathbb{N}\},$$

we also get $\|F\| = \sup\{\|f_i\| : i \in \mathbb{N}\}$. ∎

Proof of the Proposition 3.2. :Let $F\colon Q_1 \to Q$ be Q_o-linear and continuous. Then, according to *Lemma 3.2.* the sequence $f_i \in Q'; f_i \doteq F(e_i)$ is bounded and $f_i e_j + f_j e_i = 0$. To prove the *Proposition* we must show that there exists $q \in Q$ such that $f_i = qe_i$ for all $i \in \mathbb{N}$. We shall define q as the limit of a sequence of its approximations q_n constructed as follows:

i) since $f_1 e_1 = 0$ it follows (*Lemma (3.1.b.)*) that

$$f_i = q_1 e_1 \quad \text{with} \quad \{1\} \not\subset q_1 \;,$$

ii) suppose we have already constructed q_n such that $f_i = q_n e_i$ for $i \in \{1,\dots,n\}$ and $\{1,\dots,n\} \not\subset q_n$. Then, since $f_{n+1} e_i = -f_i e_{n+1} = q_n e_{n+1} e_i$ and $f_{n+1} e_{n+1} = 0$, we have

$$(f_{n+1} - q_n e_{n+1}) e_i = 0 \quad \text{for} \quad i \in \{1,\dots,n+1\} \;.$$

Therefore (*Lemma 3.1.b*) there exists v_n with $\{1,\dots,n+1\} \perp v_n$ and such that $f_{n+1} - q_n e_{n+1} = v_n e_1,\dots,e_{n+1}$. We define now $q_{n+1} = q_n + v_n e_1,\dots,e_n$. It is immediate that $q_{n+1} e_i = f_i$ for $i \in \{1,\dots,n+1\}$ and, since $\{1,\dots,n\} \not\subset q_n$, we also have $\{1,\dots,n+1\} \not\subset q_{n+1}$. Now, by *Lemma 3.1a,*

$$\|f_{n+k}\| = \|q_{n+k} e_{n+k}\| = \|q_{n+k-1} e_{n+k} + v_{n+k-1} e_1 \dots e_{n+k}\| =$$

$$+ \|q_{n+k} e_{n+k}\| + \|v_{n+k-1} e_1 \dots e_{n+k}\| >$$

$$> \|q_{n+k-1} e_{n+k}\| > \dots > \|q_n e_{n+k}\| \,.$$

Therefore $\|q_n\| < \sup\{\|f_{n+k}\| : k \in \mathbb{N}\} < \|f\|$. But $\|q_{n+1}\| = \|q_n + v_n e_1 \dots e_n\| = \|q_n\| + \|v_n\| = \dots = \|q_1\| + \sum_{k=1}^{n} \|v_k\|$, so that the series $\sum \|v_n\|$ is convergent. Since $\|q_{n+k} - q_n\| = \|\sum_{l=n}^{n+k-1} v_l\| < \sum_{l=n}^{n+k-1} \|v_l\|$, it follows that $q \doteq \lim q_n$ exists in Q. It is also evident that $f_i = qe_i$ for all $i \in \mathbb{N}$. ∎

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