GEOMETRY OF GAUGE FIELDS IN A MULTIDIMENSIONAL UNIVERSE★

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ABSTRACT. Let S be a group of automorphisms of a principal fibre bundle (U, π, E, R), both groups S and R being compact. Let I (resp. H) be the isotropy group of S (resp. $S \times R$) acting on E (resp. U), and let $N(I)$ (resp. $N(H)$) be the normalizer of I (resp. H) in S (resp. $S \times R$). We construct two principal bundles $P(M, N(I) \mid I) \subset E$ and $Q(M, N(H) \mid H) \subset U$, where $M = E/S$ is the space of orbits of S in E, and we prove that, given a connection A in P, there is a one-to-one correspondence between S-invariant connections ω in U and triples (B, Φ, ψ), where B is a connection in Q, part of which is a pullback π^*A of A, and Φ, ψ are scalars which are cross-sections of certain vector bundles associated with Q. The resulting final gauge group $N(H) \mid H$ is shown to contain as a normal subgroup the 'centralizer of I in R', known from earlier works of other authors. A dimensional reduction of the Einstein–Yang–Mills system on E is briefly discussed.

1. INTRODUCTION

In this note we study gauge fields in a higher-dimensional spacetime (multidimensional universe) E. These gauge fields are constrained to be invariant under a given global group S of transformations of E. A particular case of S acting transitively on E is analyzed in a mathematical literature (Wang's Theorem [1]), but it is too restrictive for physical applications. Indeed, we tend to believe that our four-dimensional space-time M is precisely just the space of orbits of some S on E. Generalizations of the Wang theorem were studied by many authors in the context of symmetric monopole solutions and dimensional reduction [2–7], and many models were constructed [8–10]. Both local (in terms of Lie derivatives) [2], and global (involving finite group transformations) [3–5] formulations of invariance of gauge fields were given. However, it was always assumed that the structure of E is that of the product $E = M \times S/I$ of M and of the typical orbit S/I, I being the

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isotropy group. As it is shown in a recent study by Coquereaux and Jadczyk [11], the most general S-invariant metric on E is not just a product of metrics on M and $S \mid I$, but it also involves a connection in the bundle $P(M, N(I) \mid I)$ (see Section 2.3) which is necessary to 'glue' the two metrics together. In the present paper, we give a complete description of gauge fields invariant under a given symmetry group in full generality, i.e., without the simplifying assumption of the product structure of E.

In Section 2 we analyze an action of a given global symmetry group S on a principal bundle $U(E, R)$, and introduce two important principal bundles $Q(M, N(H) \mid H)$ and $P(M, N(J) \mid I)$. The effective final gauge group is shown to be $N(H) \mid H$ where $N(H)$ is the normalizer of the isotropy subgroup $H = S \times R$. This final gauge group contains, as a normal subgroup, the gauge group (the 'centralizer') of the authors [2–7]. In Section 3, we describe S-invariant connections in terms of geometrical objects based on M. It is shown that an invariant connection ω in U gives rise to two kinds of scalar fields (Φ, ψ) and a connection B in Q, provided a connection A in P is given. In the process of dimensional reduction, a natural connection in P is provided by an S-invariant Riemannian metric on E, which is necessary for constructing an invariant action functional for ω. Comments are given in Section 4.

2. GROUP ACTION ON A PRINCIPAL BUNDLE

2.1. Let $U(E, R)$ be a principal fiber bundle with the base E and the structure group R, and let there be given a group S of bundle automorphisms acting on U from the left. We have

$$(su)r = s(ur), \quad s \in S,\ r \in R,\ u \in U,$$

where $su = L_s(u)$ and $ur = R_r(u)$ denote the images of u under $s \in S$ and $r \in R$ respectively. We assume that R and S are compact Lie groups, their actions on U being smooth. Since the actions of R and S commute, the formula

$$(s, r)\colon u \mapsto R_{(s,r)}(u) \doteq s^{-1}ur$$

defines the right action of the direct product group $G \doteq S \times R$ on U. We assume that this action is *regular*, i.e., that all the isotropy subgroups

$$H_u = \{(s, r) \in G : su = ur\}, \quad u \in U,$$

are mutually conjugated in G. Since S is fiber preserving, its action on U projects down onto E: $s\pi(u) = \pi(su)$, where $\pi\colon U \to E$ is the bundle projection. The isotropy subgroups of the induced action of S on E are denoted by I_y, $y \in E$. We shall assume that I_y are connected.

