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A Note on Conformal Field Equations[1]

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Conformal geometry is more fundamental than a Riemannian one. Whereas Riemannian geometry determines lengths and angles, a conformal one determines only angles and ratios of length. Equivalently, conformal geometry of space-time determines light cones, or causal structure. No length scale is *a priori* distinguished. It can be distinguished only *a posteriori*, given a particular solution of matter field equations. Einstein's field equations of gravitation can be thought of as describing interaction of causal structure with a matter described by a real scalar massless field of weight 1/4. Electromagnetic field equations need precisely a conformal structure. One can also write down field equations for a spin-1/2 Dirac massless field, given information about light cones only. The energy-momentum tensor density is obtained by vierbeim variations.

1. CONFORMAL STRUCTURE OF SPACE-TIME

Let M be a smooth, 4-dimensional manifold, thought of as being a model of space-time. Let $B(M)$ be the bundle of linear frames over M. Then $B(M)$ is a principal bundle with the structure group $GL(4)$. Let G be a Lie subgroup of $GL(4)$. A G structure on M is a smooth subbundle of $B(M)$, with structure group G. In many interesting cases the structure group can be described as a group leaving invariant some tensorial object on R^4. For example, to give M a pseudo-Riemannian structure is to give it an $O(1, 3)$ structure, and $O(1, 3)$ is a subgroup of $GL(4)$ leaving invariant the standard metric tensor $\eta = \eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$. Similarly, to give M a conformal structure (plus orientation) is to give it a $CO_+(1, 3)$ structure, and $CO_+(1, 3)$ is the group of all $A \in GL(4)$ leaving invariant (pseudo)tensor

$$\overset{*}{\chi}{}^{ab}_{cd} = \tfrac{1}{2}\epsilon_{cdef}\eta^{ea}\eta^{fb} \tag{1.1}$$

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Let P be a conformal structure on M. The frames in P are called conformal frames (of the first order). According to a general theorem (see, for example, Kobayashi, 1972), P is integrable (flat) iff each point of M admits a coordinate neighborhood, with local coordinates $x^0, \ldots, x^3$, with respect to which the components of χ coincide with the standard ones [equation (1.1)]. The tensor χ is nothing but a Hodge $*$ operator restricted to 2-forms. Therefore, modulo topological subtleties, to give M a conformal structure is to give it a smooth $*$ operation acting linearly on the bundle of 2-forms, and satisfying (i) $*^2 = -I$, and (ii) $*F \wedge \tilde{F} = F \wedge *\tilde{F}$ (Jadczyk, 1978). Though χ determines conformal structure completely, it is much more convenient to deal with a tensor density γ_{mn} uniquely defined by

$$\chi_{pq}^{mn} = \tfrac{1}{2}\epsilon_{pqrs}\gamma^{rm}\gamma^{sn} = -\tfrac{1}{2}\gamma_{pr}\gamma_{qs}\epsilon^{rsmn} \tag{1.2}$$

(Gûrsey, 1963, makes use of γ in his "reformulation of general relativity in accordance with Mach's principle".) An equivalent definition of γ can be described as follows: let (x^m) be a local coordinate system, and let (E_a) be a local section of the bundle of conformal frames (conformal vierbeim field). Let $E_a = E_a{}^m\partial_m$, where ∂_m are the tangents to coordinate lines. Define then

$$\gamma^{-1} = \gamma^{mn} = |\det E|^{-1/2}E\eta^{-1}({}^tE) \tag{1.3}$$

Then γ_{mn} is, in fact, independent of E, and is a symmetric tensor density of weight $W(\gamma_{mn}) = -\frac{1}{2}$, and $\det\gamma = -1$. As it was above, a conformal structure is flat iff there are local coordinate systems in which $\gamma_{mn} \equiv \eta_{mn}$. As is well known, a necessary and sufficient condition for the existence of such a coordinate system is that the Weyl conformal curvature tensor (which can be expressed in terms of γ_{mn} only) vanishes identically.

