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LIE BALLS AND RELATIVISTIC QUANTUM FIELDS

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Abstract
We investigate some aspects of the complex domain $SO(4,2)/(SO(4)\times SO(2))$ in relation with relativistic quantum mechanics and conformal invariance.

1. Geometrical aspects of Lie Balls.

In the present paper, we are mainly interested in the four dimensional (complex) Lie ball that we shall denote by $\mathcal{D}$. This smooth manifold can be written as $SO_0(4,2)/SO(4)\times SO(2)$ or as $SU(2,2)/S(U(2)\times U(2))$. Because of the local isomorphism between $G = SU(2,2)$ and $SO(4,2)$, $\mathcal{D}$ is a bounded non compact symmetric domain of type I and IV. $\mathcal{D} = G/H$ is a Kähler manifold for its G-invariant metric (which coincides with its Bergman metric) and is also Einstein. $\mathcal{D}$ is in particular a complex manifold with (integrable) complex structure j_0. It is also a non compact Hermitian homogeneous manifold for the action of the conformal group $SO(4,2)$ of Space-Time and is of rank two as a homogeneous space. Moreover it is a symmetric quaternion-Kahler manifold (hence quaternion hermitian and quaternionic) but not hyperkahlerian. As such it admits a twistor space which is also a complex manifold fibrated as a bundle above $\mathcal{D}$ with CP^1 fibers. As a topological space, $\mathcal{D}$ is homeomorphic with R^8 and is therefore a manifold without boundary. However, from its realisation as a bounded domain of C^4 or from its realisation as a subset of its compact dual, the Grassmanian $SO(6)/(SO(4)\times SO(2))$, via the Harish-Chandra embedding, it admits a weak boundary which is stratified under the action of the stabiliser $SO(4)\times SO(2)$. One of the strata is actually a singular four dimensional orbit and is of special interest for us since it is diffeomorphic with $S^3\times_{Z_2} S^1$, i.e. with compactified Minkowski Space-Time. This particular orbit is both a quotient of $S(U(2)\times U(2))$ by a diagonal $SU(2)$ and a quotient of the conformal group itself by the semi-direct product of a Poincare subgroup times the subgroup of dilations. The metric of $\mathcal{D}$ is euclidean and blows up near the boundary (as in the usual geometry of Lobatchevski) but, what is of particular interest here is that it induces a conformal Lorentz structure on the boundary. The domain $\mathcal{D}$ is a Lie ball in the

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sense that it is a ball for the Lie distance defined as follows. Let

$$\zeta = \alpha + i\beta = (\zeta_1, \zeta_2, \ldots, \zeta_4) \in C^4,$$
$$\bar{\zeta} = \alpha - i\beta = (\bar{\zeta}_1, \bar{\zeta}_2, \ldots, \bar{\zeta}_4) \in C^4,$$
$$Q(\zeta) = \zeta_1^2 + \zeta_2^2 + \ldots + \zeta_4^2 \in C,$$
$$< \zeta, a > = \bar{\zeta} a = \zeta_1 \bar{a}_1 + \zeta_2 \bar{a}_2 + \ldots + \zeta_4 \bar{a}_4,$$
$$\|\zeta\| = \bar{\zeta}\zeta = < \zeta, \zeta >^{1/2},$$