2.2. The following Lemma characterizes H_u:

LEMMA. *Given $u \in U$ there exists a unique group homomorphism $\lambda_u : I_{\pi(u)} \to R$ such that*

$H_u = \{(s, \lambda_u(s)) : s \in I_{\pi(u)}\}$.

Proof. λ_u can be read off from $su = u\lambda_u(s)$, $s \in I_{\pi(u)}$. □

COROLLARY. *S acts on E regularly.* □

2.3. Let M be the space of orbits of S in E or, equivalently, of $G \doteq S \times R$ in U. It is standard (see, e.g., [12], Ch. 6) that M is a smooth manifold. Moreover, U and E are endowed with natural fibrations over M (see [12], Ch. 2 and 6, [14] for notation concerning associated bundles; compare also [11], Sect. 2). The relevant structure is summarized in the following theorem:

THEOREM. *Denote $H \doteq H_u$, $I \doteq I_{\pi(u)}$, $\lambda \doteq \lambda_u$ for some fixed $u \in U$, and let $N(H)$ (resp. $N(I)$ be the normalizer of H in G (resp. of I in S). There exists a principal fibre bundle $Q(M, N(H) | H)$ (resp. $P(M, N(I) | I)$ with the base M and the structure group $N(H) | H$ (resp. $N(I) | I$) such that U (resp. E) is isomorphic to a fiber bundle $U(M, H\backslash G)$ (resp. $E(M, I\backslash S)$) over M with a typical fibre $H\backslash G$ (resp. $I\backslash S$) associated with Q (resp. P):*

$$U = Q \times_{N(H) | H} H\backslash G, \qquad E = P \times_{N(I) | I} I\backslash S.$$

The principal bundles Q and P can be considered as submanifolds of U and E given by the following conditions

$$Q = \{u \in U : H_u = H\}, \qquad P = \{y \in E : I_y = I\},$$

and the isomorphisms above are given by

$$Q \times_{N(H) | H} H\backslash G \ni q \cdot [(s, r)] \Leftrightarrow s^{-1}q(r) \in U,$$

$$P \times_{N(I) | I} I\backslash S \ni p \cdot [s] \Leftrightarrow s^{-1}p \in E.$$ □

2.4. The normalizer $N(H)$ of H in G is defined as the largest subgroup of G in which H is normal

$$N(H) = \{(s, r) \in G : (s, r)H(s, r)^{-1} = H\}.$$

It follows from the definition of H that $(s, r) \in N(H)$ iff

(i) $s \in N(I)$,
(ii) $r\lambda(t)r^{-1} = \lambda(sts^{-1})$ for each $t \in I$.

Let $Z \doteq \{r \in R : r\lambda(t)r^{-1} = \lambda(t)\ \forall t \in I\}$ be the centralizer of the image $\lambda(I)$ of I in R. Using (ii) we find $Z \cap H = \{e\}$ so that Z can be identified with the subgroup $Z \approx (e, Z) \subset N(H) | H$.

LEMMA. *Z is a normal subgroup of $N(H) | H$.*

2.5. Let $\mathscr{S}$, $\mathscr{R}$, $\mathscr{G}$, $\mathscr{I}$, $\mathscr{N}$, $\mathscr{N}(I)$ and $\mathscr{Z}$ be the Lie algebras of $S, R, G, I, N(H) \mid H, N(I)$ and Z respectively. They are compact and can be decomposed as follows [11]:

(i)	$\mathscr{G} = \mathscr{S} + \mathscr{R}$	direct sum of Lie algebras,
(ii)	$\mathscr{S} = \mathscr{N}(I) + \mathfrak{L}$	reductive decomposition,
(iii)	$\mathscr{N}(I) = \mathscr{I} + \mathscr{K}$	direct sum of Lie algebras,
(iv)	$\mathscr{R} = \mathscr{Z} + \mathscr{M}$	reductive decomposition.