Assume now that a conformal structure γ is given. Let φ be a scalar density of weight $W(\varphi) = 1/4$. Then $g_{mn} = \varphi^2\gamma_{mn}$ is a metric tensor on M. In this way one gets a correspondence between conformal structures and classes of conformally equivalent Riemannian metrics. It follows, in particular, that each scalar density of weight 1/4 determines a symmetric affine connection preserving the conformal structure. However, contrary to the Riemannian case, no such an affine connection is distinguished.

2. CONFORMAL FIELD EQUATIONS

Usually a field equation is said to be conformally invariant if it refers to the flat conformal structure of the Minkowski space, and is invariant under the whole 15-parameter group of local conformal automorphisms. To check whether this holds one has to specify transformation properties of the field under these automorphisms. It is then usually possible to deduce from these transformation laws what kind of a geometric object we are dealing with. It is also natural to assume that the field equations (and a Lagrangean) can be expressed in terms of the conformal structure only. However, the situation here is not as simple as in the Riemannian case, since no prescription such as "replace derivatives by the covariant ones" is possible. Indeed, if we have a differential operator D acting on sections of some fiber bundle of geometrical objects, and if D can be canonically expressed in terms of the conformal structure only, then the manifold of solutions of the equation $D(\cdot) = 0$ is invariant under all automorphisms of the conformal structure (if such automorphisms exist). In particular, in flat Minkowski space, the equation is automatically invariant under the 15-parameter conformal group, and transformation laws of the field follow then from its geometrical character. Restricting ourselves to first-order, linear objects, it is enough to specify a representation of $CO_+(1, 3)$ on some vector space. In the case of conformal spinors two-valued representations are allowed. But, since $CO_+(1, 3) = SO(1, 3) \times R^+$, to specify a geometric character of the field means to specify its tensorial or spinorial character, and a representation of dilations which commute with proper Lorentz transformations. If $\lambda \mapsto \lambda^k$ is a representation of dilations, it is usual that only for quite special values of k (canonical dimension) can one find canonical linear operators acting on sections of the corresponding associated bundle. This will be illustrated for scalar and spinorial fields. In both cases, however, the canonical operator itself has a dimension, i.e., is a map from one associated bundle to another. Nevertheless, in the discussed cases one easily finds an invariant Lagrangean, and energy-momentum tensor density can be defined by taking variation over the vierbeim

$$2T_n{}^m := \frac{\partial L}{\partial E_a{}^n}(E^{-1})^m{}_a \tag{2.1}$$

It is a 1-form with values in vector densities of weight $W = 1$, and is automatically traceless, symmetric, i.e., $\gamma_{mr}T_n{}^r = \gamma_{nr}T_m{}^r$. Moreover, T is conserved in the following sense: if X is a vector field, and if $T_X{}^m := T_n{}^m X^n$, then

$$\partial_m T_X{}^m := \tfrac{1}{2} T_{rs} \mathscr{L}_X(\gamma^{rs}) \tag{2.2}$$

where $\mathscr{L}_X$ is the Lie derivative with respect to X. In particular, $\partial_m T_X{}^m = 0$ if X is a Killing vector field of the conformal structure.

3. SCALAR FIELD

Let φ be a scalar field of weight $W(\varphi) = k$, $k \neq 0$. With respect to a vierbeim field E, φ takes the value $\varphi(E) = |\det E|^k \varphi$. It follows that for dilations $\varphi(\lambda E) = \lambda^{4k}\varphi(E)$. The simplest method to guess a Lagrangean is by observing that $g_{mn} = \varphi^{k/2}\gamma_{mn}$ is a symmetric tensor. Therefore we take

$$L = \tfrac{1}{2}|\det g|^{1/2}R(g) \tag{3.1}$$

The stationary action principle gives then the following field equation:

$$(\hat{\Box} + \tfrac{2}{3}k\hat{R})\varphi = (1 - 4k)\varphi^{-1}(\partial_m\varphi)\partial^m\varphi \tag{3.2}$$

where

$$\hat{\Box} = \partial_m\gamma^{mn}\partial_n$$

and $\hat{R}$ is obtained from γ according to the "take a scalar curvature" prescription. This equation is linear only for $k = 1/4$ (see Sigal, 1974, for a complementary statement), and reduces to a simple "wave equation":

$$(\hat{\Box} + \tfrac{1}{6}\hat{R})\varphi = 0 \tag{3.3}$$

[See Penrose, 1965, where (3.3) is also considered in slightly different language.] If the conformal structure is integrable, one can choose local coordinates in which $\gamma_{mn} = \eta_{mn}$, and one gets the familiar $\Box\varphi = 0$. It is worthwhile to note that if a "cosmological term" is added to the Lagrangean (3.1), we get $\lambda\varphi^3$ as the term on the right-hand side of (3.3). Still restricting ourselves to the case of $W(\varphi) = 1/4$, i.e., $\varphi(\lambda E) = \lambda\varphi(E)$, one easily gets the energy-momentum "tensor" (here $\varphi_m \overset{df}{=} \partial_m\varphi$, $\varphi^m = \gamma^{mn}\varphi_n$):

$$\begin{aligned} T_n{}^m = {} & \varphi^m\varphi_n - \tfrac{1}{4}\delta_n{}^m\varphi_r\varphi^r - \tfrac{1}{2}\varphi\hat{\nabla}^m\varphi_n \\ & + \tfrac{1}{2}\delta_n{}^m\varphi\hat{\Box}\varphi - \tfrac{1}{4}\varphi^2(\hat{R}_n{}^m - \tfrac{1}{2}\hat{R}\delta_n{}^m) \end{aligned} \tag{3.4}$$

Of course, one can eliminate the term containing $\hat{\Box}\varphi$ by use of the field equation. It follows, in particular, that gravitation may be considered as a result of interaction of a conformal structure with a scalar density field of weight 1/4 (canonical dimension).

4. ELECTROMAGNETIC FIELD

Electromagnetic field is described by its vector potential, i.e., a 1-form A. Then $F = dA$, and $L = -\tfrac{1}{2}F \wedge *F$. The field equation is $d*dA = 0$. Clearly, only conformal structure is involved. Observe that there is a one-to-one correspondence between conformal structures and "constitutive tensors of matter-free space" (cf. Post, 1962). The energy-momentum tensor is the standard one.

5. SPIN-1/2 MASSLESS DIRAC FIELD

To discuss spinors one has to look at representations of the covering group of $CO_+(1, 3)$. Let (γ_a) be a fixed set of γ matrices in C^4, so that

$[\gamma_a, \gamma_b]_+ = 2\eta_{ab}$. They generate the Clifford algebra of the Minkowski space. Consider the group G of all elements S of the algebra satisfying the following: (i) $S\gamma_a S^{-1} = \gamma_b A_a{}^b$, (ii) S is an even element of the algebra, and (iii) $SS^J > 0$, where $(\gamma_{a_1} \cdots \gamma_{a_N})^J = \gamma_{a_N} \cdots \gamma_{a_1}$. Define the two-to-one homomorphism ρ of G into $GL(4)$ by

$$S\gamma_a S^J = \det \Lambda(S)^{-1}\gamma_b \Lambda(S)_a{}^b$$

to which there corresponds the following representation of the Lie algebra of $CO_+(1, 3)$:

$$M \mapsto D(M) = -\tfrac{3}{8}\,\mathrm{Tr}\,(M) + \tfrac{1}{8}M^{ab}[\gamma_a, \gamma_b] \tag{5.1}$$