The Lie norm is defined as

$$L(\zeta) = [\|\zeta\|^2 + \sqrt{\|\zeta\|^4 - |Q(\zeta)|^2}]^{1/2}$$

and the Lie distance between ζ_1 and ζ_2 as $L(\zeta_1 - \zeta_2)$. The Shilov boundary (compactified Minkowski Space-Time) can be defined as the Lie sphere $\{\zeta \in C^4 / \|\zeta\|^2 = |Q(\zeta)| = 1\}$. The domain $\mathcal{D}$ also admits an unbounded realization: the future tube. It can be defined as the space of all $w = x + iy$, with $x \in R^4$ and $y \in R^4$ with the constraint $y_0^2 - y_1^2 - y_2^2 - y_3^2 > 0$. The map sending the bounded realization of $\mathcal{D}$ (the Lie ball) to the unbounded realization (the forward tube) is a generalized Cayley transform (the usual Cayley transform sending the open unit disk to the upper half plane). This last unbounded realization of the Lie ball admits a simple physical interpretation. Indeed, by computing the expression of the momentum map (the Poincare-Cartan form) on $\mathcal{D}$ viewed as a symplectic manifold with a symplectic action of the group $SO(4,2)$, one can show [3] that the imaginary part y of $z = x + iy$ can be interpreted as the inverse of a momentum (it is associated with the translation subgroup of $SO(4,2)$. it is therefore natural to set $p^\mu = y^\mu / y^2$. Points of the domain $\mathcal{D}$ describe therefore both the position (in space and time) and the momentum (with $p^2 > 0$) associated with a physical event. The domain itself becomes therefore a curved relativistic phase-space. Interpretation of $Im(z)$ as an inverse momentum is an obvious four-dimensional generalization of what is done in usual wavelet analysis (where the variable ν in $z = t + i/\nu$ is interpreted as a frequency). Special conformal transformations act both on Space-Time and inside the domain but they do not act in the same way [7]. Introducing a constant h and setting $z = x + ihp/p^2$, it can be shown [3] that the action of the conformal group G on Space-Time (variable x) and on its cotangent bundle (variable p) can be gotten from the action of G on $\mathcal{D}$ but only in the h goes to zero limit.

2. Analytical aspects of Lie balls.

The group $SO(4,2)$ defined in a purely algebraic way as the group of conformal transformations for a Lorentzian Space-Time of signature $(1,3)$ coincides with the group of analytical diffeomorphisms of the Lie ball $\mathcal{D}$. It is also interesting to notice that the same Lie ball appears as the harmonicity cell of the four-dimensional Euclidean ball [1]. Representations of the maximal compact subgroup of $G = SO(4,2)$ can be used to build equivariant vector bundles over the Lie ball $\mathcal{D}$. The space of sections of these bundles provide representation spaces for G via the induction mechanism. Is is then possible to define a scalar product in such a space of sections and consider subspaces of holomorphic (or anti-holomorphic) square-integrable sections carrying irreducible unitary representations of G. It is possible to define a Bergman kernel for each of these spaces. Let us only mention here how to construct this kernel in the following particular case: we consider holomorphic densities of weight 1 on $\mathcal{D}$, i.e., holomorphic sections of the line bundle corresponding to the representation $A \to det(A)$ of $GL(n,C)$. These sections are therefore holomorphic n-forms of type (1,0). Indeed $\Phi = \Phi(z)d^4z = \Phi'(z')d^4z'$ with $\Phi'(z') = [det(\partial z/\partial z')]\Phi(z)$. They can be defined (locally) as holomorphic functions in each coordinate system. Realizing the space $\mathcal{D}$ as a bounded domain of C^4, we build the Hilbert space $\mathcal{H}^2$ of holomorphic n-forms of type (1,0) that are square integrable w.r. to the measure $d^4zd^4\bar{z}$ of C^4. The Bergman kernel "function" of $\mathcal{D}$ associated with this line bundle, in this bounded realization, is then defined as $k(z,\bar{w}) = \Sigma\psi_n(z)\bar{\psi}_n(w)$. It is therefore itself a holomorphic (resp. anti-holomorphic) density of weight 1 w.r. to z (resp. w.r. to w). The previous scalar product can also be written as

$$(\phi,\psi)_l = \int \phi(z)\psi(\bar{z})d\mu(z) = \int \frac{\phi(z)\psi(\bar{z})}{k(z,\bar{z})}d\mu(z)$$

where $d\mu(z) = k(z,\bar{z})d^nzd^n\bar{z}$ is the intrinsic measure of $\mathcal{D}$ associated with its Kähler metric. The Bergman kernel has the reproducing property