Observe that the Lie algebras of $K \doteq N(I) \mid I$ and of $N(H) \mid (H \times Z)$ are both isomorphic to $\mathscr{K}$.

PROPOSITION. *The Lie algebra $\mathscr{N}$ of $N(H) \mid H$ is a direct sum of two ideals $\mathscr{N} = \mathscr{K} + \mathscr{Z}$.*

Proof. Let $[(\xi, \eta)] \in \mathscr{N}$, then by 2.4(i) and 2.5(iii) we can make $\xi \in \mathscr{K}$. Then 2.4(ii) implies $\eta \in \mathscr{Z}$. □

The isomorphism described above can be extended to the components of the identity.

COROLLARY. $(N(H) \mid H)_e \simeq K_e \times Z_e$. □

2.6. LEMMA. *Two points $q, q' \in Q \subset U$ are in the same fibre of U (i.e., $\pi(q) = \pi(q')$) iff $q' = qr$ for some $r \in Z$.*

Proof. $\pi(q) = \pi(q')$ iff $q' = qr$, $r \in R$, and the result follows from the observation that by (2.4) we have $N(H) \cap (e, R) = Z$. □

2.7. As a result of 2.6 we get

PROPOSITION. $\pi(Q) \simeq Q/Z \subset P$.

Observe that $\pi(Q)$ is a principal bundle with the structure group $N(H) \mid (H \times Z) \subset K$. It follows, in particular, that if Q can be reduced to Z then $\pi(Q)$ and, *a fortiori*, P are trivial.

3. S-INVARIANT CONNECTIONS IN $U(E, R)$.

3.1. Let ω be a principal connection in $U(E, R)$. ω is a 1-form on the total space U with values in the Lie algebra $\mathscr{R}$ of the initial gauge group R, satisfying

(a) $\omega(\xi_u) = \xi \quad \forall \xi \in \mathscr{R}, u \in U,$ where ξ_u is the fundamental vector generated by ξ at u,

(b) $R_r^* \omega = \mathrm{Ad}(r^{-1})\omega, \quad r \in R.$

Since every $s \in S$ is a bundle automorphism it follows that $L_s^* \omega$ is also a connection form. The connection ω is said to be S-invariant if $L_s^* \omega = \omega$ for all $s \in S$.

PROPOSITION. *An S-invariant connection in U is fully determined by its values at points of the* *bundle* $Q \subset U$.

Proof. Follows from the fact that $QG = U$. □

3.2. To describe S-invariant connections in terms of geometrical objects on M, we have to introduce two vector bundles associated with Q with typical fibers $\mathscr{F}_1$ and $\mathscr{F}_2$ defined as follows

DEFINITION. $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) consists of linear mappings $\phi: \mathscr{K} \to \mathscr{R}$ (resp. $\phi: \mathcal{L} \to \mathscr{R}$) such that

$$\phi \circ \operatorname{Ad}(s) = \operatorname{Ad}(\lambda(s)) \circ \phi, \quad s \in I.$$

Owing to 2.5(iii) and the connectedness of I, every $\phi \in \mathscr{F}_1$ maps $\mathscr{K}$ into $\mathscr{Z}$. It follows now that $\phi \to (s, a)\phi \doteq \operatorname{Ad}(a) \circ \phi \circ \operatorname{Ad}(s^{-1})$ defines a representation of $N(H) | H$ on $\mathscr{F}_i$ $(i = 1, 2)$, so that we can construct associated vector bundles $F_i \doteq Q \times_{N(H) | H} \mathscr{F}_i$.