Assume now that F is a complex vector bundle over M with a typical fiber C^4, and let P be a reduction to G of the bundle of complex frames of F. Finally, let q be a homomorphism of P into $B(M)$, which commutes with the representation ρ. If $\tilde{E}$ is a local section of P, then $q(\tilde{E}) \coloneqq E$ is a vierbeim field, and q is uniquely determined by E, if $\tilde{E}$ is fixed. Let us fix q. Then define

$$\gamma(\tilde{E})^m = \det(E)^{-1/4} E_a{}^m \gamma^a \tag{5.2}$$

It is easy to see that γ^m does not, in fact, depend on $\tilde{E}$. Under coordinate transformations $x^m \mapsto x^{m'}$ one has

$$\gamma^{m'} = \left|\frac{\partial x}{\partial x'}\right|^{1/4} \frac{\partial x^{m'}}{\partial x^m}\,\gamma^m$$

so that

$$\gamma^{mn} \coloneqq \tfrac{1}{2}[\gamma^m, \gamma^n]_+$$

determines a conformal structure on M (which, of course depends on q). There is no distinguished linear connection in M (however, it is easy to see that there is a canonical nonlinear one). Let φ be any scalar density of weight 1/4, and let $g_{mn} = \varphi^2\gamma_{mn}$. Consider the Levi-Civitta connection of g_{mn}. By continuity this connection lifts by the bundle homomorphism q to the bundle P, and thus to F. Let ∇ be the corresponding covariant derivative acting on spinor fields (i.e., sections of F). ∇ depends now on φ. However, owing to the fact that we have chosen a power "-1" in the definition of a group homomorphism ρ, the operator $D \coloneqq \gamma^m\nabla_m$ happens to be independent of φ. Explicitly,

$$(D\psi)(\tilde{E}) = \gamma^m\partial_m\psi(\tilde{E}) + \tfrac{1}{8}|E|^{-1/4}E^{am}(E^{-1})_n{}^b(\partial_m E^{cn})\gamma_{abc}\psi(\tilde{E})$$

and D is independent of $\tilde{E}$, and of the coordinate system. Here $\psi(\tilde{E})$ are the components of ψ with respect to the complex basis $\tilde{E}$, i.e., $\psi(\tilde{E}) \in C^4$. The Dirac equation now reads $D\psi = 0$. Observe that D gives to ψ an additional weight 1/4. To find an invariant Lagrangean we have to define a scalar product between sections of F. The following definition

$$(\psi, \psi') := |E|^{-3/4}\overline{\psi(\tilde{E})}\psi'(\tilde{E})$$

is independent of $\tilde{E}$, and is a scalar density of weight 3/4. It follows that $(\psi, \psi)^{-1/3}D\psi$ is again a section of F, so that we can define

$$L = -\operatorname{Im} (\psi, \psi)^{1/3}(\psi, (\psi, \psi)^{-1/3}D\psi)$$

which is a density of weight +1, as it should be. Clearly one can always add a term proportional to $(\psi, \psi)^{4/3}$ (compare Gürsey, 1956). The energy-momentum tensor corresponding to this Lagrangean has the form

$$T_m{}^n = -L\delta_m{}^n + \frac{i}{2}|E|^{3/4}[\bar{\psi}\gamma^n\partial_m\psi - \gamma^n(\partial_m\bar{\psi})\psi]$$

$$+ \frac{i}{16}|E|^{-3/4}\bar{\psi}[\gamma_r[\partial_m\gamma^r, \gamma^n]]_+\psi$$

$$- \frac{i}{16}\partial_r(|E|^{-3/4}\bar{\psi}[\gamma_m, [\gamma^n, \gamma^r]]_+\psi)$$

Specifying to a flat case one should put $E = I$, and take for γ_m standard Dirac matrices. It is worthwhile to observe that if we take $(\psi, \psi)^{1/3}$ for φ, a theory of gravitation with gravitational "potential" of spinorial character can be obtained. It is, however, hopelessly nonlinear.

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