$$f(z) = \int_{\mathcal{D}} f(\xi)k(z,\xi)dvol(\xi),$$

where f is in $\mathcal{H}^2$. The Bergman kernel can be also constructed for other choices of associated vector bundles. In the present situation, there is (up to a normalization factor) a unique vector in $\mathcal{H}^2$, that is orthogonal to all the ψ that vanish at the point z of $\mathcal{D}$. The coherent state (of unit norm) at the point z is therefore defined precisely as this unique vector. It will be denoted by $|z>$. This is actually a very general definition of coherent states, it works whenever the evaluation functional vanishes on a hyperplane and is continuous. This definition was given in [3]. The basic relations of the calculus with coherent states can be written using the traditional (and very convenient) notations of quantum mechanics. They read as follows. Calling $|\psi>$ an arbitrary element of $\mathcal{H}^2$, $|\phi_n>$ an orthonormal basis and $\psi(z)$ the evaluation of ψ at the point z, we have:

$$k(\bar{z},z) = \Sigma\phi_n(\bar{z})\phi_n(z)$$

$$<z|\psi> = \psi(z)/k(\bar{z},z)^{1/2}$$

$$<\psi|z> = \psi(\bar{z})/k(\bar{z},z)^{1/2}$$

$$<\phi_n|z> = \phi_n(z)/k(\bar{z},z)^{1/2}$$

$$1 = |\phi_n><\phi_n|$$

$$|z> = 1|z> = \Sigma|\phi_n><\phi_n|z> = \Sigma\phi_n(\bar{z})/k(\bar{z},z)^{1/2}$$

$$<z|z> = 1$$

$$<z_2|z_1> = \Sigma <z_2|\phi_n><\phi_n|z_1> = k(\bar{z}_2,z_1)/k(\bar{z}_2,z_2)^{1/2}k(\bar{z}_1,z_1)^{1/2}$$

$$1 = \int |z><z|\,k(\bar{z},z)\,d\bar{z}dz$$

The last equation displays the reproducing kernel property. To each function $f \in \mathcal{L}^2$ (not necessarily in $\mathcal{H}^2$, i.e. not necessarily holomorphic) we associate an operator F acting in $\mathcal{H}^2$ defined as $F = \int |z> f(z) <z|d\mu$ where $d\mu = k(\bar{z},z)d\bar{z}dz$ is the natural invariant measure

on the Lie ball. Calculations are very similar to those made in non relativistic quantum mechanics using the Bargman coherent states associated with the harmonic oscillator. The Bargman kernel $exp(-\bar{z}z)$ is here replaced by the Bergman kernel of $\mathcal{D}$ (calculated by [6]) i.e. $k(\bar{z},z) = (2^3 4!/\pi^4)1/[1+\bar{z}^2 z^2 - 2\bar{z}z]^4$. This approach to the pseudo-differential calculus on classical domains has been investigated already in [2], (cf. also [8]) but what makes it into a new and open subject is the physical identification of Space-Time with the Shilov boundary of $\mathcal{D}$ together with the interpretation of the imaginary part of the complex variable $z_\mu = x_\mu + iy_\mu$, in the unbounded representation, as the inverse of a momentum($y_\mu = p_\mu/p^2$). Notice that, in this representation, the Bergman kernel for holomorphic densities of weight 1 is equal (up to a constant) to $1/[(z_1 - \bar{z}_2)^2]^4$. Here the square is computed with a Lorentz metric of signature $+---$. We should recall here the analytic definition of the Shilov boundary S, namely, the smallest closed subset of the boundary where every element of any set of (non constant) holomorphic functions in the domain reaches its maximum (in module). Elements of the various Hilbert spaces of interest usually approach a distribution when their argument tends to the Shilov boundary S. Only elements of a small subspace of these Hilbert spaces approach square-integrable functions on S. Conversely, distributions (or hyperfunctions) on S can be extended to $\mathcal{D}$. Notice in particular that any holomorphic bounded function in $\mathcal{D}$ can be written as

$$\psi(z) = \int_S \psi(\zeta)s(z,\zeta)d\mu(\zeta),$$

where $d\mu$ is a measure on S, and $s(z,\zeta)$ is holomorphic in z and integrable with respect to $d\mu$ in ζ. The kernel function $s(z,\zeta)$ is called the Szegő kernel of the domain. This suggests that a relativistic analogue of the wavelet transform [4],[5] should be defined via the use of the Szegő kernel.