3.3. PROPOSITION. *There is a* 1–1 *correspondence between S-invariant connections* ω *on* U *and pairs* (μ, Φ) *where*

(a) μ *is a* $\mathscr{Z}$*-valued* 1*-form on* Q *such that*

(a₁) $\mu(\zeta_q) = \zeta, \quad \zeta \in \mathscr{Z}, q \in Q$

(a₂) $R^*_{(s, r)}\mu = \operatorname{Ad}(r^{-1})\mu, \quad (s, r) \in N(H)$

(b) Φ *is a cross-section of the associated bundle* F_2.

Proof. The tangent space T_qU at $q \in Q$ canonically decomposes into a direct sum $T_qU = T_qQ + \mathcal{L}_q + \mathscr{M}_q$, where $\mathcal{L}_q$ and $\mathscr{M}_q$ are the subspaces of fundamental vectors generated by $\mathcal{L}$ and $\mathscr{M}$ at q. Since $\mathscr{M} \subset \mathscr{R}$, $\omega | \mathscr{M}_q$ is fixed by 3.1(a). We define μ as the restriction of ω to T_qQ so that the conditions in (a) are satisfied owing to 3.1(a), (b). On the other hand, the restriction of ω to $\mathscr{M}_q$ gives a value $\Phi_q \in \mathscr{F}_2$ of a unique cross-section Φ of $\mathscr{F}_2$. □

3.4. It is interesting to see whether μ can be related to a connection form on Q. Observe that the embedding of $\mathscr{Z}$ in $N(H) | H$ allows us to interpret the condition (a₂) (ii) in Proposition 3.3 as

$$R^*_g u = \operatorname{Ad}(g^{-1}), \quad g \in N(H) | H,$$

i.e., μ is a 1-form on Q of type Ad.

3.4. LEMMA. *Let* A *be a connection form in the principal bundle* P, *let* $\pi; Q \to P$ *be the restriction of the bundle projection* $\pi: U \to E$, *and let* μ *be as in Proposition* 3.3. *Then the* 1*-forms* $\pi^*A: v \mapsto A(d\pi(v))$ *and* $\mu\pi^*A: v \to \mu((A(d\pi(v)))_q)$, $v \in T_qQ$, *with values in* $\mathscr{K}$ *and* $\mathscr{Z}$ *respectively, are both of type* Ad *on* Q.

Proof. Follows since A is of type Ad (with respect to K) on P. □

In the Proposition below we denote by $B_{\mathscr{K}}$ the part $\operatorname{pr}_{\mathscr{K}} \circ B$ of B corresponding to the direct sum decomposition $\mathscr{N} = \mathscr{K} \oplus \mathscr{Z}$.

3.5. PROPOSITION. *Let* A *be a fixed connection in* P. *There is a natural* 1–1 *correspondence* *between 1-forms μ on Q satisfying the conditions* (a_1), (a_2) *of Proposition* 3.3 *and pairs* (B, ψ), *where B is a principal connection in Q such that* $B_{\mathscr{K}} = \pi^*A$, *and* ψ *is a cross-section of the vector bundle* F_1. The correspondence *is given by*

$$B \doteq \mu + \pi^*A - \mu\pi^*A, \qquad \psi_q(\xi) \doteq \mu(\xi_q), \quad \xi \in \mathscr{K},$$

where ξ_q *is the fundamental vector generated by* ξ *at* $q \in Q$.

Proof. By Lemma 3.4. B is of type Ad on Q. One easily checks that for every $\xi \in \mathscr{N} = \mathscr{K} \oplus \mathscr{Z}$ we have $B(\xi_q) = \xi$ so that B is a connection form on Q. Clearly $B_{\mathscr{K}} = \pi^*A$ since $\mu_{\mathscr{K}} = 0$. Conversely, given B and ψ one gets back the 1-form μ from the formula $\mu = B - \pi^*A + \psi\pi^*A$. It is easy to see that the connection form B constructed above out of ω and A is natural in the following sense: the horizontal lift $\chi_B: TM \to TQ \subset TU$ defined by B is the composition $\chi_B = \chi_\omega \circ \chi_A$ of the horizontal lift $\chi_A: TM \to TP \subset TE$ of A and $\chi_\omega: TE \to TU$ of ω. □

3.6. COROLLARY. *Given a connection A in P there is a* 1–1 *correspondence between S-invariant connections* ω *in U and triples* (B, ϕ, ψ), *where B is a connection in Q such that* $B_{\mathscr{K}} = \pi^*A$ *and* ϕ, ψ *are cross-sections of the bundles* F_2 *and* F_1 *respectively.*

4. CONCLUDING REMARKS AND EXAMPLES

4.1. As was observed in 2.7, in general there is no natural reduction which gives a bundle with the structure group Z – the centralizer of $\lambda(I)$ in R. Such a reduction is possible only if P is trivial. In many cases (in particular in the six-dimensional model of Manton [8]), P is a bundle with a discrete structure group (see [11]), but then it need not be trivial (as it is, for example, with the Moebius strip). This global topological structure is important for monopole-like solutions.

4.2. To construct a connection in Q, according to Corollary 3.6, one must choose a connection A in P. As was shown recently [11], such a connection is provided by a reduction of S-invariant metric on E. Then one finds that the reduction of the Yang–Mills action for ω produces a correct action for B and a number of scalar interactions, provided one adds a kinetic term for A. This is automatically achieved by a simultaneous reduction of Yang–Mills and Einstein actions of the metric. Such a simultaneous reduction has been suggested in [13].

4.3. The same results can be also obtained through the Kaluza–Klein ansatz. Given an S-invariant metric and an S-invariant gauge field on E, we can use the Killing form on R to construct an $S \times R$ invariant metric on U. Then the analysis given in [11] can be applied to give as an output a metric on M, a connection B in Q, and a certain number of scalar fields. Some of these scalars are our ϕ and ψ, the others being either trivial (constants) or coming from the reduction of the Einstein action on E. Both methods, that of the present paper as well as the one of [11], lead to the same final gauge group $N(H) \mid H$. One should, however, remember that a further reduction may occur as a result of the Higgs mechanism.

4.4. To illustrate our results we will analyze briefly a simple example. Let us consider an Einstein–Yang–Mills theory with the gauge group R = SU(5) over d = 11 multidimensional universe E, with S = Usp(4) as a symmetry group. S contains SU(2) × SU(2) as a maximal subgroup [15], and we assume that the isotropy subgroup I = SU(2) of S acting on E is one of the SU(2) factors in this maximal subgroup. Then, locally, $E = M \times S_7$, where M is the final four-dimensional spacetime. The maximal subgroup of R is SU(3) × SU(2) × U(1). Let us choose the homomorphism $A: I \to R$ as an isomorphism onto the SU(2) factor of the maximal subgroup in R. Now one easily finds the relevant groups: G = USp(4) × SU(5), H = diag(SU(2) × SU(2)), $N(I) \mid I$ = SU(2), Z = SU(3) × U(1). The resulting gauge group $N(H) \mid H$ = SU(3) × SU(2) × U(1). (These equalities should be understood as modulo discrete subgroups. However, it is sufficient as far as the field content of the final theory is concerned.) The symmetric gauge field together with the gauge field coming from the symmetric metric give rise to a gauge field on M with the gauge group SU(3) × SU(2) × U(1), 30 scalar fields of type $\mathscr{F}_1$ (see 2.5 and 3.2) which are given as a linear mapping from $\mathscr{K}$ = su(2) into $\mathscr{Z}$ = su(3) + u(1), and 12 scalar fields of type $\mathscr{F}_2$ (see 2.5 and 3.2) which are intertwining mappings between equivalent two-dimensional irreducible components of the representation of I = SU(2) on $\mathcal{L}$ and $\mathscr{R}$. The detailed analysis of the action for this model, and other examples of symmetric metrics and gauge fields will be given elsewhere [16].

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