The study of analyticity properties of n-points functions is a rather traditional field of research in Relativistic Quantum Mechanics and in Quantum Field Theory. Many results are known thanks to the work of a generation of theoretical physicists (and in particular thanks to the efforts of Raymond Stora [9-12] who has always been a master in this area (and in others..!)). What we suggest here is a kind of different game: analytic continuation from the real line to the complex plane, with its flat euclidean geometry, is not the same as going from the real line (or from the circle) to the Poincaré upper half-plane (or to the disk), with its curved Lobatchevskian geometry. This one-dimensional (complex) comparison (and contrast) sits at the roots of the message carried by the authors of [3]. Our belief is that Physics is "simple" (and euclidean) in the domain $\mathcal{D}$ and that many of the difficulties of classical or quantum physics arise because we try to go to the "boundary" and to formulate the laws of Physics there. Many mathematical properties of the space $\mathcal{D}$ (and of other classical domains) are already known but the use of those properties in Physics is a new and open subject. In particular, new physical intuitions have to be developed (for instance, everybody knows what the Fourier transformation is, and understands its physical interpretation, but here, what we have is rather a (Radon)-Gelfand-Graev transformation -i.e. integration over horocycles- and its physical interpretation is quite different and not necessarily familiar...). Much remains to be done. Here, we have only sketched a few properties of the appropriate mathematical structures (more can be found in [3]). The present article is also a kind of invitation to further study.

References

[1] V. Avanissian. "Cellule d'harmonicite et prolongement analytique complexe". Coll. Travaux en Cours. Hermann, Paris 1985.

[2] F.A. Berezin, "Quantization", Math. USSR Izvestja Vol. 8 (1974), No. 5. "Quantization in complex symmetric spaces", Math. USSR Izvestja Vol. 9 (1975), No. 2.

[3] R. Coquereaux, A. Jadczyk, "Conformal Theories, Curved Phase Spaces, Relativistic Wavelets and the Geometry of Complex Domains", CPT-89/P.2302, to appear in Reviews in Math. Phys.

[4] A.Grossmann, J.Morlet, T.Paul, "Transforms associated to square integrable representations" I. J. Math. Phys. 27, 1985, pp 2473-79

[5] A.Grossmann, J.Morlet, T.Paul, "Transforms associated to square integrable representations II. Ann. Inst Henri Poincaré 45,1986,pp 293-309.

[6] L.K.Hua. "Harmonic analysis of functions of several variables in the classical domains" Translation of mathematical monographs 6, Amer. Math. Soc., Providence, R.I., 1963.

[7] A. Odzijewicz. "A model of Conformal Kinematics." Inter. Jour. of Theor. Phys. Vol 15, No 8 (1976) pp. 575-593

[8] Unterberger. "Analyse harmonique et analyse pseudo-differentielle du cne de lumiere". Asterisque 156. 1987.

[9] H. Epstein, V. Glaser, R. Stora. "General properties of the N Point functions in local quantum field theory". In Les Houches 1975, Proceedings.

[10] H. Epstein, V. Glaser, R. Stora. "Geometry of the N Point space functions of quantum field theory". In Nice 1973, Lecture Notes in Mathematics, Vol. 449, Berlin 1975.

[11] R. Stora. "Some applications of the Jost-Lehmann-Dyson theorem to the study of global analytic structure of the N-point function of quantum field theory". In Nice 1973, Proceedings, Lecture Notes in Mathematics, Vol 449, Berlin 1975.

[12] J. Bros, H. Epstein, V. Glaser, R. Stora. "Some global aspects of the edge of the wedge problems".In Nice 1973, Proceedings, Lecture Notes in Mathematics, Vol 449, Berlin 1975.